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A Weak Order One Stochastic Runge-Kutta Method

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Abstract

In this paper, we derive new two stage explicit SRK methods with weak order 1 for SDEs with one. With two test problems,the absolute error and the CPU time of our method present and compare with the Euler method.

Keywords: Stochastic differential equations, weak approximation, Stochastic Runge–Kutta methods

1 Introduction

Consider the autonomous stochastic differential equations(SDEs), given by:

$$dy(t) = g_0(y(t))dt + \sum_{j=0}^{m} g_j(y(t))dW_j \qquad y(t_0) = y_0, t \in [t_0, T]$$

where $g_j(y), j = 0, ..., m$ are,d–vector valued functions and the $W_j(t), j = 1, ..., m$, are independent Wiener processes. The above equation can be written in the integral form:

$$y(t) = y_0 + \int_{t_0}^{t} g_0(y(t))ds + \sum_{j=1}^{m} \int_{t_0}^{t} g_j(y(s))dW_j \ .$$

The variance and the mean of the Wiener process are t and zero, respectively.Hence, typical sample paths of a Wiener process obtain large value in magnitude as time progress.Therefore,the sample paths are not of bounded variation and the above stochastic integral cannot be calculated by the usual Reiman-Stelitjes rules and are defined as

$$\sum_{i=1}^{N} g_j(y(\zeta_i))\ (W_j(t_i) - W_j(t_{i-1}))\ ,$$

where $\zeta_i = \theta t_i + (1-\theta)t_{i-1}$, for $0 \leq \theta \leq 1$ and, for $\{t_0, \cdots, t_N\}$ be partition of $[t_0, t]$, with $t_i = t_0 + \frac{i(t-t_0)}{N}$, $i = 0, \cdots, N$ .The most common choices are $\theta = 0$ and $\theta = \frac{1}{2}$, which gives the It$\hat{o}$ and stratnovich integrals(see[4] for further details), respectively.

Definition 1: Suppose $\bar{y}_N$ be the numerical approximation to $y(t_N)$ after N step with $h = \frac{t_N - t_0}{N}$,then $\bar{y}$ is converge weakly with order q ,if for each G with $2(q+1)$ times continuously differentiable $\Re$-valued function on $\Re^d$,there exist $c > 0$ (independent of h) and $\delta > 0$ such that

$$|E[G(y(t_N))] - E[G(y_N)]| \leq ch^q \qquad h \in (0, \delta). \tag{1}$$

2 Stochastic Runge–Kutta Methods

In order to derive stochastic Runge–Kutta(SRK) methods to approximate of the exact solution $y(t_{n+1})$ by y_{n+1}, when y_n be given, consider the following family:

$$\begin{aligned} y_{n+1} &= y_n + \sum_{i=1}^{s} \sum_{j_a, j_b=0}^{m} c_i^{(j_a,j_b)} Y_i^{(j_a,j_b)} \qquad (2) \\ Y_i^{(j_a,j_b)} &= \widetilde{\eta}_i^{(j_a,j_b)} \{ g_{j_b}(y_n + \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \widetilde{\alpha}_{i,i_b}^{(j_a,j_b,j_c,j_d)} Y_{i_b}^{(j_c,j_d)}) \\ &+ g_{j_b}(y_n) \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \widetilde{\gamma}_{i,i_b}^{(j_a,j_b,j_c,j_d)} Y_{i_b}^{(j_c,j_d)} \} \end{aligned}$$

where $\widetilde{\eta}_i^{(j_a,j_b)}$ is a random variable independent of $y_n$with the following property:

$$E[(\widetilde{\eta}_i^{(j_a,j_b)})^{2k}] = \begin{cases} K_1 h^{2k} & j_b = 0 \\ K_2 h^k & j_b \neq 0 \end{cases}$$

for constant K_1 and K_2 and $k = 1, 2, \ldots$.
For deriving explicit SRK, assume ,$\widetilde{\gamma}_{i,i_b}^{(j_a,j_b,j_c,j_d)} = 0$ (for any j_a, j_b, j_c, j_d) and $\alpha_{i,i_b}^{(j_a,j_b,j_c,j_d)} = 0$ (for any $i \leq i_b$ and j_a, j_b, j_c, j_d) (see for example, [5],[6]). Hence (2)

can be rewritten as follows:

$$y_{n+1} = y_n + \sum_{i=1}^{s} \sum_{j_a,j_b=0}^{m} c_i^{(j_a,j_b)} Y_i^{(j_a,j_b)} \tag{3}$$
$$Y_i^{(j_a,j_b)} = \widetilde{\eta}_i^{(j_a,j_b)} \{ g_{j_b}(y_n + \sum_{i_b=1}^{i-1} \sum_{j_c,j_d=0}^{m} \alpha_{i,i_b}^{(j_a,j_b,j_c,j_d)} Y_{i_b}^{(j_c,j_d)}) \}$$

2.1 SRK Methods and the Butcher Array

In this section ,the weak order conditions with multi–colored rooted trees(MRTs)be studied. An MRT with a root $ⓚ$ $(0 \leq k \leq m$ and k is the colored label)is a tree recursively defined such that:

- $\tau^{(j)}$ is the primitive tree having a vertex with root $ⓙ$.
- if $t_1, t_2, ..., t_k$ are MRTs then $[t_1, t_2, ..., t_k]^{(j)}$ is MRT with root $ⓙ$ and for $i = 1, 2, ..., k$, t_i are its sub-MRTs.

First,we consider only deterministic case($m = 0$),therefore $j_a = j_b = j_c = j_d = 0$ and (3)may be rewritten as follows:

$$y_{n+1} = y_n + \sum_{i=1}^{s} c_i^{(0,0)} Y_i^{(0,0)} \tag{4}$$
$$Y_i^{(0,0)} = \widetilde{\eta}_i^{(0,0)} \{ g_0(y_n + \sum_{i_b=1}^{i-1} \alpha_{i,i_b}^{(0,0,0,0)} Y_{i_b}^{(0,0)}) \}$$

with comparing (4) and the normal formula of the Runge–Kutta methods:

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i \overline{Y}_i \tag{5}$$
$$\overline{Y}_i = h g_0(y_n + \sum_{i=1}^{s} a_{ij} \overline{Y}_j)$$

we have:

$$\widetilde{\eta}_i^{(0,0)} = h \quad , \quad c_i^{(0,0)} = b_i \quad , \quad \alpha_{i,j}^{(0,0,0,0)} = a_{i,j}$$

with the "Butcher tabular",

$$\begin{array}{c|c} & \alpha^{(0,0,0,0)} \\ \hline & c^{(0,0)} \end{array}$$

For stochastic case ($m \neq 0$) , consider the general form of explicit stochastic Runge–Kutta methods with s–stage given by:

$$y_{n+1} = y_n + \sum_{i=1}^{s}\sum_{k=0}^{m} z_i^{(k)} g_k(Y_i) \qquad (6)$$

$$Y_i = y_n + \sum_{j=1}^{i-1}\sum_{k=0}^{m} Z_{ij}^{(k)} g_k(Y_j)$$

with "Butcher tabular"

$$\begin{array}{c|cccc} & Z^{(0)} & Z^{(1)} & \cdots & Z^{(m)} \\ \hline & z^{(0)T} & z^{(1)T} & \cdots & z^{(m)T} \end{array}$$

where $Z^{(k)} = (Z_{ij}^{(k)})$ for $i,j = 1,2,\cdots,s$ and $z^{(k)^T} = (z_1^{(k)},\cdots,z_s^{(k)})$ for $k = 0,1,\cdots,m$. Since the method is explicit, hence $Z^{(k)}$ is strictly lower triangular matrix.
By substituting of $Y_i^{(j_a,j_b)}$ to y_{n+1} in (3)the following expression be obtained:

$$y_{n+1} = y_n + \sum_{i=1}^{s}\sum_{j_a,j_b=0}^{m} \beta_i^{(j_a,j_b)} g_{j_b}(k_i^{(j_a,j_b)}) \qquad (7)$$

$$k_i^{(j_a,j_b)} = y_n + \sum_{i_b=1}^{i-1}\sum_{j_c,j_d=0}^{m} \lambda_{i\ i_b}^{(j_a,j_b,j_c,j_d)} g_{j_b}(k_{i_b}^{(j_c,j_d)})$$

where $\beta_i^{(j_a,j_b)} = c_i^{(0,0)}\widetilde{\eta}_i^{(j_a,j_b)}$ and $\lambda_{i\ i_b}^{(j_a,j_b,j_c,j_d)} = \alpha_{i,i_b}^{(j_a,j_b,j_c,j_d)}\widetilde{\eta}_i^{(j_a,j_b)}$.
For simplicity, we assume

$$\alpha_{ij}^{(0,j_b,j_c,j_d)} = \alpha_{ij}^{(1,j_b,j_c,j_d)} = \cdots = \alpha_{ij}^{(m,j_b,j_c,j_d)}$$

and consequently,

$$k_i^{(0,j_b)} = k_i^{(1,j_b)} = \cdots = k_i^{(m,j_b)}.$$

By expansion of (7) on indices j_b and j_d with the above assumption, the following result be derived:

$$y_{n+1} = y_n + \sum_{i=1}^{s}(\sum_{j_a=0}^{m} \beta_i^{(j_a,0)})g_0(k_i^{(j_a,0)}) + \cdots + \sum_{i=1}^{s}(\sum_{j_a=0}^{m} \beta_i^{(j_a,m)})g_m(k_i^{(j_a,m)}) \qquad (8)$$

$$k_i^{(j_a,j_b)} = y_n + \sum_{i_b=1}^{i-1}(\sum_{j_c=0}^{m} \lambda_{i\ i_b}^{(j_a,j_b,j_c,0)})g_0(k_{i_b}^{(j_c,0)}) + \cdots + \sum_{i_b=1}^{i-1}(\sum_{j_c=0}^{m} \lambda_{i\ i_b}^{(j_a,j_b,j_c,m)})g_m(k_{i_b}^{(j_c,m)})$$

and from (6) and (8) we have:

$$\sum_{j_a=0}^{m} \beta_i^{(j_a,\ell)} = \sum_{j_a=0}^{m} c_i^{(j_a,\ell)} \widetilde{\eta}_i^{(j_a,\ell)} = z_i^{(\ell)}$$
$$\sum_{j_c=0}^{m} \lambda_{ii_b}^{(j_a,j_b,j_c,\ell)} = \sum_{j_c=0}^{m} \alpha_{ii_b}^{(j_a,j_b,j_c,\ell)} \widetilde{\eta}_{i_b}^{(j_c,\ell)} = Z_{ii_b}^{(\ell)} \quad .$$

Therefore the "Butcher tabular" of the method, for example, if $s = 2$ and $m = 1$ will be:

0	0	0	0
$\sum_{j=0}^{m=1} \alpha_{21}^{(1,0,j,0)} \widetilde{\eta}_1^{(j,0)}$	0	$\sum_{j=0}^{m=1} \alpha_{21}^{(1,0,j,1)} \widetilde{\eta}_1^{(j,1)}$	0
0	0	0	0
$\sum_{j=0}^{m=1} \alpha_{21}^{(1,1,j,0)} \widetilde{\eta}_1^{(j,0)}$	0	$\sum_{j=0}^{m=1} \alpha_{21}^{(1,1,j,1)} \widetilde{\eta}_1^{(j,1)}$	0
$\sum_{j=0}^{m} c_1^{(j,0)} \widetilde{\eta}_1^{(j,0)}$	$\sum_{j=0}^{m} c_2^{(j,0)} \widetilde{\eta}_2^{(j,0)}$	$\sum_{j=0}^{m} c_1^{(j,1)} \widetilde{\eta}_1^{(j,1)}$	$\sum_{j=0}^{m} c_2^{(j,1)} \widetilde{\eta}_2^{(j,1)}$

Table 1: The Butcher tabular for $s = 2, m = 1$

3 Approximation of stochastic differential equation with one Wiener process

Consider the stratonovich SDEs with one Wiener process given by:

$$dy = g_0(y(t))dt + g_1(y(t)) \circ dW.$$

In this subsection,we are going to approximate this equation by bi–colored rooted tree,when $s = 2$. Here, we show the color,associated with deterministic case,with zero and another color,associated with stochastic case,with 1. komori[5] has given the analysis of weak order conditions of a SRK family for SDEs. The following trees are satisfying in weak order 1 condition:

$$\tau^{(0)}\ ,\ \tau^{(1)}\ ,\ [\tau^{(1)}]^{(1)}\ ,\ [\tau^{(0)}]^{(1)}\ ,\ [\tau^{(1)}]^{(0)}\ ,\ [[\tau^{(1)}]^{(1)}]^{(1)}$$
$$[\tau^{(1)}, \tau^{(1)}]^{(1)}, \left(\tau^{(1)}, \tau^{(1)}\right)$$

therefore the order conditions corrosponding to these trees, will be:

$$\begin{cases} \sum_{i_1=1}^{2}\sum_{j_1=0}^{1} c_{i_1}^{(j_1,0)} E(\widetilde{\eta}_{i_1}^{(j_1,0)}) = h \\ \sum_{i_1=1}^{2}\sum_{j_1=0}^{1} c_{i_1}^{(j_1,1)} E(\widetilde{\eta}_{i_1}^{(j_1,1)}) = 0 \\ \sum_{i_1,i_2=1}^{2}\sum_{j_1,j_2=0}^{1} c_{i_1}^{(j_1,1)} \alpha_{i_1 i_2}^{(j_1,1,j_2,1)} E(\widetilde{\eta}_{i_1}^{(j_1,1)}\widetilde{\eta}_{i_2}^{(j_2,1)}) = \frac{h}{2} \\ \sum_{i_1,i_2=1}^{2}\sum_{j_1,j_2=0}^{1} c_{i_1}^{(j_1,1)} c_{i_1}^{(j_1,1)} E(\widetilde{\eta}_{i_1}^{(j_1,1)}\widetilde{\eta}_{i_2}^{(j_2,1)}) = h \\ \sum_{i_1,i_2,i_3=1}^{2}\sum_{j_1,j_2,j_3=0}^{1} c_{i_1}^{(j_1,1)} \alpha_{i_1 i_2}^{(j_1,1,j_2,1)} \alpha_{i_2 i_3}^{(j_1,1,j_2,1)} E(\widetilde{\eta}_{i_1}^{(j_1,1)}\widetilde{\eta}_{i_2}^{(j_2,1)}\widetilde{\eta}_{i_3}^{(j_3,1)}) = 0 \\ \sum_{i_1,i_2,i_3=1}^{2}\sum_{j_1,j_2,j_3=0}^{1} c_{i_1}^{(j_1,1)} \alpha_{i_1 i_2}^{(j_1,1,j_2,1)} \alpha_{i_1 i_3}^{(j_1,1,j_2,1)} E(\widetilde{\eta}_{i_1}^{(j_1,1)}\widetilde{\eta}_{i_2}^{(j_2,1)}\widetilde{\eta}_{i_3}^{(j_3,1)}) = 0 \\ \sum_{i_1,i_2=1}^{2}\sum_{j_1,j_2=0}^{1} c_{i_1}^{(j_1,0)} \alpha_{i_1 i_2}^{(j_1,0,j_2,1)} E(\widetilde{\eta}_{i_1}^{(j_1,0)}\widetilde{\eta}_{i_2}^{(j_2,1)}) = 0 \\ \sum_{i_1,i_2=1}^{2}\sum_{j_1,j_2=0}^{1} c_{i_1}^{(j_1,1)} \alpha_{i_1 i_2}^{(j_1,1,j_2,0)} E(\widetilde{\eta}_{i_1}^{(j_1,1)}\widetilde{\eta}_{i_2}^{(j_2,0)}) = 0 \end{cases} \quad (9)$$

Note that our method is explicit, therefore the fifth equation is always holds. We assume:

$$\begin{aligned} &\widetilde{\eta}_1^{(1,0)} = 0 \ , \ \widetilde{\eta}_2^{(0,0)} = \widetilde{\eta}_2^{(1,0)} \ , \ \widetilde{\eta}_2^{(0,1)} = \widetilde{\eta}_2^{(1,1)} \ , \ \widetilde{\eta}_1^{(0,1)} = \widetilde{\eta}_1^{(1,1)} \\ &A_1 = \alpha_{21}^{(1,0,0,0)} \ , \ A_2 = \alpha_{21}^{(1,1,0,0)}, B_1 = \sum_{j=0}^{m} \alpha_{21}^{(1,0,j,1)} \\ &B_2 = \sum_{j=0}^{m} \alpha_{21}^{(1,1,j,1)} \ , \ d_1 = \sum_{j=0}^{m} c_1^{(j,1)} \ , \ d_2 = \sum_{j=0}^{m} c_2^{(j,1)} \\ &c_2 = \sum_{j=0}^{m} c_2^{(j,0)}. \end{aligned} \quad (10)$$

Therefore the Runge–Kutta methods formula and the Butcher tabular corresponding to these assumptions, will be:

$$\begin{cases} y_{n+1} = y_n + c_1^{(0,0)} Y_1^{(0,0)} + c_2 Y_2^{(0,0)} + d_1 Y_1^{0,1)} + d_2 y_2^{(0,1)} \\ Y_1^{(0,0)} = \widetilde{\eta}_1^{(0,0)} \ g_0(y_n) \\ Y_1^{(0,1)} = \widetilde{\eta}_1^{(0,1)} \ g_1(y_n) \\ Y_2^{(0,0)} = \widetilde{\eta}_2^{(0,0)} \ g_0(y_n + A_1 y_1^{(0,0)} + B_1 Y_1^{(0,1)}) \\ Y_2^{(0,1)} = \widetilde{\eta}_2^{(0,1)} \ g_1(y_n + A_2 y_1^{(0,0)} + B_2 Y_1^{(0,1)}) \end{cases} \quad (11)$$

and

$$\begin{array}{c|cc|cc} & 0 & 0 & 0 & 0 \\ & A_1\widetilde{\eta}_1^{(0,0)} & 0 & B_1\widetilde{\eta}_1^{(0,1)} & 0 \\ \hline & 0 & 0 & 0 & 0 \\ & A_2\widetilde{\eta}_1^{(0,0)} & 0 & B_2\widetilde{\eta}_1^{(0,1)} & 0 \\ \hline & c_1^{(0,0)}\widetilde{\eta}_1^{(0,0)} & c_2\widetilde{\eta}_2^{(0,0)} & d_1\widetilde{\eta}_1^{(0,1)} & d_2\widetilde{\eta}_2^{(0,1)} \end{array}$$

Table 2: Butcher tableau for $s = 2, m = 2$ with assumption(10)

The explicit Runge–Kutta coefficients in (9)will be reduced,for example:

$$\widetilde{\eta}_1^{(0,0)} = \widetilde{\eta}_2^{(0,0)} = h \quad , \quad c_1^{(0,0)} = c_2^{(0,0)} = \frac{1}{2} \quad , \quad A_1 = 1, \tag{12}$$

also,we can assume that $\widetilde{\eta}_1^{(0,1)} = \widetilde{\eta}_2^{(0,1)} = \Delta W = J_1$.
By substituting of (10) and (12) in (9),the following system be obtained:

$$\begin{cases} (1 + c_2^{(1,0)})h = h \ \rightarrow c_2^{(1,0)} = 0 \ \rightarrow c_2 = c_2^{(0,0)} \\ (d_1 + d_2) \times 0 = 0 \\ B_2.d_2 = \frac{1}{2} \qquad\qquad (13) \\ (d_1 + d_2)^2 = 1 \\ (B_2^2.d_2) \times 0 = 0 \\ B_1 \times 0 = 0 \\ (A_2.d_2) \times 0 = 0 \end{cases}$$

One solution of (13),is:

$$A_2 = B_1 = B_2 = 1$$
$$d_1 = d_2 = \frac{1}{2}$$

Therefore,The Runge–Kutta formula will be:

$$\begin{cases} y_{n+1} = y_n + \frac{1}{2}\left(Y_1^{(0,0)} + Y_2^{(0,0)} + Y_1^{0,1)} + Y_2^{(0,1)}\right) \\ Y_1^{(0,0)} = h \ g_0(y_n) \\ Y_1^{(0,1)} = J_1 \ g_1(y_n) \\ Y_2^{(0,0)} = h \ g_0(y_n + y_1^{(0,0)} + Y_1^{(0,1)}) \\ Y_2^{(0,1)} = J_1 \ g_1(y_n + y_1^{(0,0)} + Y_1^{(0,1)}) \qquad\qquad (14) \end{cases}$$

that can be presented by the following tableau where be named by "WEM1":

0	0	0	0
h	0	J_1	0
0	0	0	0
h	0	J_1	0
$\frac{1}{2}h$	$\frac{1}{2}h$	$\frac{1}{2}J_1$	$\frac{1}{2}J_1$

Table 3: The WEM1 Method

4 Numerical results

In this section, numerical results from the implementation of the above methods are compared with their exact solution.The methods are (14) correspond to one Wiener processes.

It will be implemented with constant stepsize.Since $J_i \sim N(0,h)$, $i = 1, 2, ...,$ hence for generating the wiener increments J_i, we use the random number generator $randn(\#traj, \#numstep)$ in Matlab, such that at each call `randn`$(\#traj, \#numstep)$ a $\#traj \times \#numstep$ matrix of independent $N(0,1)$ samples, be created.

In this paper, the solutions are computed by averaging the results on 1000 trajectory of simulation.

The average of error for each stepsize at the end of the interval of integration, for $G(x) = x$ in (1), is defined by:

$$AE = \frac{1}{K}\sum_{i=1}^{K} |y^i(t_N) - y_N^i|$$

where y_N^i and $y_{(t_N)}^i$ are the numerical approximation and the exact solution of SDE at t_N on the $i-$th simulation path over K simulations,respectively.

We have considered two test problems. The first problem is a pure SDE without deterministic part with one Wiener process. The second example has both deterministic and stochastic parts with two unknown parameter α and β. This problem be solved numerically with $\alpha = -1$ and different values $\beta = 1$ and 0.001.

Test Problem 1. Consider

$$dy = a(1 - y^2) \circ dW \qquad y(0) = 0, \ t \in [0, 1]$$

with the exact solution $y(t) = tanh(aW(t) + arctanh(y_0))$.

Test Problem 2. Consider

h	*Error*		CPU Time	
	$WEM1$	$Euler$	$WEM1$	$Euler$
$\frac{1}{25}$	0.0238	0.1863	1.0625	0.0313
$\frac{1}{50}$	0.0120	0.1865	1.9844	0.0469
$\frac{1}{100}$	0.0063	0.1800	4.0938	0.0938
$\frac{1}{200}$	0.0031	0.1738	7.9219	0.1094
$\frac{1}{400}$	0.0016	0.1805	15.8594	0.2031
$\frac{1}{800}$	0.0007592	0.1814	31.7813	0.3906

Table 4: Absolute errors and CPU time for the test Problem 1, with $a = 1,\ K = 1000$

$$dy = -\alpha(1-y^2)dt + \beta(1-y^2) \circ dW \qquad y(0) = 0.5,\ t \in [0,1]$$

with the exact solution :

$$y(t) = \frac{(1+y_0)exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1+y_0)exp(-2\alpha t + 2\beta W(t)) + 1 - y_0}.$$

In this example α and β are the coefficients of the deterministic and stochastic parts respectively. When $\beta = 1$ then maximum of the absolute error is about 0.0205 for $h = \frac{1}{25}$. It is clear that the exact and numerical solution of all equations have been calculated on the same simulation paths.

h	*Error*		CPU Time	
	$WEM1$	$Euler$	$WEM1$	$Euler$
$\frac{1}{25}$	0.0205	0.1158	1.7188	0.9219
$\frac{1}{50}$	0.0102	0.1137	3.3438	1.6719
$\frac{1}{100}$	0.0048	0.1120	6.5469	3.2031
$\frac{1}{200}$	0.0024	0.1231	12.9375	6.2500
$\frac{1}{400}$	0.0012	0.1149	25.75	12.3281
$\frac{1}{800}$	0.00062467	0.1235	51.4219	24.8281

Table 5: Absolute errors and CPU time for the test Problem 2, with $\alpha = -1, \beta = 1$ and $K = 1000$

Tables $4, 5$ show the absolute errors and CPU time of the WEM1 methods and the Euler method. All numerical results confirm the accuracy of this method with respect to the Euler method.

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