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Orthodox Semigroups With Inverse Transversals*

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Abstract: A new construction theorem for orthodox semigroups with inverse transversals was given.

Key words: regular semigroup; inverse transversal; orthodox semigroup

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0 Introduction and preliminaries

In 1982, Blyth and McFadden proposed the concept of inverse transversal[1]. If S is a regular semigroup, then an inverse transversal of S is an inverse subsemigroup S° such that there exists a unique inverse in S° for every $x \in S$. In what follows we write the unique inverse of x as x°, and denote $(x^\circ)^\circ$ by $x^{\circ\circ}$. Let $L = \{a \in S: a = aa^\circ a^{\circ\circ}\} = \{aa^\circ a^{\circ\circ}: a \in S\}$, $R = \{a \in S: a = a^{\circ\circ}a^\circ a\} = \{a^{\circ\circ}a^\circ a: a \in S\}$, $I = \{a \in S: a = aa^\circ\}$, $\Lambda = \{a \in S: a = a^\circ a\}$. Then $E(L) = I$, $E(R) = \Lambda$.

We list the following known results.

Let S be a regular semigroup with an inverse transversal S°, and let $E^\circ = E(S^\circ)$, then

(a) $(xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ(xyy^\circ)^\circ$ for any $x, y \in S$;[2]

$(xy^\circ)^\circ = y^{\circ\circ}x^\circ$, $(x^\circ y)^\circ = y^\circ x^{\circ\circ}$ for any $x, y \in S$;[3]

I and Λ are left and right regular bands with a common semilattice transversal E°, respectively; L and R are left and right inverse semigroups with a common inverse transversal S°, respectively[4];

I (resp. Λ) is a semilattice of left (resp. right) zero semigroups $\{I_{e^\circ}: e^\circ \in E^\circ\}$ (resp. $\{\Lambda_{e^\circ}: e^\circ \in E^\circ\}$), where $I_{e^\circ} = \{a \in I: a^\circ = e^\circ\}$ (resp. $\Lambda_{e^\circ} = \{a \in \Lambda: a^\circ = e^\circ\}$)[4,5].

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(b) S is orthodox iff $(xy)^\circ = y^\circ x^\circ$ for any $x, y \in S$[5];

S is orthodox iff $(li)^\circ = l^\circ i^\circ$ for any $l \in \Lambda$, $i \in I$[6];

S is orthodox iff S is quasi-orthodox and S° is weakly multiplicative[7].

(c) S° is weakly multiplicative iff for any $e \in E(S)$, $e^\circ \in E^\circ$[8].

Recall that a regular semigroup S is orthodox if the set of idempotents $E(S)$ is a band.

Proposition 1 Let S be a regular semigroup with an inverse transversal S°. Then the following statements are equivalent:

(1) S is orthodox;

(2) For any $l \in \Lambda$ and $i \in I$, $(li)^2 = li$, $(il)^2 = il$.

Proof It is clear that (1)⇒(2).

(2)⇒(1): For any $l \in \Lambda$ and $i \in I$, we have $lii^\circ l^\circ li = lili = li$ and $i^\circ l^\circ lii^\circ l^\circ = i^\circ lil^\circ = i^\circ ilill^\circ = i^\circ ill^\circ = i^\circ l^\circ$, thus $(li)^\circ = i^\circ l^\circ = l^\circ i^\circ$, it follows by (b) that S is orthodox.

1 A construction theorem for orthodox semigroups with inverse transversals

The full transformation semigroup of a semigroup S acting on the right side is denoted by $\mathscr{T}(S)$, and $\mathscr{T}^*(S)$ denotes the dual semigroup of $\mathscr{T}(S)$.

Theorem 1 Let L (resp. R) be a left (resp. right) inverse semigroup with an inverse transversal S°, and let $I = E(L) = \cup_{e^\circ \in E^\circ} I_{e^\circ}$ be a semilattice of left zero semigroups with a semilattice transversal E°, $\Lambda = E(R) = \cup_{e^\circ \in E^\circ} \Lambda_{e^\circ}$ be a semilattice of right zero semigroups with a semilattice transversal E°, where I_{e° is a left zero semigroup containing e°, and Λ_{e° is a right zero semigroup containing e°. If there exist homomorphisms $\lambda: R \to \mathscr{T}^*(I)$, $a \mapsto \lambda_a$, $\mu: L \to \mathscr{T}(\Lambda)$, $x \mapsto \mu_x$ satisfying the following conditions:

(1) $(\forall e^\circ \in E^\circ), (\forall a \in R), (\forall x \in L)$

$$\lambda_a I_{e^\circ} \subseteq I_{a^{\circ\circ} e^\circ a^\circ}, \quad \Lambda_{e^\circ}\mu_x \subseteq \Lambda_{x^\circ e^\circ x^{\circ\circ}}, \quad \lambda_a E^\circ \subseteq E^\circ, \quad E^\circ \mu_x \subseteq E^\circ;$$

(2) $(\forall e, g \in I), (\forall f, h \in \Lambda), (\forall a \in R), (\forall x \in L)$

$$\lambda_a(eg) = (\lambda_a e)(\lambda_a(\lambda_{(a^\circ a)\mu_e} g)), \quad (fh)\mu_x = ((f\mu_{\lambda_h(xx^\circ)})\mu_x)(h\mu_x);$$

(3) $(\forall g, u \in I), (\forall f, h \in \Lambda), (\forall b \in S^\circ)$, where $g^\circ = bb^\circ$, $h^\circ = b^\circ b$

$$(\lambda_{bh} u)b = b(\lambda_h u), \quad b(f\mu_{gb}) = (f\mu_g)b.$$

Define a multiplication on the set $I * S^\circ * \Lambda = \{(e, a, f) \in I \times S^\circ \times \Lambda: e \in I_{aa^\circ}, f \in \Lambda_{a^\circ a}\}$ by $(e, a, f)(g, b, h) = (e(\lambda_{af} g), ab, (f\mu_{gb})h)$. Then $I * S^\circ * \Lambda$ is an orthodox semigroup with an inverse transversal $(I * S^\circ * \Lambda)^\circ = \{(a^\circ a, a^\circ, aa^\circ): a \in S^\circ\}$.

Proof Let $(e, a, f), (g, b, h), (u, c, v) \in I * S^\circ * \Lambda$. By Condition (1), we have $\lambda_{af} g \in \lambda_{af} I_{g^\circ} = \lambda_{af} I_{bb^\circ} \subseteq I_{(af)^{\circ\circ} bb^\circ (af)^\circ}$, so $(\lambda_{af} g)^\circ = (af)^{\circ\circ} bb^\circ (af)^\circ = a^{\circ\circ} f^\circ bb^\circ f^\circ a^\circ = af^\circ bb^\circ a^\circ = aa^\circ abb^\circ a^\circ = abb^\circ a^\circ$, and so $(e(\lambda_{af} g))^\circ = (\lambda_{af} g)^\circ e^\circ = abb^\circ a^\circ e^\circ = abb^\circ a^\circ aa^\circ = abb^\circ a^\circ = ab(ab)^\circ$. Similarly, we have $((f\mu_{gb})h)^\circ = (ab)^\circ ab$. Thus $e(\lambda_{af} g) \in I_{ab(ab)^\circ}$, $(f\mu_{gb})h \in \Lambda_{(ab)^\circ ab}$. Thus the multiplication is well-defined.

At the same time,

$$
\begin{aligned}
&((e,a,f)(g,b,h))(u,c,v)\\
&=(e(\lambda_{af}g),\ ab,\ (f\mu_{gb})h)(u,c,v)\\
&=(e(\lambda_{af}g)(\lambda_{ab(f\mu_{gb})h}u),\ abc,\ (((f\mu_{gb})h)\mu_{uc})v)\\
&=(e(\lambda_{af}g)(\lambda_{ab(f\mu_{gb})h}u),\ abc,\ (((f\mu_{gb})\mu_{\lambda_h(uc(uc)^\circ)})\mu_{uc})(h\mu_{uc})v) && (\text{by } (2))\\
&=(e(\lambda_{af}g)(\lambda_{ab(f\mu_{gb})h}u),\ abc,\ (((f\mu_{gb})\mu_{\lambda_h u})\mu_{uc})(h\mu_{uc})v) && (\text{since } uc(uc)^\circ=u)\\
&=(e(\lambda_{af}g)(\lambda_{ab(f\mu_{gb})h}u),\ abc,\ (f\mu_{gb(\lambda_h u)uc})(h\mu_{uc})v) && (\text{since } \mu \text{ is homomorphic})\\
&=(e(\lambda_{af}g)(\lambda_{ab(f\mu_{gb})h}u),\ abc,\ (f\mu_{gb(\lambda_h u)(\lambda_h u)^\circ uc})(h\mu_{uc})v) && (\text{since } \lambda_h u\in I)\\
&=(e(\lambda_{af}g)(\lambda_{ab(f\mu_{gb})h}u),\ abc,\ (f\mu_{gb(\lambda_h u)h^\circ u^\circ uc})(h\mu_{uc})v) && (\text{since } \lambda_h u\in I_{h^\circ u^\circ})\\
&=(e(\lambda_{af}g)(\lambda_{af^\circ b(f\mu_{gb})h}u),\ abc,\ (f\mu_{gb(\lambda_h u)h^\circ u^\circ c})(h\mu_{uc})v)\\
&=(e(\lambda_{af}g)(\lambda_{aff^\circ bb^\circ b(f\mu_{gb})h}u),\ abc,\ (f\mu_{gb(\lambda_h u)c})(h\mu_{uc})v)\\
&=(e(\lambda_{af}g)(\lambda_{afbb^\circ f^\circ b(f\mu_{gb})h}u),\ abc,\ (f\mu_{g(\lambda_{bh}u)bc})(h\mu_{uc})v) && (\text{by } (3))\\
&=(e(\lambda_{af}g)(\lambda_{afb(f\mu_{gb})^\circ(f\mu_{gb})h}u),\ abc,\ (f\mu_{g(\lambda_{bh}u)bc})(h\mu_{uc})v) && (\text{by } (1),\ b^\circ f^\circ b=(f\mu_{gb})^\circ)\\
&=(e(\lambda_{af}g)(\lambda_{afb(f\mu_{gb})h}u),\ abc,\ (f\mu_{g(\lambda_{bh}u)bc})(h\mu_{uc})v) && (\text{since } f\mu_{gb}\in\Lambda)\\
&=(e(\lambda_{af}g)(\lambda_{af(f\mu_g)bh}u),\ abc,\ (f\mu_{g(\lambda_{bh}u)bc})(h\mu_{uc})v) && (\text{by } (3))\\
&=(e(\lambda_{af}g)(\lambda_{af}(\lambda_{f\mu_g}(\lambda_{bh}u))),\ abc,\ (f\mu_{g(\lambda_{bh}u)bc})(h\mu_{uc})v) && (\text{since } \lambda \text{ is homomorphic})\\
&=(e(\lambda_{af}g)(\lambda_{af}(\lambda_{((af)^\circ af)\mu_g}(\lambda_{bh}u))),\ abc,\ (f\mu_{g(\lambda_{bh}u)bc})(h\mu_{uc})v) && (\text{since } (af)^\circ af=f)\\
&=(e(\lambda_{af}(g(\lambda_{bh}u))),\ abc,\ (f\mu_{g(\lambda_{bh}u)bc})(h\mu_{uc})v) && (\text{by } (2))\\
&=(e,a,f)(g(\lambda_{bh}u),\ bc,(h\mu_{uc})v)\\
&=(e,a,f)((g,b,h)(u,c,v)).
\end{aligned}
$$

So $I*R$ is a semigroup.

Let $(e,a,f)\in I*S^\circ*\Lambda$ and $a^2=a$, then by Condition (1), we have $\lambda_{af}e\in\lambda_{af}I_{e^\circ}\subseteq I_{(af)^{\circ\circ}e^\circ(af)^\circ}=I_{a^{\circ\circ}f^\circ e^\circ f^\circ a^\circ}=I_{aa^\circ aaa^\circ a^\circ}=I_a$, similarly, $f\mu_{ea}\in\Lambda_a$. We also have that $e\in I_a$ and $f\in\Lambda_a$. It follows by the fact that I_a and Λ_a are respectively left and right zero semigroups containing a that $(e,a,f)(e,a,f)=(e(\lambda_{af}e),a^2,(f\mu_{ea})f)=(e,a,f)$. Conversely, let $(e,a,f)\in E(I*S^\circ*\Lambda)$, then $a^2=a$. Therefore $E(I*S^\circ*\Lambda)=\{(e,a,f)\in I*S^\circ*\Lambda:\ a^2=a\}$.

Let $(e,a,f),(g,b,h)\in I*S^\circ*\Lambda$, $b\in V(a)$. Then $aba=a$, $bab=b$. So $a^\circ=b$. Thus $\lambda_{af}g\in\lambda_{af}I_{g^\circ}\subseteq I_{(af)^{\circ\circ}g^\circ(af)^\circ}=I_{a^{\circ\circ}f^\circ g^\circ a^\circ}=I_{aa^\circ abb^\circ a^\circ}=I_{aa^\circ}$, $f\mu_{gb}\in\Lambda_{(gb)^\circ f^\circ(gb)^{\circ\circ}}=\Lambda_{b^\circ g^\circ f^\circ b^{\circ\circ}}=\Lambda_{b^\circ bb^\circ a^\circ ab}=\Lambda_{b^\circ b}$. Since I_{aa° is a left zero semigroup and $e\in I_{aa^\circ}$, $\Lambda_{b^\circ b}$ is a right zero semigroup and $h\in\Lambda_{b^\circ b}$, then $(e,a,f)(g,b,h)=(e(\lambda_{af}g),\ ab,\ (f\mu_{gb})h)=(e,ab,h)$. In the same manner we can verify that $(e,a,f)(g,b,h)(e,a,f)=(e,ab,h)(e,a,f)=(e,a,f)$. Similarly, $(g,b,h)(e,a,f)(g,b,h)=(g,b,h)$. So $(g,b,h)\in V((e,a,f))$. Conversely, let $(g,b,h)\in V((e,a,f))$, then $b\in V(a)$. Therefore $(g,b,h)\in V((e,a,f))$ in $I*S^\circ*\Lambda$ if and only if $b\in V(a)$.

Let $(I*S^\circ*\Lambda)^\circ=\{(a^\circ a,\ a^\circ,aa^\circ):a\in S^\circ\}$. Let $a,b\in S^\circ$, by Condition (1), we have $\lambda_{a^\circ}(b^\circ b)\in I_{a^\circ b^\circ ba}\cap E^\circ$ and $(aa^\circ)\mu_{b^\circ}\in\Lambda_{baa^\circ b^\circ}\cap E^\circ$, so $\lambda_{a^\circ}(b^\circ b)=a^\circ b^\circ ba$ and $(aa^\circ)\mu_{b^\circ}=$

$baa^\circ b^\circ$, and so

$$(a^\circ a, a^\circ, aa^\circ)(b^\circ b, b^\circ, bb^\circ) = (a^\circ a(\lambda_{a^\circ}(b^\circ b)), a^\circ b^\circ, ((aa^\circ)\mu_{b^\circ})bb^\circ)$$
$$= (a^\circ b^\circ ba, a^\circ b^\circ, baa^\circ b^\circ)$$
$$= ((ba)^\circ ba, (ba)^\circ, ba(ba)^\circ),$$

thus $(I * S^\circ * \Lambda)^\circ$ is a subsemigroup of $I * S^\circ * \Lambda$.

For any $(e,a,f) \in I * S^\circ * \Lambda$, we prove that $(a^\circ a, a^\circ, aa^\circ)$ is the unique inverse of (e,a,f) in $(I * S^\circ * \Lambda)^\circ$.

By Condition (1) and since I_{aa° is a left zero semigroup, Λ_{aa° and $\Lambda_{a^\circ a}$ are right zero semigroups, we have $\lambda_{af}(a^\circ a) \in I_{(af)^{\circ\circ}a^\circ a(af)^\circ} = I_{af^\circ a^\circ af^\circ a^\circ} = I_{aa^\circ}$, $f\mu_{a^\circ} \in \Lambda_{af^\circ a^\circ} = \Lambda_{aa^\circ}$, $(aa^\circ)\mu_{ea} \in \Lambda_{aa^\circ}\mu_{ea} \subseteq \Lambda_{(ea)^\circ aa^\circ(ea)^{\circ\circ}} = \Lambda_{a^\circ a}$ and $\lambda_{aa^\circ}e \in I_{(aa^\circ)^{\circ\circ}e^\circ(aa^\circ)^\circ} = I_{aa^\circ}$.

Again since $e \in I_{aa^\circ}$, $f \in \Lambda_{a^\circ a}$, we have

$$(e,a,f)(a^\circ a, a^\circ, aa^\circ)(e,a,f) = (e(\lambda_{af}(a^\circ a)), aa^\circ, (f\mu_{a^\circ})aa^\circ)(e,a,f)$$
$$= (e, aa^\circ, aa^\circ)(e,a,f)$$
$$= (e(\lambda_{aa^\circ}e), a, ((aa^\circ)\mu_{ea})f)$$
$$= (e,a,f).$$

Similarly, $(a^\circ a, a^\circ, aa^\circ)(e,a,f)(a^\circ a, a^\circ, aa^\circ) = (a^\circ a, a^\circ, aa^\circ)$. So $(a^\circ a, a^\circ, aa^\circ) \in V((e,a,f))$. Let $(b^\circ b, b^\circ, bb^\circ) \in V((e,a,f))$, then $ab^\circ a = a$, $b^\circ ab^\circ = b^\circ$, so $a = b$. It follows that $(a^\circ a, a^\circ, aa^\circ)$ is the unique inverse of (e,a,f) in $(I * S^\circ * \Lambda)^\circ$. Therefore $(I * S^\circ * \Lambda)^\circ$ is an inverse transversal of $I * S^\circ * \Lambda$.

Denote $X = \{x \in I * S^\circ * \Lambda : x = xx^\circ\}$, $Y = \{x \in I * S^\circ * \Lambda : x = x^\circ x\}$.

Let $a \in E^\circ$, $e \in I_a$, then $(e,a,a)(e,a,a)^\circ = (e,a,a)(a,a,a) = (e(\lambda_a a), a, (a\mu_a)a) = (ea,a,a) = (ee^\circ,a,a) = (e,a,a)$, so $(e,a,a) \in X$. Conversely, let $(e,a,f) \in X$, then $(e,a,f) = (e,a,f)(e,a,f)^\circ = (e,a,f)(a^\circ a, a^\circ, aa^\circ) = (e(\lambda_{af}(a^\circ a)), aa^\circ, (f\mu_{a^\circ})aa^\circ)$, so $a = aa^\circ \in E^\circ$, again since $f\mu_{a^\circ} \in \Lambda_a$, then $f = a$. Thus $X = \{(e,a,a) \in I \times E^\circ \times E^\circ : e \in I_a\}$.

Similarly, $Y = \{(a,a,f) \in E^\circ \times E^\circ \times \Lambda : f \in \Lambda_a\}$. It is clear that $E((I * S^\circ * \Lambda)^\circ) = \{(a,a,a) \in E^\circ \times E^\circ \times E^\circ\}$.

For any $(b,b,h) \in Y$, $(e,a,a) \in X$, we have $((b,b,h)(e,a,a))^\circ = (b(\lambda_{bh}e), ba, (h\mu_{ea})a)^\circ = (ba,ba,ba) = (b,b,b)(a,a,a) = (b,b,h)^\circ(e,a,a)^\circ$, by (b), $I * S^\circ * \Lambda$ is orthodox.

Theorem 2　Let S be an orthodox semigroup with an inverse transversal S°. Then $S \backsimeq I * S^\circ * \Lambda$, where $I = \{e \in S : e = ee^\circ\}$, $\Lambda = \{f \in S : f = f^\circ f\}$.

Proof　By (a), $L = \{a \in S : a = aa^\circ a^{\circ\circ}\}$ and $R = \{a \in S : a = a^{\circ\circ}a^\circ a\}$ are respectively left and right inverse semigroups with the common inverse transversal S°. $I = \cup_{e^\circ \in E^\circ} I_{e^\circ}$ is a semilattice of left zero semigroups, where I_{e° is a left zero semigroup containing e°, and $\Lambda = \cup_{e^\circ \in E^\circ}\Lambda_{e^\circ}$ is a semilattice of right zero semigroups, where Λ_{e° is a right zero semigroup containing e°. For any

$a, b \in R, x, y \in L, s \in I, t \in \Lambda$, we have $asa^\circ(asa^\circ)^\circ = asa^\circ a^{\circ\circ} s^\circ a^\circ = asa^\circ \in I$, $(x^\circ tx)^\circ x^\circ tx = x^\circ t^\circ x^{\circ\circ} x^\circ tx = x^\circ t^\circ tx = x^\circ tx \in \Lambda$. Let $\lambda_a : I \to I$ defined by $\lambda_a s = asa^\circ$, $\mu_x : \Lambda \to \Lambda$ by $t\mu_x = x^\circ tx$. Since $(\lambda_a \lambda_b)s = absb^\circ a^\circ = abs(ab)^\circ = \lambda_{ab} s$, $t(\mu_x \mu_y) = y^\circ x^\circ txy = (xy)^\circ txy = t\mu_{xy}$, then the mappings $\lambda : R \to \mathscr{T}^*(I)$, $a \mapsto \lambda_a$ and $\mu : L \to \mathscr{T}(\Lambda)$, $x \mapsto \mu_x$ are homomorphisms.

(1) For any $e^\circ \in E^\circ$, $a \in R$, $x \in L$, let $s \in I_{e^\circ}$, $t \in \Lambda_{e^\circ}$, then $(\lambda_a s)^\circ = (asa^\circ)^\circ = a^{\circ\circ} s^\circ a^\circ = a^{\circ\circ} e^\circ a^\circ$, $(t\mu_x)^\circ = (x^\circ tx)^\circ = x^\circ t^\circ x^{\circ\circ} = x^\circ e^\circ x^{\circ\circ}$, it follows by (a) that $\lambda_a s \in I_{a^{\circ\circ} e^\circ a^\circ}$ and $t\mu_x \in \Lambda_{x^\circ e^\circ x^{\circ\circ}}$, therefore $\lambda_a I_{e^\circ} \subseteq I_{a^{\circ\circ} e^\circ a^\circ}$ and $\Lambda_{e^\circ}\mu_x \subseteq \Lambda_{x^\circ e^\circ x^{\circ\circ}}$. It is clear that $\lambda_a E^\circ \subseteq E^\circ$ and $E^\circ \mu_x \subseteq E^\circ$.

(2) For any $e, g \in I$, $f, h \in \Lambda$, $a \in R$, $x \in L$, we have

$$
\begin{aligned}
(\lambda_a e)(\lambda_a(\lambda_{(a^\circ a)\mu_e} g)) &= aea^\circ a((a^\circ a)\mu_e) g((a^\circ a)\mu_e)^\circ a^\circ \\
&= aea^\circ ae^\circ a^\circ aeg(e^\circ a^\circ ae)^\circ a^\circ \\
&= aea^\circ aege^\circ(a^\circ a)^\circ e^\circ a^\circ \\
&= aea^\circ aega^\circ \\
&= aa^\circ aea^\circ aega^\circ \\
&= aega^\circ = \lambda_a(eg).
\end{aligned}
$$

Dually, $((f\mu_{\lambda_h(xx^\circ)})\mu_x)(h\mu_x) = (fh)\mu_x$.

(3) For any $g, u \in I$, $f, h \in \Lambda$, $b \in S^\circ$, where $g^\circ = bb^\circ$, $h^\circ = b^\circ b$, we have $(\lambda_{bh} u)b = bhu(bh)^\circ b = bhuh^\circ b^\circ b = bhuh^\circ = b(\lambda_h u)$. Dually, $b(f\mu_{gb}) = (f\mu_g)b$.

Thus Conditions (1), (2), (3) in Theorem 1 hold. So by Theorem 1, $I * S^\circ * \Lambda$ is an orthodox semigroup with an inverse transversal $(I * S^\circ * \Lambda)^\circ = \{(a^\circ a, a^\circ, aa^\circ) : a \in S^\circ\}$.

Define $\psi : S \to I * S^\circ * \Lambda$ by $x\psi = (xx^\circ, x^{\circ\circ}, x^\circ x)$ for any $x \in S$. It is obvious that ψ is injective. For any $(e, a, f) \in I * S^\circ * \Lambda$, since $(eaf)\psi = (eaf(eaf)^\circ, (eaf)^{\circ\circ}, (eaf)^\circ eaf) = (eaff^\circ a^\circ e^\circ, e^\circ af^\circ, f^\circ a^\circ e^\circ eaf) = (eaf^\circ a^\circ, a, a^\circ e^\circ af) = (eaa^\circ, a, a^\circ af) = (ee^\circ, a, f^\circ f) = (e, a, f)$, then ψ is surjective. For any $x, y \in S$, we have

$$
\begin{aligned}
(xy)\psi &= (xy(xy)^\circ, (xy)^{\circ\circ}, (xy)^\circ xy) \\
&= (xyy^\circ x^\circ, x^{\circ\circ} y^{\circ\circ}, y^\circ x^\circ xy) \\
&= (xx^\circ x^{\circ\circ} x^\circ xyy^\circ(x^{\circ\circ} x^\circ x)^\circ, x^{\circ\circ} y^{\circ\circ}, (yy^\circ y^{\circ\circ})^\circ x^\circ xyy^\circ y^{\circ\circ} y^\circ y) \\
&= (xx^\circ(\lambda_{x^{\circ\circ} x^\circ x}(yy^\circ)), x^{\circ\circ} y^{\circ\circ}, ((x^\circ x)\mu_{yy^\circ y^{\circ\circ}})y^\circ y) \\
&= (xx^\circ, x^{\circ\circ}, x^\circ x)(yy^\circ, y^{\circ\circ}, y^\circ y) \\
&= (x\psi)(y\psi).
\end{aligned}
$$

Therefore ψ is an isomorphism.

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具有逆断面的纯正半群

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摘要：本文给出了具有逆断面的纯正半群的一个新的构造定理.

关键词：正则半群；逆断面；纯正半群