

# Two-Scale Homogenization of a Robin Problem in Perforated Media

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## Abstract

The two scale convergence of the solution to a Robin's type-like problem of a stationary diffusion problem in a periodically perforated domain is investigated. It is shown that the Robin's problem converges to a problem associated to a new operator which is the sum of a standard homogenized operator plus an extra first order "strange" term; its appearance is due to the non-symmetry of the diffusion matrix and to the non rescaled resistivity.

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## 1 Introduction

Periodic homogenization in perforated media with Robin boundary conditions prescribed on the boundary of the holes has been extensively studied by many authors and we refer for instance to [5], [7], [13],... In this paper we study the stationary diffusion equation in a periodic perforated body where the heat flow is proportional to the temperature field on the boundary of the holes with a resistivity having zero average value on the boundary of the reference hole. In [5], the authors studied a model problem for a second-order symmetric elliptic operator in a periodically perforated domain with the Robin boundary condition prescribed on the boundary of the holes. They use the asymptotic expansion technique [6], [14], [15] to obtain the homogenized problem and they construct correctors to justify the expansion and then estimate the error. In

this paper, we consider a problem but with another configuration, namely the holes may or may not be connected and the boundary of the holes may intersect the exterior boundary of the body. Moreover we assume that the matrix diffusion of the second-order operator may or may not be symmetric. We use the two-scale convergence technique [3], [9], [11], [12] to obtain the two-scale limit system. After the decoupling technique, we show that the homogenized problem contains a convective term. Its appearance is due essentially to the general character of the matrix diffusion and on the fact that the resistivity function is not rescaled as usually assumed when dealing with two-scale convergence on periodic surfaces, see for instance [2], [4], [10],...

The paper is organized as follows: in section 2, we define the geometry of the perforated body and we give the Robin boundary-value problem setting. Section 3 is aimed at showing the existence and unicity of the solution of the Robin problem and obtaining a priori estimates of the solution. The asymptotic limit via two-scale convergence procedure is analyzed in section 4. We obtain the homogenized boundary-value problem which is a second order elliptic operator containing first and zero order terms. The latter term is classical see e.g. [4]. The first order term is a convection one and it is null when the diffusion matrix is symmetric or with constant coefficients.

## 2 Setting of the Problem

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  of variable  $x = (x_i)_{1 \leq i \leq n}$  ( $n \geq 2$ ) with a smooth boundary  $\Gamma$  and  $\varepsilon$  a real parameter taking values in a sequence of positive numbers tending to zero. As usual in periodic homogenization let  $Y = [0, 1]^n$  be the generic unit cell of periodicity in the auxiliary space  $\mathbb{R}^n$  of variable  $y = (y_i)_{1 \leq i \leq n}$ . The cell  $Y$  is identified to the unit torus  $\mathbb{R}^n/\mathbb{Z}^n$ . A function defined on  $\mathbb{R}^n$  is said to be  $Y$ -periodic if it is periodic of period 1 in each  $y_i$  variable with  $1 \leq i \leq n$ . In the sequel we will suppose that any function defined on  $Y$  is extended periodically to the whole space  $\mathbb{R}^n$ . If  $E(Z)$  is a function space (where  $Z$  is a subset of  $Y$ ) we denote  $E_{\#}(Z) := \{w \in E(Z); w \text{ is extended periodically to } \mathbb{R}^n\}$ .

Let  $H$ , the reference hole, be an open subset of  $Y$  with a smooth boundary  $\Sigma$  and set  $Y_s = Y \setminus cl(H)$  where  $cl(\cdot)$  denotes the closure. Thus  $Y$  is partitioned as  $Y = Y_s \cup \Sigma \cup H$ . Note that we do not require that  $H$  is strictly included in  $Y$ . As a consequence the periodic extension of  $H$  may or may not be connected. Let us denote  $\chi(y)$  the characteristic function of  $Y_s$  in  $Y$ . We define the perforated material

$$\Omega_\varepsilon = \left\{ x \in \Omega; \chi\left(\frac{x}{\varepsilon}\right) = 1 \right\}$$

and the one codimensional periodic surface

$$\Sigma_\varepsilon = \left\{ x \in \Omega; \frac{x}{\varepsilon} \in \Sigma \right\}.$$

Here  $\Omega_\varepsilon$  represents the matrix or the solid part of  $\Omega$ , by opposition to the holes or the void part that is represented by the open subset  $H_\varepsilon := \Omega \setminus cl(\Omega_\varepsilon)$ . By construction, all of these holes are identical and they are periodically distributed in  $\Omega$  with period  $\varepsilon$  in each  $x_i$ -direction. Since we use the two-scale convergence method we do not require that the boundary  $\Sigma_\varepsilon = \partial H_\varepsilon$  does not intersect  $\Gamma$ . As in [3], We shall use the natural extension by zero of any function defined on  $\Omega_\varepsilon$ .

Let  $f_\varepsilon$  be a given function in  $L^2(\Omega_\varepsilon)$ ,  $g_\varepsilon$  be given in  $L^2(\Sigma_\varepsilon)$  such that

$$\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon}\|g_\varepsilon\|_{L^2(\Sigma_\varepsilon)} \leq C. \tag{1}$$

Here and throughout this paper  $C$  denotes a positive constant independent of  $\varepsilon$ .

Let  $A(x, y) = (a_{ij})_{1 \leq i, j \leq n}$  be a real-valued matrix function defined on  $\Omega \times Y$ ,  $Y$ -periodic in the second variable  $y$  such that there exists two positive constants  $m, M$  independent of  $\varepsilon$  satisfying the following inequality:

$$m |\zeta|^2 \leq (A\zeta, \zeta) \leq M |\zeta|^2 \tag{2}$$

for all  $\zeta \in \mathbb{R}^n$ . We suppose that the matrix  $A$  lies in  $C(\Omega; L^\infty_\#(Y))^{n^2}$ . We note that no symmetry condition on  $A$  is assumed. Let  $\mu(y) \in L^\infty_\#(Y_s)$  such that  $\int_{Y_s} \mu(y) \geq \mu_0 > 0$  where  $\mu_0$  is independent of  $\varepsilon$ . Let  $\alpha$  be a  $Y$ -periodic measurable bounded function defined on  $\Sigma$  such that

$$\int_\Sigma \alpha(y) d\sigma(y) = 0. \tag{3}$$

Let us decompose the function  $\alpha$  into its positive and negative parts as follows:

$$\alpha = \alpha^+ - \alpha^-, \quad \alpha^+ = \max(\alpha, 0), \quad \alpha^- = \max(-\alpha, 0).$$

Assume that the positive part of  $\alpha$  satisfies the condition:

$$\alpha^+(y) \geq \alpha_0 > 0 \text{ a.e. in } \Sigma. \tag{4}$$

Let us consider the following Robin boundary value problem:

$$-div(A_\varepsilon \nabla u_\varepsilon) + \mu_\varepsilon u_\varepsilon = f_\varepsilon \text{ in } \Omega_\varepsilon, \tag{5}$$

$$(A_\varepsilon \nabla u_\varepsilon) \cdot \nu_\varepsilon + \alpha_\varepsilon u_\varepsilon = \varepsilon g_\varepsilon \text{ on } \Sigma_\varepsilon, \tag{6}$$

$$u_\varepsilon = 0 \text{ on } \Gamma \tag{7}$$

where

$$A_\varepsilon(x) = A\left(x, \frac{x}{\varepsilon}\right), \quad \mu_\varepsilon(x) = \mu\left(\frac{x}{\varepsilon}\right), \quad \alpha_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right)$$

and  $\nu_\varepsilon$  is the unit outward normal to  $\Omega_\varepsilon$ .

This problem can be regarded as a simplified model of the condensation of steam in a periodic cooling structure (see [4]). We can also consider as a model for treatment planning hyperthermia in microvascular tissue, see [8]. The boundary condition (6) means that the heat flow  $(A_\varepsilon \nabla u_\varepsilon) \cdot \nu_\varepsilon$  is proportional to the temperature  $u_\varepsilon$  with a periodic resistivity given by the function  $\alpha_\varepsilon$ . In many situations, the resistivity function is taken to be  $\varepsilon^m \alpha_\varepsilon$ . Since the operator is of order 2 the interesting cases are then  $m = -2, -1, 0, 1$  and  $2$ . The case  $m = 2$  is trivial since we obtain the classical homogenized equation. This can be seen easily by using the asymptotic expansion method. The case  $m = 1$  with  $\alpha_\varepsilon \geq 0$  has been studied by [4], [10] using the two-scale convergence technique. In this situation  $\alpha_\varepsilon$  is rescaled since the surface  $\Sigma_\varepsilon$  is of codimension 1. Here we study the case  $m = 0$ , i.e. a non rescaled resistivity. We use the same technique but with  $\alpha_\varepsilon$  changing sign. We show that the assumptions (3), (4) and a non-symmetric  $A_\varepsilon(x)$  contribute to the description of the effective thermal conductivity with convection. The case  $m = -1$  will be studied in a forthcoming paper. Note that the case  $m = -2$  is also trivial since it yields that the effective thermal conductivity is 0.

### 3 Study of the Problem and A priori Estimates

Let

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon); v = 0 \text{ on } \Gamma\}$$

equipped with the scalar product

$$(u, v)_{V_\varepsilon} = \int_{\Omega_\varepsilon} \nabla u(x) \nabla v(x) dx$$

and the associated norm  $\|u\|_{V_\varepsilon} = (u, u)_{V_\varepsilon}^{1/2}$  which is equivalent to the  $H^1$ -norm thanks to the Poincaré inequality. The variational formulation of the boundary-value problem (5)-(7) reads as follows:

$$\begin{cases} \text{For each } \varepsilon > 0, \text{ find } u_\varepsilon \in V_\varepsilon \text{ such that} \\ a_\varepsilon(u_\varepsilon, v) = L_\varepsilon(v) \text{ for any } v \in V_\varepsilon, \end{cases} \quad (8)$$

where  $a_\varepsilon(\cdot, \cdot)$  is the bilinear form defined on  $V_\varepsilon \times V_\varepsilon$  by:

$$a_\varepsilon(u, v) = \int_{\Omega_\varepsilon} A_\varepsilon(x) \nabla u(x) \nabla v(x) dx + \int_{\Omega_\varepsilon} \mu_\varepsilon(x) u(x) v(x) dx + \int_{\Sigma_\varepsilon} \alpha_\varepsilon(x) u(x) v(x) d\sigma_\varepsilon(x)$$

and  $L_\varepsilon(\cdot)$  is the linear form defined on  $V_\varepsilon$  by:

$$L_\varepsilon(v) = \int_{\Omega_\varepsilon} f_\varepsilon(x) v(x) dx + \varepsilon \int_{\Sigma_\varepsilon} g_\varepsilon(x) v(x) d\sigma_\varepsilon(x).$$

**Lemma 3.1** *There exists a positive constant  $C_s$  independent of  $\varepsilon$  such that for every  $v \in V_\varepsilon$  and for every  $\delta > 0$  we have*

$$\|v\|_{L^2(\Sigma_\varepsilon)}^2 \leq C_s \left[ (\delta\varepsilon)^{-1} \|v\|_{L^2(\Omega_\varepsilon)}^2 + (\delta\varepsilon) \|\nabla v\|_{(L^2(\Omega_\varepsilon))^n}^2 \right]. \tag{9}$$

**Proof.** Let us introduce the notation

$$v_\varepsilon^k(x) = v(\varepsilon(k + y))$$

where  $k \in K_\varepsilon = \{k \in \mathbb{Z}^n; \varepsilon(Y + k) \cap \Omega \neq \emptyset\}$ . By the change of variable  $x = \varepsilon(k + y)$  we have

$$\int_{\Sigma_\varepsilon} v^2(x) d\sigma_\varepsilon(x) = \sum_{k \in K_\varepsilon} \int_{\varepsilon(\Sigma+k)} v^2(x) d\sigma_\varepsilon(x) = \varepsilon^{n-1} \sum_{k \in K_\varepsilon} \int_\Sigma [v_\varepsilon^k(y)]^2 d\sigma(y).$$

From the trace theorem we see that for every  $\delta > 0$

$$\begin{aligned} \int_\Sigma [v_\varepsilon^k(y)]^2 d\sigma(y) &\leq C_s \left[ \delta^{-1} \int_{Y_s+k} [v_\varepsilon^k(y)]^2 dy + \delta \int_{Y_s+k} |\nabla_y v_\varepsilon^k(y)|^2 dy \right] \\ &\leq \frac{C_s}{\varepsilon^n} \left[ \delta^{-1} \int_{\varepsilon(Y_s+k)} v(x)^2 dx + \delta\varepsilon^2 \int_{\varepsilon(Y_s+k)} |\nabla_x v(x)|^2 dx \right]. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\Sigma_\varepsilon} v^2(x) d\sigma_\varepsilon(x) &\leq \varepsilon^{n-1} \frac{C_s}{\varepsilon^n} \left[ \delta^{-1} \sum_{k \in K_\varepsilon} \int_{\varepsilon(Y_s+k)} v(x)^2 dx + \right. \\ &\quad \left. \delta\varepsilon^2 \sum_{k \in K_\varepsilon} \int_{\varepsilon(Y_s+k)} |\nabla_x v(x)|^2 dx \right] \\ &\leq C_s \left[ (\delta\varepsilon)^{-1} \int_{\Omega_\varepsilon} v^2(x) dx + (\delta\varepsilon) \int_{\Omega_\varepsilon} |\nabla v(x)|^2 dx \right]. \end{aligned}$$

The Lemma is proved. ■

**Lemma 3.2** *Let  $\sqrt{\mu_0 m} > C_s \|\alpha\|_{L^\infty(\Sigma)}$  where  $C_s$  is the constant given in Lemma 3.1. Then  $a_\varepsilon(\cdot, \cdot)$  is coercive on  $V_\varepsilon$ .*

**Proof.** Let  $v \in V_\varepsilon$ . Then using (2), we have

$$a_\varepsilon(v, v) \geq m \int_{\Omega_\varepsilon} |\nabla v(x)|^2 dx + \mu_0 \int_{\Omega_\varepsilon} v(x)^2 dx - \|\alpha\|_{L^\infty(\Sigma)} \int_{\Sigma_\varepsilon} v(x)^2 d\sigma_\varepsilon(x).$$

By (9) we see that for every  $\delta > 0$

$$a_\varepsilon(v, v) \geq (m - \delta\varepsilon C_s \|\alpha\|_{L^\infty(\Sigma)}) \int_{\Omega_\varepsilon} |\nabla v(x)|^2 dx + (\mu_0 - (\delta\varepsilon)^{-1} C_s \|\alpha\|_{L^\infty(\Sigma)}) \int_{\Omega_\varepsilon} [v(x)]^2 dx.$$

Choosing  $\delta = \frac{1}{\varepsilon} \sqrt{\frac{m}{\mu_0}}$ . Then

$$a_\varepsilon(v, v) \geq c_0 \left[ \|\nabla v\|_{(L^2(\Omega_\varepsilon))^n}^2 + \|v\|_{L^2(\Omega_\varepsilon)}^2 \right]$$

where  $c_0$  is the positive constant given by

$$c_0 = \left( 1 - \frac{C_s \|\alpha\|_{L^\infty(\Sigma)}}{\sqrt{m\mu_0}} \right) \min(m, \mu_0) > 0.$$

This completes the proof. ■

In the sequel we shall assume that the condition  $\sqrt{\mu_0 m} > C_s \|\alpha\|_{L^\infty(\Sigma)}$  is fulfilled.

**Proposition 3.3** *The variational formulation (8) admits a unique solution  $u_\varepsilon \in V_\varepsilon$ . Moreover we have the a priori estimates.*

$$\|\nabla u_\varepsilon\|_{(L^2(\Omega_\varepsilon))^n} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Sigma_\varepsilon)} \leq C. \tag{10}$$

**Proof.** The existence and uniqueness is a straightforward application of Lemma 3.1 and the Lax-Milgram Lemma. It remains to prove the a priori estimates (10). Take  $v = u_\varepsilon$  in (8). We have

$$\begin{aligned} & \int_{\Omega_\varepsilon} (A_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon + \mu_\varepsilon u_\varepsilon^2) dx + \int_{\Sigma_\varepsilon} \alpha_\varepsilon^+ u_\varepsilon^2 d\sigma_\varepsilon(x) \\ &= \int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon dx + \int_{\Sigma_\varepsilon} (\varepsilon g_\varepsilon + \alpha_\varepsilon^- u_\varepsilon) u_\varepsilon d\sigma_\varepsilon(x). \end{aligned}$$

Let us denote

$$A_\varepsilon(u_\varepsilon) := \|\nabla u_\varepsilon\|_{(L^2(\Omega_\varepsilon))^n}^2 + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|u_\varepsilon\|_{L^2(\Sigma_\varepsilon)}^2.$$

Then using (2) and (4) we obtain

$$A_\varepsilon(u_\varepsilon) \leq \frac{1}{c_1} \left( \int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon dx + \int_{\Sigma_\varepsilon} (\varepsilon g_\varepsilon + \alpha_\varepsilon^- u_\varepsilon) u_\varepsilon d\sigma_\varepsilon(x) \right) \tag{11}$$

where  $c_1 = \min(m, \mu_0, \alpha_0) > 0$ . Applying Young’s inequality on the right hand side of (11), we get

$$\begin{aligned} A_\varepsilon(u_\varepsilon) \leq & \frac{1}{c_1} \left[ \frac{\beta^2}{2} \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{2\beta^2} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \right. \\ & \left. + \frac{\gamma^2 \varepsilon^2}{2} \|g_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \left( \frac{1}{2\gamma^2} + \|\alpha\|_{L^\infty(\Sigma)} \right) \|u_\varepsilon\|_{L^2(\Sigma_\varepsilon)}^2 \right]. \end{aligned} \tag{12}$$

But in view of (9) inequality (12) becomes

$$\begin{aligned} A_\varepsilon(u_\varepsilon) \leq & \frac{1}{c_1} \left[ \left( \frac{1}{2\beta^2} + \left( \frac{\varepsilon}{2\gamma^2} + \|\alpha\|_{L^\infty(\Sigma)} \right) \frac{C_s}{\varepsilon\delta} \right) \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \right. \\ & \left. + \left( \frac{\varepsilon}{2\gamma^2} + \|\alpha\|_{L^\infty(\Sigma)} \right) C_s \varepsilon \delta \|\nabla u_\varepsilon\|_{(L^2(\Omega_\varepsilon))^n}^2 \right] + C. \end{aligned}$$

Now, appropriate choice of  $\beta, \gamma, \delta$  yields

$$A_\varepsilon(u_\varepsilon) = \|\nabla u_\varepsilon\|_{(L^2(\Omega_\varepsilon))^n}^2 + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|u_\varepsilon\|_{L^2(\Sigma_\varepsilon)}^2 \leq C.$$

The Proposition is now proved. ■

One is led to determine the homogenized problem of (5)-(7). Namely we study the limiting behavior of the solutions  $u_\varepsilon$  as  $\varepsilon$  tends to zero. This the subject of the next section.

## 4 Homogenization Procedure

We shall use the well-known two-scale convergence method that we briefly recall here the definition and the main results.

### 4.1 Two-scale Convergence

**Definition 4.1** 1. A sequence  $v_\varepsilon$  in  $L^2(\Omega)$  two-scale converges to  $v_0(x, y) \in L^2((\Omega \times Y))$  and we denote this  $v_\varepsilon \rightharpoonup v_0$  if for any  $\varphi(x, y) \in L^2(\Omega; C_\#(Y))$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y v_0(x, y) \varphi(x, y) dy dx.$$

2. A sequence  $v_\varepsilon$  in  $L^2(\Sigma_\varepsilon)$  two-scale converges to  $v_0(x, y) \in L^2((\Omega \times \Sigma))$  and we denote this  $v_\varepsilon \overset{S}{\rightrightarrows} v_0$  if for any  $\varphi(x, y) \in C(\overline{\Omega}; C_\#(Y))$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} \varepsilon v_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x) = \int_\Omega \int_\Sigma v_0(x, y) \varphi(x, y) d\sigma(y) dx.$$

**Proposition 4.2** 1. For any uniformly bounded sequence  $v_\varepsilon$  in  $L^2(\Omega)$  one can extract a subsequence still denoted by  $\varepsilon$  and a two-scale limit  $v_0 \in L^2((\Omega \times Y))$  such that  $v_\varepsilon \overset{S}{\rightrightarrows} v_0$ .

2. If  $v_\varepsilon$  is in  $L^2(\Sigma_\varepsilon)$  such that

$$\varepsilon \|v_\varepsilon\|_{L^2(\Sigma_\varepsilon)}^2 \leq C,$$

then one can extract a subsequence still denoted by  $\varepsilon$  and a two-scale limit  $v_0 \in L^2((\Omega \times \Sigma))$  such that  $v_\varepsilon \overset{S}{\rightrightarrows} v_0$ .

### 4.2 Two-scale limit system

By virtue of the estimate (1) and the proposition 4.2, there exists  $f \in L^2(\Omega \times Y)$  and  $g \in L^2(\Omega \times \Sigma)$  such that, up to a subsequence, one has

$$\chi\left(\frac{x}{\varepsilon}\right) f_\varepsilon(x) \overset{S}{\rightrightarrows} \chi(y) f(x, y), \quad g_\varepsilon(x) \overset{S}{\rightrightarrows} g(x, y). \tag{13}$$

Furthermore we have .

**Lemma 4.3** [3], [12], [4]. Let  $u_\varepsilon$  be the solution of (8). Then there exists a subsequence still denoted by  $\varepsilon$  and two functions  $u(x) \in H_0^1(\Omega)$ ,  $u_1(x, y) \in L^2(\Omega; H_\#^1(Y_s)/\mathbb{R})$  such that

$$\chi\left(\frac{x}{\varepsilon}\right) u_\varepsilon(x) \overset{S}{\rightrightarrows} \chi(y) u(x), \tag{14}$$

$$\chi\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \overset{S}{\rightrightarrows} \chi(y) (\nabla u(x) + \nabla_y u_1(x, y)). \tag{15}$$

Moreover we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} \varepsilon u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x) = \int_\Omega \int_\Sigma u(x) \varphi(x, y) d\sigma(y) dx \tag{16}$$

for every  $\varphi \in C(\overline{\Omega}; C_\#(Y_s))$ .



**Lemma 4.4** *We have*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} u_\varepsilon(x) \varphi(x) \alpha\left(\frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x) = \int_\Omega \int_\Sigma u_1(x) \varphi(x) \alpha(y) d\sigma(y) dx$$

for all  $\varphi(x) \in C(\overline{\Omega})$ .

**Proof.** Define a function  $\psi(y) \in H^1_\#(Y_s)/\mathbb{R}$  solution of the problem

$$\begin{cases} -\Delta\theta(y) = 0 \text{ in } Y_s, \\ (\nabla\theta(y)) \cdot \nu(y) = \alpha(y) \text{ on } \Sigma, \\ y \mapsto \theta(y) \text{ Y-periodic.} \end{cases} \tag{17}$$

Such a function exists since  $\alpha(y)$  satisfies (3) which is the compatibility condition for the solvability of the problem (17). Set  $\psi(y) = \nabla\theta$  and consider the function  $\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right)$ . Then we have

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \psi_\varepsilon(x) dx &= \int_{\Sigma_\varepsilon} u_\varepsilon(x) \psi_\varepsilon(x) \cdot \nu\left(\frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x) \\ &= \int_{\Sigma_\varepsilon} u_\varepsilon(x) \alpha\left(\frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x). \end{aligned} \tag{18}$$

Passing to the limit in the left hand side of (18) and taking into account (15) we find

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} u_\varepsilon(x) \alpha\left(\frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x) = \int_\Omega \int_Y \chi(y) (\nabla u(x) + \nabla_y u_1(x, y)) \psi(y) dy dx. \tag{19}$$

Since  $u \in H^1_0(\Omega)$  we have

$$\int_\Omega \int_Y \chi(y) \nabla u(x) \psi(y) dy dx = \left( \int_\Omega \nabla u(x) dx \right) \int_Y \chi(y) \psi(y) dy = 0.$$

Hence the right hand side of (19) becomes

$$\int_\Omega \int_Y \chi(y) \nabla_y u_1(x, y) \psi(y) dy dx.$$

On the other hand, we have

$$\begin{aligned} \int_\Omega \int_Y \chi(y) \nabla_y u_1(x, y) \psi(y) dy dx &= - \int_\Omega \int_{Y_1} u_1(x, y) \operatorname{div}_y \psi(y) dy dx \\ &\quad + \int_\Omega \int_\Sigma u_1(x, y) \psi(y) \cdot \nu(y) d\sigma(y) dx \\ &= \int_\Omega \int_\Sigma u_1(x, y) \alpha(y) d\sigma(y) dx \end{aligned}$$

which proves the Lemma. ■

Now we are able to give the two-scale limit system:

**Proposition 4.5** *The couple  $(u, u_1) \in H_0^1(\Omega) \times L^2(\Omega; H_{\#}^1(Y_s)/\mathbb{R})$  is the solution of the following two-scale homogenized system :*

$$-div_y (A (\nabla u + \nabla_y u_1)) = 0 \quad \text{in } \Omega \times Y_s, \tag{20}$$

$$(A (\nabla u + \nabla_y u_1) \cdot \nu) + \alpha u = 0 \quad \text{on } \Omega \times \Sigma, \tag{21}$$

$$y \longmapsto u_1 \quad Y - \text{periodic}, \tag{22}$$

$$-div_x \left( \int_{Y_1} A (\nabla u + \nabla_y u_1) dy \right) + \tilde{\mu} u + \int_{\Sigma} \alpha u_1 d\sigma(y) = F \quad \text{in } \Omega, \tag{23}$$

$$u = 0 \quad \text{on } \Gamma \tag{24}$$

where  $\tilde{\mu} = \int_{Y_s} \mu(y) dy$  and  $F(x) = \int_Y \chi(y) f(x, y) dy + \int_{\Sigma} g(x, y) d\sigma(y)$ .

**Proof.** Let  $\varphi(x) \in \mathcal{D}(\Omega)$  and  $\varphi_1(x, y) \in \mathcal{D}(\Omega; C_{\#}^{\infty}(Y_s))$ . Choosing  $v(x) = \varphi(x) + \varepsilon\varphi_1(x, \frac{x}{\varepsilon})$  as a test function in problem (8), we have

$$\begin{aligned} & \int_{\Omega} \nabla u(x) \chi\left(\frac{x}{\varepsilon}\right)^t A\left(\frac{x}{\varepsilon}\right) \left( \nabla \varphi(x) + \varepsilon \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y \varphi_1\left(x, \frac{x}{\varepsilon}\right) \right) dx \\ & \quad + \int_{\Omega_{\varepsilon}} \mu_{\varepsilon}(x) u(x) \left[ \varphi(x) + \varepsilon \varphi_1\left(x, \frac{x}{\varepsilon}\right) \right] dx \\ & \quad + \int_{\Sigma_{\varepsilon}} \alpha_{\varepsilon}(x) u(x) \left[ \varphi(x) + \varepsilon \varphi_1\left(x, \frac{x}{\varepsilon}\right) \right] d\sigma_{\varepsilon}(x) = \tag{25} \\ & \quad \int_{\Omega} \chi\left(\frac{x}{\varepsilon}\right) f_{\varepsilon}(x) \left[ \varphi(x) + \varepsilon \varphi_1\left(x, \frac{x}{\varepsilon}\right) \right] dx \\ & \quad \varepsilon \int_{\Sigma_{\varepsilon}} g_{\varepsilon}(x) \left[ \varphi(x) + \varepsilon \varphi_1\left(x, \frac{x}{\varepsilon}\right) \right] d\sigma_{\varepsilon}(x). \end{aligned}$$

By virtue of (15) the first two terms of the left hand side of (25) converges to

$$\begin{aligned} & \int_{\Omega} \int_{Y_s} [A(y) [\nabla u(x) + \nabla_y u_1(x, y)] [\nabla \varphi(x) + \nabla_y \varphi_1(x, y)] \\ & \quad + \mu(y) u(x) \varphi(x)] dy dx. \tag{26} \end{aligned}$$

Taking into account (16), and the lemma 4.4, the third term of the left hand side of (25) tends to

$$\int_{\Omega} \int_{\Sigma} \alpha(y) u_1(x, y) \varphi(x) d\sigma(y) dx + \int_{\Omega} \int_{\Sigma} \alpha(y) u(x) \varphi_1(x, y) d\sigma(y) dx. \tag{27}$$

Thanks to (13) the right hand side of (25) converges to

$$\int_{\Omega} \left[ \int_{Y_s} f(x, y) dy + \int_{\Sigma} g(x, y) d\sigma(y) \right] \varphi(x) dx = \int_{\Omega} F(x) \varphi(x) dx. \tag{28}$$

By the density of  $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega; C^\infty(Y_s))$  in  $H_0^1(\Omega) \times L^2(\Omega; H_\#^1(Y_s)/\mathbb{R})$  we get from the limits (26)-(28) the following two-scale weak formulation system:

$$\left\{ \begin{array}{l} (u, u_1) \in H_0^1(\Omega) \times L^2(\Omega; H_\#^1(Y_s)/\mathbb{R}) \text{ is such that} \\ \int_\Omega \int_{Y_s} A[\nabla u + \nabla_y u_1][\nabla v + \nabla_y v_1] dy dx + \\ \tilde{\mu} \int_\Omega u v dx + \int_\Omega \int_\Sigma \alpha u_1 v d\sigma(y) dx + \int_\Omega \int_\Sigma \alpha u v_1 d\sigma(y) = \int_\Omega F v dx \end{array} \right. \quad (29)$$

for all  $(v, v_1) \in H_0^1(\Omega) \times L^2(\Omega; H_\#^1(Y_s)/\mathbb{R})$ . Integration by parts in (29) with respect to  $v_1$  ( $v = 0$ ) gives (20)-(22) and with respect to  $v$  ( $v_1 = 0$ ) yields (23)-(24). The Proposition is now proved. ■

Thanks to the linearity of the first equation of (20) we can compute  $u_1(x, y)$  in terms of  $u(x)$  as follows:

$$u_1(x, y) = \sum_{k=1}^n \zeta_k(y) \frac{\partial u}{\partial x_k}(x) + \gamma(y)u(x) + \tilde{u}(x) \quad (30)$$

where for each  $k$  the function  $\zeta_k(y)$  satisfies the auxiliary problem:

$$\begin{aligned} -div(A(y) \nabla \zeta_k(y)) &= div(A(y) e_k) \text{ in } \Omega \times Y_s, \\ A(y) \nabla \zeta_k(y) \cdot \nu &= -A(y) e_k \cdot \nu \text{ on } \Omega \times \Sigma, \\ y &\rightarrow \zeta_k(y) \text{ } Y\text{-periodic, } x \in \Omega. \end{aligned}$$

where  $e_k = (\delta_{ik})_{1 \leq i \leq n}$ ,  $\delta_{ik}$  is the Krönecker symbol.

The function  $\gamma(y)$  satisfies

$$\begin{aligned} -div(A(y) \nabla \gamma(y)) &= 0 \text{ in } \Omega \times Y_s, \\ A(y) \nabla \gamma(y) \cdot \nu &= -\alpha(y) \text{ on } \Omega \times \Sigma, \\ y &\rightarrow \gamma(y) \text{ } Y\text{-periodic, } x \in \Omega. \end{aligned}$$

Finally, inserting the relation (30) into the equation (23) yields to the homogenized equation

$$-div(A^{\text{hom}} \nabla u(x)) + B \cdot \nabla u(x) + \lambda u(x) = F(x) \quad (31)$$

where  $A^{\text{hom}}$  is the matrix with coefficients

$$a_{ij}^{\text{hom}} = \sum_{k=1}^n \int_{Y_s} \left[ a_{ij}(y) \left( \delta_{kj} + \frac{\partial \zeta_j}{\partial y_k}(y) \right) \right] dy$$

$B$  is the vector with components:

$$\begin{aligned} b_i &= \int_{\Sigma} \alpha(y) \zeta_i(y) d\sigma(y) - \sum_{k=1}^n \int_{Y_s} a_{ik}(y) \frac{\partial \gamma}{\partial y_k}(y) dy \\ &= \int_{\Sigma} \alpha(y) \zeta_i(y) d\sigma(y) - \int_{Y_s} A(y) e_i \nabla \gamma(y) dy \end{aligned}$$

$\lambda$  is the real number:

$$\begin{aligned} \lambda &= \int_{\Sigma} \alpha(y) \gamma(y) d\sigma(y) + \tilde{\mu} \\ &= - \int_{Y_s} A(y) \nabla \gamma(y) \nabla \gamma(y) dy + \tilde{\mu} \end{aligned}$$

Thus we have proved the following result

**Theorem 4.6** *Let  $u_\varepsilon$  be the solution in  $V_\varepsilon$  of the Robin boundary problem (5)-(7). Then  $\chi_\varepsilon(x) u_\varepsilon(x)$  two-scale converges to  $\chi(y) u(x)$  where  $u(x)$  is a solution in  $H_0^1(\Omega)$  of the homogenized problem:*

$$\begin{cases} -\operatorname{div}(A^{\text{hom}} \nabla u(x)) + B \cdot \nabla u(x) + \lambda u(x) = F(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \quad (32)$$

**Remark 4.7** *We observe that the limit equation (32) contains an extra strange term of order 1. Namely the convection term  $B \cdot \nabla u$ . The vector  $B$  depends closely on the matrix  $A$  and the resistivity function  $a$ . For example, if  $A$  is symmetric then  $B = 0$ . Indeed*

$$\int_{\Sigma} \alpha(y) \zeta_i(y) d\sigma = - \int_{Y_s} A(y) \nabla \gamma(y) \nabla \zeta_i(y) dy = - \int_{Y_s} A(y) \nabla \zeta_i(y) \nabla \gamma(y) dy$$

and since  $A$  is symmetric

$$\int_{\Sigma} \alpha(y) \zeta_i(y) d\sigma = \int_{Y_s} A(y) e_i \nabla \gamma(y) dy.$$

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