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Modelling Shocks and Solitons with a Modified Third Order High Resolution Semi-Discrete Scheme for Soliton and Shock Wave Simulations

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Abstract

The Kurganov and Levy scheme [SIAM J.Sci.Comput.22, p1467,2000] which is a semi-discrete numerical solution of hyperbolic systems is applied to the system of one dimensional electrostatic fluid equations and modified to model perturbations and shockwaves in an electrostatic plasma. For an initial density perturbations in the system, we illustrate how the method captures the formation and evolution of ion-acoustic solitons and shock waves.

Mathematics Subject Classification: 65M06

Keywords: Semi-discrete, hyperbolic, solitons, shock wave, high resolution schemes

1. Introduction

Shocks and solitons occur in various fluids such as electrostatic fluids consisting of electrons and ions. Theory, experiments and simulations [1] show that solitons and shocks occur both in the laboratory and space.To model the shocks and the solitons we use a modified version of the Kurganov Levy third order semi-discrete scheme (SD3). In a previous study [4] a fully discrete NNT scheme which is characterised by a limiter, produced nonoscillatory

pertubations and shock solutions whilst the SD3 produced oscillations. Hence the limiter was used in the CWENO reconstruction part of the SD3 scheme. We then compared the SD3 scheme to the another recently modified Nonstaggered Nessyahu Tadmor high resolution scheme (NNT).

2. The numerical integration scheme

We study the system of equations in the conservative form

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = g(u(x,t)) \qquad (1)$$

which is a one dimensional hyperbolic system of partial differential equations. Here $u(x,t)$ is the unkown m-dimensional vector function, $f(u)$ is the flux vector and $g(u)$ is a continuous source vector function on the right hand side (RHS), with x the single spatial coordinate and t the temporal coordinate.Such equations can be used to model many physical systems,including fluids and various types of gases,including electrical plasmas.

We apply a recent high-resolution semi-discrete numerical scheme due to Kurganov and Levy (2000) which can be applied to (1) as outlined in (Naidoo and Baboolal 2004). In applying this method we employ uniform spatial and temporal grids with spacing, $\Delta x = x_{j+1} - x_j$;

$\Delta t = t^{n+1} - t^n$ (with j and n being suitable integer indices) together with the semi-discrete scheme ("SD3") (R. Naidoo and Baboolal 2004),

$$\frac{d\bar{u}_j}{dt} = -\frac{1}{2\Delta x}\left[f\left(u^+_{j+\frac{1}{2}}(t)\right) + f\left(u^-_{j-\frac{1}{2}}(t)\right) - f\left(u^+_{j-\frac{1}{2}}(t)\right) - f\left(u^-_{j-\frac{1}{2}}(t)\right)\right]$$

$$-\frac{a_{j+\frac{1^1}{2}}(t)}{2\Delta x}\left[u^+_{j+\frac{1}{2}}(t) - u^-_{j+\frac{1}{2}}(t)\right] - \frac{a_{j-\frac{1}{2}}(t)}{2\Delta x}\left[u^+_{j-\frac{1}{2}}(t) - u^-_{j-\frac{1}{2}}(t)\right] + g(u_j(t)) \qquad (2)$$

The construction of this scheme is described in detail in (Kurganov and Levy 2000) and (R. Naidoo and Baboolal 2004).We note in particular that the solution is updated by fitting on already computed or known cell average values $\left\{\bar{U}^n_j\right\}$ at time level n, piecewise polynomials of degree two on cells of size Δx central at x_j namely

$$P_j(x,t^n) = A_j + B_j(x - x_j) + C_j(x - x_j)^2, \qquad (3)$$

Where the constants are $(A_j .....)$ are specified later.

$$\text{Here } u^{+}_{j\pm\frac{1}{2}} := P_{j+1}(x_{j\pm\frac{1}{2}}, t^n); u^{-}_{j\pm\frac{1}{2}} := P_j\left(x_{j\pm\frac{1}{2}}, t^n\right) \tag{4}$$

$$a^n_{j\pm\frac{1}{2}} = \max\left(\rho\left(\frac{\partial f}{\partial u}(u^{-}_{j\pm\frac{1}{2}}(t))\right), \rho\left(\frac{\partial f}{\partial u}(u^{+}_{j\pm\frac{1}{2}}(t))\right)\right) \tag{5}$$

Where the forms (4) are respectively the left and right intermediate values at $x_{j+\frac{1}{2}}$ and $\rho(.)$ denotes

the spectral radii of the respective flux Jacobian, defining the maximum local propagation speeds $a^n_{j\pm\frac{1}{2}}$. This scheme has been tested on problems involving shock propagation in various gas fluids (R. Naidoo and Baboolal 2004) and is known to give accurate results. Also, as a comparison we shall make use of a fully discrete scheme ("NNT") for hyperbolic systems with source terms, recently derived by the authors, as a modification of the Nessyahu and Tadmor scheme(R. Naidoo and Baboolal, 2003):

$$\bar{u}^{n+1}_j = \frac{1}{4}(\bar{u}^n_{j+1} + 2\bar{u}^n_j + \bar{u}^n_{j-1}) - \frac{1}{16}(u^n_{xj+1} - u^n_{xj-1}) - \frac{1}{8}\left[u^{n+1}_{xj+\frac{1}{2}} - u^{n+1}_{xj-\frac{1}{2}}\right]$$

$$+\frac{\Delta t}{8}[g(u^n_{j+1}) + 2g(u^n_j) + g(u^n_{j-1}) + g(u^{n+1}_{j+1}) + 2g(u^{n+1}_j) + g(u^{n+1}_{j-1})]$$

$$-\frac{\Delta t}{4\Delta x}[(f^n_{j+1} - f^n_{j-1}) + (f^{n+1}_{j+1} - f^{n+1}_{j-1})] \tag{6}$$

Again this scheme has been tested on similar problems as noted above, as well as on the electrostatic fluid equation to model ion-acoustic soliton propagation (R. Naidoo and Baboolal, 2003). Here we are interested in applying the SD3 scheme (2), together with a modification (11) on the electrostatic fluid equations, and in comparing its performance on them with the NNT scheme (6).

3. Some implementation details of the numerical schemes

The implementation of the NNT scheme above follows the report (R. Naidoo and Baboolal, 2005), where in particular we mention that the source term can make the scheme implicit. Then it requires fixed-point iterations to convergence at each grid point at every time level. On the other hand, the SD3 scheme is explicit in time. Thus the implementation of it follows closely the prescription given in (Kurganov and Levy 2000) and (R. Naidoo and Baboolal ,2004) where in particular we use for the non-oscillatory piece-wise polynomial (4) the "CWENO" reconstruction (Kurganov and Levy 2000) in which form (3) is determined by

$$A_j = \bar{u}^n_j - \frac{W_c}{12}(\bar{u}^n_{j+1} - 2\bar{u}^n_j + \bar{u}^n_{j-1}), \tag{7}$$

$$B_j = \frac{1}{\Delta x}\left[W_R(\bar{u}^n_{j+1} - \bar{u}^n_j) + \frac{W_c}{2}(\bar{u}^n_{j+1} - \bar{u}^n_{j-1}) + W_L(\bar{u}^n_j - \bar{u}^n_{j-1}) \right] \quad (8)$$

$$\text{And}\, C_j = \frac{W_j}{\Delta x^2}(\bar{u}^n_{j-1} - 2\bar{u}^n_j + \bar{u}^n_{j+1}) \quad (9)$$

The constants W_L, W_C, W_R and are calculated as in Kurganov and Levy (2000) and involve heuristic factors which have a bearing on the sharpness of the slopes near discontinuities. In addition, it is required to compute at every time step the spectral radii (5) of the Jacobian of the flux terms, which we obtained exactly for the case to follow. In some cases (at or near discontinuities) we shall find that solutions can be improved, by employing the so called nonlinear limiters, in the calculation of the derivative terms. Thus we employ the min-mod limiter function $MM(.)$ given by (Nessyahu and Tadmor 1990).

$$MM(s_1, s_2, ..) = \begin{Bmatrix} \min\{s_j\}\, if\ s_j > 0\ \forall_j \\ \max\{s_j\}\, if\ s_j < 0\ \forall_j \\ 0, otherwise \end{Bmatrix} \quad (10)$$

Then for (8) we obtain the improved form,

$$B_j = \frac{1}{\Delta x}\left[W_R MM(\bar{u}^n_{j+1} - \bar{u}^n_j, \bar{u}^n_j - \bar{u}^n_{j-1}) + \frac{W_c}{2} MM(\bar{u}^n_{j+1} - \bar{u}^n_j, \bar{u}^n_j - \bar{u}^n_{j-1}) \right]$$
$$+ \frac{1}{\Delta x} W_L [MM(\bar{u}^n_j - \bar{u}^n_{j-1}, \bar{u}^n_{j-1} - \bar{u}^n_{j-2})] \quad (11)$$

4.The electrostatic coupled fluid equations

The one-dimensional Euler-Poisson equations for an unmagnetized electrostatic system consisting of electrons and ions taken as ideal fluids together with the ideal gas law are:

$$\text{Continuity:} \frac{\partial n_k}{\partial t} + \frac{\partial (n_k v_k)}{\partial x} = 0 \quad (12)$$

$$\text{Momentum conservation:} m_k n_k \left(\frac{\partial v_k}{\partial t} + v_k \frac{\partial v_k}{\partial x} \right) + \frac{\partial p_k}{\partial x} = -q_k n_k \frac{\partial \phi}{\partial x} \quad (13)$$

$$\text{Equation of state: } p_k n_k^{-yk} = \text{constant} \quad (14)$$

Poisson's equation for the electric potential:

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi \sum_{k=e,i} q_k n_k \quad (15)$$

The index k=e(i) denotes electrons (ions) respectively and n_k, v_k, p_k, y_k, m_k and q_k are the respective component densities, flow velocities, partial pressures, adiabatic indices (=1 for isothermal electrons and =3 adiabatic

ions),particle masses and charges and ϕ is the electric potential.T_e and T_i are the respective electron and ion temperatures and $\mu_e = \frac{m_e}{m_i}$ is the electron to ion mass ratio. The final equations are suitably normalized to time and spatial scales appropriate for the observation of ion-acoustic wave structures.
With the above completed, the model equations may be written in the respective conservation and quasilinear forms (Baboolal, 2001).

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial F(U)}{\partial x} = G(U,\phi) \tag{16}$$

$$\frac{\partial U(x,t)}{\partial t} + A(U)\frac{\partial U}{\partial x} = G(U,\phi) \tag{17}$$

With the Poisson equation cast as

$$\frac{\partial^2 \phi}{\partial x^2} = u_1 - u_2 \equiv S(U) \tag{18}$$

In the above, $U = [u_1, u_2, u_3, u_4]^T$ is the 4-vector of conserved quantities (densities and currents), $F(U)$ is the flux vector function with $A(U)$ its Jacobian and $G(U,\phi)$ the vector function of the RHS of (12)-(13). In the flux Jacobian $A(U)$ the eigenvalues are real and distinct with linearly independent eigenvectors and hence justify the classification of the plasma fluid equations as hyperbolic, in the local sense. The coupled fluid-Poisson equations are solved as a system. For the numerical integration we employ a system length $L_x = 256\lambda_{de}$, with the number of grid points per Debye length $N_{px} = 10$, giving $\Delta x = 0.1$, ion/electron mass ratio of 50, an ion/electron temperature ratio of 1/100 together with $\Delta t = 0.01$. This choice satisfies the CFL condition (Baboolal 2001, Hirsch 1997) for linear stability given in terms of the spectral radius Λ_m of the Jacobian $A(U)$, $\Lambda_m \frac{\Delta t}{\Delta x} \le 1$.
In our application, we illustrate how solitons can be generated from an initial Gaussian density perturbation of the form

$$n_k(x,0) = 1.0 + 0.1\exp(-\frac{1}{2}(x - x_c)^2; 0 \le x \le L_x, \tag{19}$$

where x_c is the system centre and L_x is its length (See Baboolal, 2001) for more details).The initial velocities of the ions and electrons are set to zero for all x and reflective boundary conditions are employed (Baboolal and R. Naidoo, 2003).

5. Initial calculations using NNT and SD3

Employing both the NNT scheme and the SD3 scheme (2) with (7)-(9), we allowed the system to evolve from the initial equilibrium state, but with a

Gaussian density profile as given above. As noted in previous simulations (R. Naidoo and Baboolal, 2005, Baboolal, 2001), solitons form and evolve and get reflective from the boundaries. In the figure 1 below we observe the situation when some 20000 time steps have been reached. Two effects that are noticeable are that there is significant dissipation in the soliton structures in the NNT results. This behaviour is due to inherent dissipation noted by Kurganov and Levy (2000) of such schemes, an effect which manifests when $\Delta t \sim (\Delta x)^2$ as is the case here. Secondly, whilst the SD3 scheme captures the structures with apparently negligible dissipation, there appear to be unacceptable large random of high frequency oscillations in the curves, particularly noticeable in the electron fluid velocity profile. We believe that this effect no physical origin and is merely numerical noise.

6. Initial calculations using SD3 and SD3 with limiter

Using the same conditions for the SD3 scheme as above but using (10) and (11) in the CWENO reconstruction in the SD3 we again allowed the system to evolve from the initial equilibrium state, with a Gaussian density profile as given above. In figure 2 the SD3 scheme with limiter shows stability over a longer time period. The non physical oscillations have been considerably reduced in the SD3 limiter scheme. These results agree in the physics with previous simulations or no dissipation over long integration times. In the figure 2 below we see that there is less numerical noise in the SD3 scheme with limiter than the SD3 scheme. However the amplitude remains the same. In Table 1 below the CPU times for SD3 with limiter scheme is 10% and 8% larger than the NNT and SD3 schemes respectively.

Table 1

CPU times (s) for the NNT, SD3 and SD3 with limiter.
The solitons were computed for 20000 time units.

Scheme	Time(s)
NNT	698.21
SD3	718.40
SD3 with limiter	778.81

7. Initial Calculations for electrostatic shock waves

The initial calculations for the shock wave was taken as a Riemann (Shock Tube) problem for general gas problems. Here we use $\Delta t = 0.001, \Delta x = 0.1, x_c = 100\ grids$ where Δt is the time interval, Δx is the space interval and x_c is the discontinuity. The initial conditions are given as follows:

$$\begin{cases} n_{e(i)} = 2.5 \\ n_{e(i)} v_{e(i)} = 1.0 \end{cases}, x_c < 100 \; grids$$

$$\begin{cases} n_{e(i)} = 1.0 \\ n_{e(i)} v_{e(i)} = 0.4 \end{cases}, x_c > 100 \; grids$$

In fig 3 we observe shockwaves moving to the left and right together with the contact discontinuity moving to the right. Sharp electron waves are observed at the discontinuity whilst the ion waves remains relatively smooth. For larger times the density of the electron and ion waves tend to break-up at the discontinuity due to the fluctuation of the electric fields. The density and momentum propagation tend to be stable as time progresses.

8. Conclusion

We have illustrated how the electrostatic fluid equations can be numerically integrated with a modern high-resolution semi-discrete scheme. It has been applied to demonstrate ion-acoustic soliton and shock wave formation and propagation. The characteristics features of such solitons are adequately depicted with the SD3 scheme, as compared with results from a fully discrete NNT scheme, but fare poorly with the latter in suppressing non-physical noise. However, when the SD3 scheme is combined with a nonlinear limiter for the derivative terms, the non-physical oscillations are considerably reduced, resulting in superior results to those of the NNT scheme although the CPU in table 1 is larger. The SD3 shock wave solution is also very stable and comparable with other known results. It is thus expected that the SD3 scheme will be of great advantage in the numerical investigation of other nonlinear plasma fluid structures.

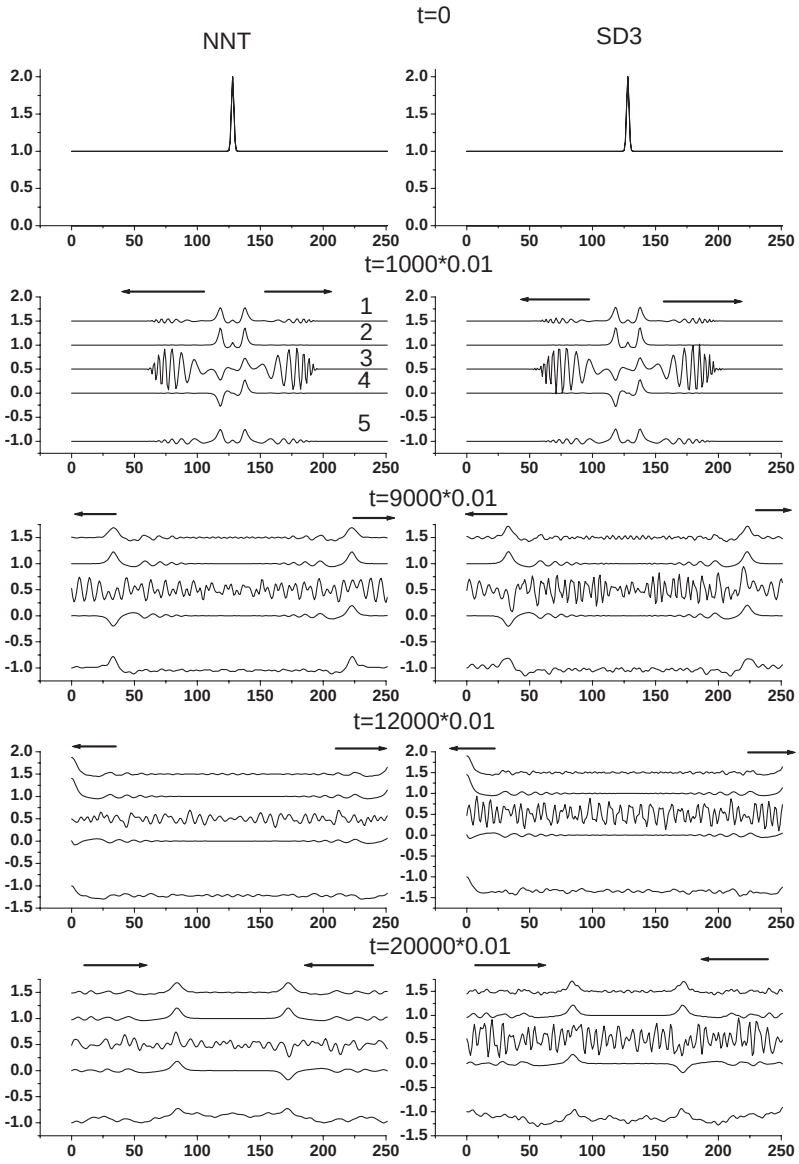

Fig 1: Formation and time evolution of a Gaussian-perturbation induced solitons in a plasma fluid with a NNT and SD3 schemes. Here the curve labelled $1 \sim n_e + 0.5, 2 \sim n_i\ 3 \sim v_{\mathrm{e}} + 0.5, 4 \sim v_i, 5 \sim \phi$ -1 where $n_e(n_i)$ is the electron/ion density, $v_e(v_i)$ is the electron/ion flow velocity, ϕ is the electrostatic potential all in normalised units.

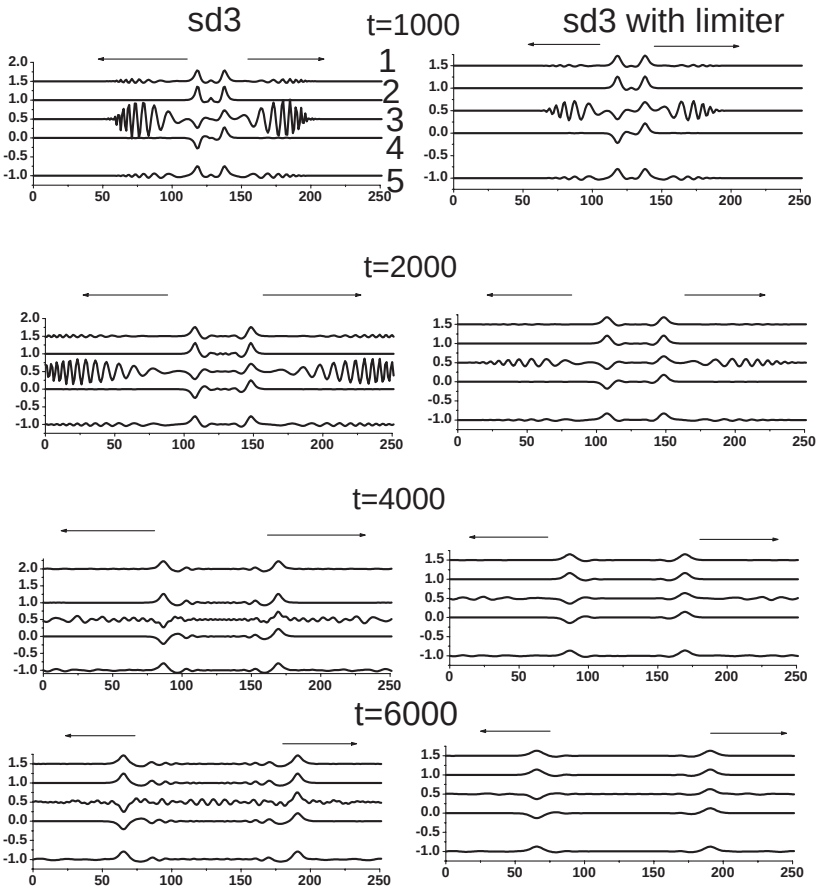

Fig 2 : Formation and time evolution of a Gaussian-perturbation induced solitons in a plasma fluid with an sd3 and sd3 limiter schemes. Here the curve labelled $1 \sim n_e + 0.5, 2 \sim n_i\ 3 \sim v_e + 0.5, 4 \sim v_i, 5 \sim \phi$ -1 where $n_e(n_i)$ is the electron/ion density, $v_e(v_i)$ is the electron/ion flow velocity, ϕ is the electrostatic potential all in normalised units.

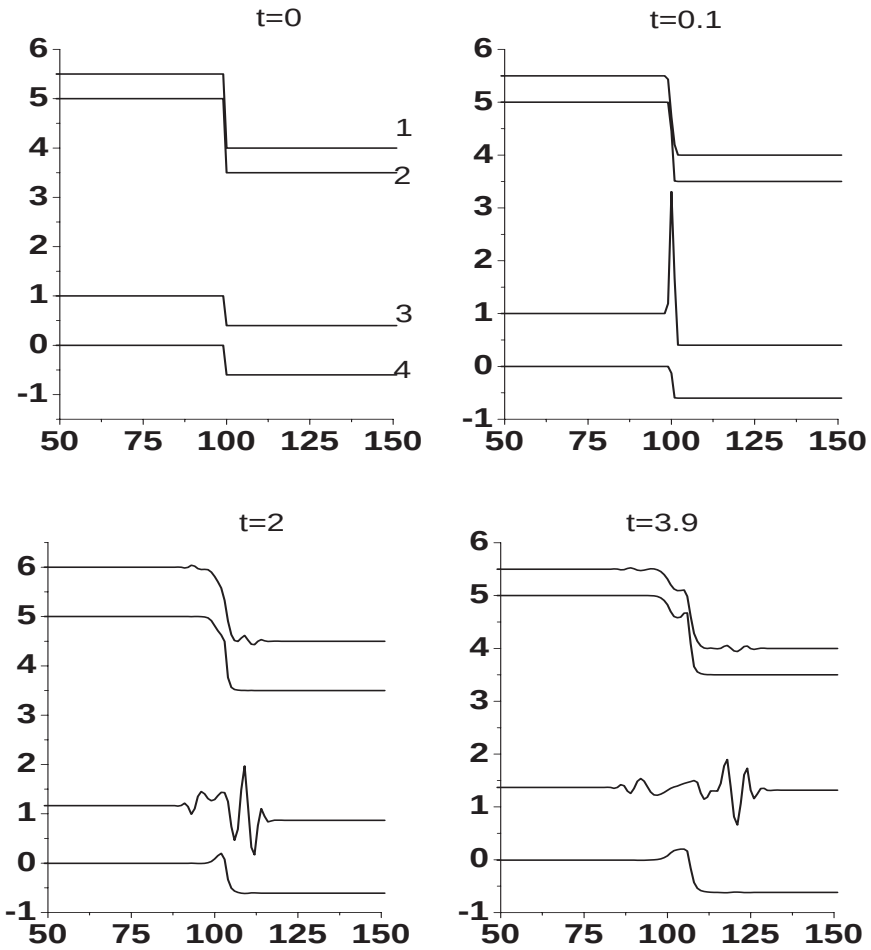

Fig 3: Formation and time evolution of shock waves in a plasma fluid. Here the curve labelled $1 \sim n_e + 3.5,\ 2 \sim n_i + 2.5,\ 3 \sim v_{\mathrm{e}}\ , 4 \sim v_i - 1$ where $n_e(n_i)$ is the electron/ion density, $v_e(v_i)$ is the electron/ion flow velocity, all in normalised units.

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