Applied Mathematical Sciences, Vol. 1, 2007, no. 39, 1903 - 1913

A Method of Solving Singular Boundary Value Problems

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Abstract

In this paper,we present a new method for numerically solving singular two-point boundary value problems for certain ordinary differential equation having singular coefficients. The analytic solution is represented in the form of series in reproducing kernel space $W_2[0,1]$. Some numerical examples have been studied to demonstrate the accuracy of the present method.

Keywords: Numerical Solution; Singular Boundary Value Problems; Reproducing Kernel Space

1 Introduction

Singular boundary value problems for ordinary differential equation arise very frequently in several areas of science and engineering. Singular boundary value problems have been studied by several authors. To mention a few, Jamet[1] has discussed existence and uniqueness of solutions and presented finite difference method for numerically solving such problems. Gustafsson [2] has treated the problem by first writing the series solution in the neighborhood of the singularity and employing several compact and non-compact difference schemes in the remaining part of the interval. Cohen and Jones [3] have used an economized expansion to overcome the slow convergence of the Taylor series solution for the problems and employed deferred correction outside the range of economized expansion. They considered these polynomials on the whole interval where the polynomials are valid, neglecting the effect of singularity. Reddien [4] has

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studied collocation method for the numerical solution of such problems. There are results about solving algorithm on singular boundary value problems are reported, for details, see [8-14] and their references therein.

In this paper, we will consider a homogeneous second order linear differential equation having regular singularity given in [5] by :

$$u''(x) + m(x)u'(x) + n(x)u(x) = 0, 0 \le x \le 1 \tag{1.1}$$

subject to boundary conditions

$$u(0) = \alpha, u(1) = \beta \tag{1.2}$$

where the coefficient function $m(x)$ and $n(x)$ fail to be analytic at $x = 0$, and α, β are finite constants. Through transformation of function, (1.1), (1.2) can be converted into following equivalent form:

$$\begin{cases} p(x)u''(x) + f(x)u'(x) + g(x)u(x) = w(x) \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

where $p(0) = 0$.

Put $Lu(x) \equiv p(x)u''(x) + f(x)u'(x) + g(x)u(x)$, then Eqs.(1.3) can further be converted into following form:

$$\begin{cases} Lu(x) = w(x) \\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

It is easy to prove $L : W_2[0, 1] \to W_1[0, 1]$ is bound linear operator. We will give a new method in order to solve Eq.(1.4), the method of this paper is still effective for $p(x) = \varepsilon$. The representation of the exact solution is given in the reproducing kernel space. We only need choose appropriate reproducing kernel space according to boundary condition to solve the approximate solution.In the last section of the paper we also illustrate the numerical experiment . It shows our method is effective.

2 Several reproducing kernel spaces

In the section , several reproducing kernel spaces needed are introduced .

1. The reproducing kernel space $W_2[0, 1]$

$$\begin{aligned} W_2[0,1] = \{u(x)|u, u', u'' &\text{ are absolutely continuous real value functions,} \\ u, u', u'', u^{(3)} &\in L^2[0,1], u(0) = u(1) = 0\,\} \end{aligned} \tag{2.1}$$

and endowed it with the inner product and norm, respectively,

$$< u, v >_{W_2} = \int_0^1 36u(x)v(x) + 49u'(x)v'(x) + 14u''(x)v''(x) + u^{(3)}v^{(3)}dx \tag{2.2}$$

for $u(x), v(x) \in W_2[0,1]$, $\|u\|_{W_2[0,1]} =< u, u >^{\frac{1}{2}}$, $W_2[0,1]$ is a complete reproducing kernel space, that is, for any $u(y) \in W_2[0,1]$ and each fixed $x \in [0,1]$, there exists $R_x(y) \in W_2[0,1], y \in [0,1]$, such that $< u(y), R_x(y) >_{W_2} = u(x)$, the reproducing kernel $R_x(y)$ can be denoted by

$$R_x(y) = \begin{cases} c_1 e^y + c_2 e^{-y} + c_3 e^{2y} + c_4 e^{-2y} + c_5 e^{3y} + c_6 e^{-3y}, & y \le x, \\ d_1 e^y + d_2 e^{-y} + d_3 e^{2y} + d_4 e^{-2y} + d_5 e^{3y} + d_6 e^{-3y}, & y > x, \end{cases} \tag{2.3}$$

The coefficients of the reproducing kernel and the process of obtaining them see[6] and $W_2[0,1]$ is a complete space .

2 The reproducing kernel space $W_1[0,1]$

The inner space $W_1[0,1]$ is defined by $W_1[0,1] = \{u(x) \mid u$ is absolutely continuous real value function, $u' \in L^2[0,1]\}$. The inner product and norm in $W_1[0,1]$ are given respectively by

$$< u(x), v(x) >_{W_1} = \int_0^1 (uv + u'v')dx, \ \| u \|_{W_1} = \sqrt{< u, u >_{W_1}},$$

where $u(x), v(x) \in W_1[0,1]$. In Ref.[7], the author had proved that $W_1[0,1]$ is a reproducing kernel space and its reproducing kernel is

$$Q_x(y) = \frac{1}{2\sinh(1)}[\cosh(x + y - 1) + \cosh(|x - y| - 1)].$$

3 Solving of equation (1.4)

In this section,we shall give a representation of the exact solution of the Eqs.(1.4)

Lemma 3.1. *Suppose $\varphi_i(x) = R_{y_i}(x)$, $\{y_i\}_{i=1}^{\infty}$ is the maximum point of $|\rho_i(x)|$, $\psi_i(x) = L^*\varphi_i(x)$, then $\| \psi_i(x) \|_{W_2} \le \| L \| M, i = 1, 2, \dots,$ where L^* denotes the conjugated operator of L, M is a constant.*

Proof. $\| \psi_i(x) \|_{W_2}^2 = \| L^*\varphi_i(x) \|_{W_1}^2 \le \| L \| \| \varphi_i(x) \|_{W_1}^2 \le \| L \| (\varphi_i(x), \varphi_i(x))_{W_1} = \| L \| (\varphi_i(x), R_{y_i}(x))_{W_1}$
$= \| L \| \varphi_i(y_i) = \| L \| R_{y_i}(y_i) \le \| L \| M$. □

Suppose $u(x)$ is solution of Eqs.1.4, Let

$$r_0(x) = u(x), \rho_0(x) = w(x),$$

$$r_1(x) = r_0(x) - P_0 r_0(x), \dots, r_k(x) = r_{k-1}(x) - P_{k-1} r_{k-1}(x),$$

$$\rho_i(x) = \rho_0(x) - LP_0 r_0(x), \dots, \rho_k(x) = \rho_{k-1}(x) - LP_{k-1} r_{k-1}(x),$$

then $Lr_i(x) = \rho_i(x), i = 0, 1, 2, \dots$.

In order to discuss conveniently, we let:
(1) $P_i, i = 0, 1, \ldots$ is projective operator from $W_2[0, 1]$ to $\overline{\psi}_i(x)$.
(2)$\{y_i\}_{i=1}^{\infty}$ is the maximum point of $|\rho_i(x)|$.
(3) From lemma 3.1, we may get $B_i = \frac{1}{\|\psi_i(x)\|} \geq \frac{1}{M\|L\|} \overset{def}{=} M'$, $\overline{\psi_i}(x) = \frac{\psi_i(x)}{\|\psi_i(x)\|} = B_i\psi_i(x)$,.

Lemma 3.2. *let* $Lr_i(x) = \rho_i(x), i = 0, 1, \ldots,$ *then* $P_i r_i(x) = B_i^2 \rho_i(y_i)\psi_i(x), i = 0, 1, 2, \ldots.$

Proof.

$$\begin{aligned} P_i r_i(x) &= (r_i(x), \overline{\psi_i}(x))\overline{\psi_i}(x) \\ &= (r_i(x), \tfrac{\psi_i(x)}{\|\psi_i(x)\|})\tfrac{\psi_i(x)}{\|\psi_i(x)\|} \\ &= B_i^2(r_i(x), L^*\varphi_i(x))\psi_i(x) \\ &= B_i^2(Lr_i(x), \varphi_i(x))\psi_i(x) \\ &= B_i^2(\rho_i(x), R_{y_i}(x))\psi_i(x) \\ &= B_i^2\rho_i(y_i)\psi_i(x) \end{aligned}$$

□

Theorem 3.1. *Assume that* $r_0(x)$ *is the solution of Eqs.*(1.4)*, then* $r_k(x)$ *is monotone decreasing in the sense of* $\|.\|_{W_2}$.

Proof. From lemma 3.2, we have

$$\begin{aligned} &\|r_{k+1}(x)\|^2 \\ &= (r_k(x) - P_k r_k(x), r_k(x) - P_k r_k(x)) \\ &= \| r_k(x) \|^2 - 2(r_k(x), P_k r_k(x)) + (P_k r_k(x), P_k r_k(x)) \\ &= \| r_k(x) \|^2 - 2(r_k(x), B_k^2 \rho_k(y_k)\psi_k(x)) + B_k^4 \rho_k^2(y_k)(\psi_k(x), \psi_k(x)) \\ &= \| r_k(x) \|^2 - 2B_k^2 \rho_k(y_k)(r_k(x), \psi_k(x)) + B_k^2 \rho_k^2(y_k)(\overline{\psi_k}(x), \overline{\psi_k}(x)) \\ &= \| r_k(x) \|^2 - 2B_k^2 \rho_k(y_k)(r_k(x), L^*\varphi_k(x)) + B_k^2 \rho_k^2(y_k). \end{aligned}$$

Since $(r_k(x), L^*\varphi_k(x)) = (Lr_k(x), \varphi_k(x)) = (\rho_k(x), \varphi_k(x)) = \rho_k(y_k)$, it follows that

$$\| r_{k+1}(x) \|^2 = \| r_k(x) \|^2 - B_k^2 \rho_k^2(y_k). \tag{3.1}$$

Therefore $r_k(x)$ is monotone decreasing. □

Repeat (3.1) , we obtain

$$\| r_{k+1}(x) \|^2 = \| r_0(x) \|^2 - \sum_{i=0}^{k} B_i^2 \rho_i^2(y_i). \tag{3.2}$$

In order to obtain the representation of the exact solution of (1.4), we give the following theorems.

Theorem 3.2. *Suppose solution of Eqs.*(1.4) *is existential, then* $\rho_i(x) \in l^2, i = 0, 1, 2, \ldots.$

Proof. If solution of $Eqs.$(1.4) is existential, then

$$Lu(x) = Lr_0(x) = \rho_0(x) = w(x).$$

From Theorem 3.1, $\| r_k \|$ is convergent. By the (3.2) , we know $\sum_{i=0}^{\infty} B_i^2 \rho_i^2(y_i)$ is convergent. By the Lemma 3.1, $B_i > M' > 0$, then

$$\rho_i(y_i) \in l^2, i = 0, 1, 2, \ldots .$$

Note that $\{y_i\}_{i=1}^{\infty}$ is maximum point of $|\rho_i(x)|$, it shows that $\rho_i(x) \in l^2, i = 0, 1, 2, \ldots$. □

Corollary 3.1. *$\rho_i(x)$ is uniformly convergent to 0 with respect to x.*

Theorem 3.3. *Suppose $\rho_i(x) \in l^2$, for any $x \in [0, 1]$, then*

$$\rho_0(x) = \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i) L\psi_i(x). \tag{3.3}$$

Proof. By the assumption, we have

$$\rho_{i+1}(x) = \rho_i(x) - LP_i r_i(x) = \rho_i(x) - B_i^2 \rho_i(y_i) L\psi_i(x), i = 0, 1, 2, \ldots . \tag{3.4}$$

we use (3.4) repeatedly, we obtain

$$\rho_{i+1}(x) = \rho_0(x) - \sum_{i=0}^{k} LP_i r_i(x) = \rho_0(x) - \sum_{i=0}^{k} B_i^2 \rho_i(y_i) L\psi_i(x), i = 0, 1, 2, \ldots \tag{3.5}$$

Taking limits for k on both sides of (3.5) and by the Corollary 3.1, it follows that

$$\rho_0(x) = \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i) L\psi_i(x), \forall x \in [0, 1].$$

□

Finally, we give the main theorem of this paper.

Theorem 3.4. *Suppose that the solution of Eqs(1.4) exists and is unique, $\rho_{i+1}(x) = \rho_i(x) - B_i^2 \rho_i(y_i) L\psi_i(x), \rho_i(x) \in l^2, \forall x \in [0, 1]$, and $B_i, i = 0, 1, 2, \ldots$ is bounded, then solution of Eqs.(1.4) can be expressed as follows*

$$u(x) = r_0(x) = \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i) \psi_i(x), \tag{3.6}$$

where y_i is the maximum point of $| \rho_i(x) |, x \in [0, 1]$.

Proof. Let M_0 is a sufficient large number and let

$$\overline{r}_n(x) = \sum_{i=n}^{M_0} B_i^2 \rho_i(y_i)\psi_i(x), n = 0, 1, 2, \dots, M_0,$$

then

$$\overline{r}_{n+1}(x) = \overline{r}_n(x) - B_n^2 \rho_n(y_n)\psi_n(x).$$

Since

$$L\overline{r}_i(x) = \rho_i(x), i = 0, 1, 2, \dots, M_0,$$

by the way of Theorem 3.1, we also obtain

$$\| \overline{r}_{m+1} \|_{W_2}^2 = \| \overline{r}_m \|_{W_2}^2 - B_m^2 \rho_m^2(y_m), m = 0, 1, 2, \dots, M_0 - 1, \tag{3.7}$$

then

$$\| \overline{r}_{M_0} \|_{W_2}^2 = \| \overline{r}_m \|_{W_2}^2 - \sum_{i=0}^{M_0-1} B_i^2 \rho_i^2(y_i),$$

namely

$$\| \overline{r}_{M_0} \|_{W_2}^2 = \| \sum_{i=0}^{M_0} B_i^2 \rho_i(y_i)\psi_i(x) \|_{W_2}^2 - \sum_{i=0}^{M_0-1} B_i^2 \rho_i^2(y_i),$$

and

$$\| \overline{r}_{[\frac{M_0}{2}]} \|_{W_2}^2 = \| \sum_{i=0}^{[\frac{M_0}{2}]} B_i^2 \rho_i(y_i)\psi_i(x) \|_{W_2}^2 - \sum_{i=0}^{[\frac{M_0}{2}]-1} B_i^2 \rho_i^2(y_i).$$

If $\overline{r}_0(x)$ is replaced by $\overline{r}_n(x)$, we have

$$\| \overline{r}_{[\frac{M_0}{2}]+n} \|_{W_2}^2 = \| \sum_{i=n}^{[\frac{M_0}{2}]} B_i^2 \rho_i(y_i)\psi_i(x) \|_{W_2}^2 - \sum_{i=n}^{[\frac{M_0}{2}]-1} B_i^2 \rho_i^2(y_i), n \le [\frac{M_0}{2}] - 1 \tag{3.8}$$

and

$$\| \overline{r}_{[\frac{M_0}{2}]+n+1} \|_{W_2}^2 = \| \sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2 \rho_i(y_i)\psi_i(x) \|_{W_2}^2 - \sum_{i=n+1}^{[\frac{M_0}{2}]-1} B_i^2 \rho_i^2(y_i), n \le [\frac{M_0}{2}] - 1 \tag{3.9}$$

Making deviation on the both sides of (3.8) and (3.9), then

$$\begin{aligned} & \| \overline{r}_{[\frac{M_0}{2}]+n} \|_{W_2}^2 - \| \overline{r}_{[\frac{M_0}{2}]+n+1} \|_{W_2}^2 \\ = & \| \textstyle\sum_{i=n}^{[\frac{M_0}{2}]} B_i^2 \rho_i(y_i)\psi_i \|_{W_2}^2 - \| \textstyle\sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2 \rho_i(y_i)\psi_i(x) \|_{W_2}^2 - B_n^2 \rho_n(y_n), \end{aligned} \tag{3.10}$$

where

$$
\begin{aligned}
&\| \sum_{i=n}^{[\frac{M_0}{2}]} B_i^2\rho_i(y_i)\psi_i \|_{W_2}^2\\
= &< B_n^2\rho_n(y_n)\psi_n(x) + \sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2\rho_i(y_i)\psi_i(x), B_n^2\rho_n(y_n)\psi_n(x)\\
&+ \sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2\rho_i(y_i)\psi_i(x) >_{W_2}\\
= &\| \sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2\rho_i(y_i)\psi_i(x) \|_{W_2}^2 + \| B_n^2\rho_n(y_n)\psi_n \|_{W_2}^2\\
&+2 < B_n^2\rho_n(y_n)\psi_n(x), \sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2\rho_i(y_i)\psi_i(x) >_{W_2}\\
= &\| \sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2\rho_i(y_i)\psi_i(x) \|_{W_2}^2 + B_n^2\rho_n(y_n)\\
&+2 < B_n^2\rho_n(y_n)\psi_n(x), \sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2\rho_i(y_i)\psi_i(x) >_{W_2} .
\end{aligned}
\tag{3.11}
$$

Taking above equality into (3.10), we have

$$
\begin{aligned}
&2 < B_n^2\rho_n(y_n)\psi_n(x), \sum_{i=n+1}^{[\frac{M_0}{2}]} B_i^2\rho_i(y_i)\psi_i(x) >_{W_2}\\
= &\| \bar{r}_{[\frac{M_0}{2}]+n} \|_{W_2}^2 - \| \bar{r}_{[\frac{M_0}{2}]+n+1} \|_{W_2}^2,
\end{aligned}
\tag{3.12}
$$

from (3.7),

$$
\| \bar{r}_{[\frac{M_0}{2}]+n} \|_{W_2}^2 - \| \bar{r}_{[\frac{M_0}{2}]+n+1} \|_{W_2}^2 = B^2_{[\frac{M_0}{2}]+n}\rho^2_{[\frac{M_0}{2}]+n}(y_{[\frac{M_0}{2}]+n}),
$$

Since B_i is bounded and $\rho_i(x) \to 0$, we obtain

$$
2 < B_n^2\rho_n(y_n)\psi_n(x), \sum_{i=n}^{\infty} B_i^2\rho_i(y_i)\psi_i(x) >_{W_2} = 0. \tag{3.13}
$$

Equality (3.13) holds for any $n \in N$, thus

$$
B_i^2 B_j^2 \rho_i(y_i)\rho_j(y_j) < \psi_i(x), \psi_j(x) >_{W_2} = 0, (i \neq j). \tag{3.14}
$$

Finally, by (3.14), it follows that

$$
\begin{aligned}
&\| \sum_{i=n}^{M_0} B_i^2\rho_i(y_i)\psi_i \|_{W_2}^2\\
= &< \sum_{i=n}^{M_0} B_i^2\rho_i(y_i)\psi_i(x), \sum_{i=n}^{M_0} B_i^2\rho_i(y_i)\psi_i(x) >_{W_2}\\
= &\sum_{i=n}^{M_0} \| B_i^2\rho_i(y_i)\psi_i(x) \|_{W_2}^2\\
&+ \sum_{i=n,j=n}^{M_0} \sum_{j\neq i}^{M_0} B_i^2 B_j^2 \rho_i(y_i)\rho_j(y_j) < \psi_i(x), \psi_j(x) >_{W_2}\\
= &\sum_{i=n}^{M_0} B_i^2\rho_i^2(y_i),
\end{aligned}
\tag{3.15}
$$

where we used $B_i \| \psi_i \|_{W_2} = 1$.
Since $B_i < M$, series $\sum_{i=n}^{M_0} B_i^2\rho_i(y_i) \to 0, M_0, n \to \infty$, namely

$$
\| \sum_{i=n}^{M_0} B_i^2\rho_i(y_i)\psi_i \|_{W_2}^2 \to 0, M_0, n \to \infty,
$$

hence series $\sum_{i=0}^{\infty} B_i^2 \rho_i(y_i)\psi_i(x)$ is convergent. Let

$$u(x) = \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i)\psi_i(x),$$

we have

$$Lu(x) = L\sum_{i=0}^{\infty} B_i^2 \rho_i(y_i)\psi_i(x) = \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i)L\psi_i(x).$$

By the Theorem 3.3, the right of above equality amounts to $\rho_0(x)$, so

$$Lu(x) = \rho_0(x),$$

as

$$u(x) = \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i)\psi_i(x)$$

is solution of $Eqs.$(1.4). □

The approximate solution

$$u_n(x) = \sum_{i=0}^{n} B_i^2 \rho_i(y_i)\psi_i(x) \tag{3.16}$$

is obtained by truncating the series (3.6).

Summarizing above discussion, by Theorem 3.2, if solution of $Eqs.$(1.4) is existential, then $\rho_i(x) \in l^2, i = 0, 1, 2, \ldots$, by Theorem 3.4, if $\rho_i(x) \in l^2, i = 0, 1, 2, \ldots$,, then

$$u(x) = \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i)\psi_i(x)$$

is solution of $Eqs.$(1.4).

Corollary 3.2. *The solution of* $Eqs.$(1.4) *is*

$$u(x) = \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i)\psi_i(x)$$

if and only if $\rho_i(x) \in l^2$, *where* $\rho_{i+1}(x) = \rho_i(x) - B_i^2 \rho_i(y_i)L\psi_i(x), i = 0, 1, 2, \ldots$.

Theorem 3.5. $r_k(x) \to 0$ *as* $k \to \infty$ *in the sense of* $\| \cdot \|_{W_2}$

Proof. From Theorem 3.1, $\| r_k \|_{W_2}$ is convergent, we suppose that $\| r_k \|_{W_2} \to c$,(c is constant). Taking linits for k on both sides of (3.2), we have

$$c = \| r_0 \|_{W_2} - \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i),$$

where by (3.14),

$$\begin{array}{rl} & \| r_0 \|_{W_2} \\ = & \| \sum_{i=0}^{\infty} B_i^2 \rho_i(y_i)\psi_i(x) \|_{W_2} \\ = & < \sum_{i=n}^{M_0} B_i^2 \rho_i(y_i)\psi_i(x), \sum_{i=n}^{M_0} B_i^2 \rho_i(y_i)\psi_i(x) >_{W_2} \\ = & \sum_{i=0}^{\infty} B_i^2 \rho_i^2(y_i) + \sum_{i=0,j=0}^{\infty} \sum_{j\neq i}^{\infty} B_i^2 B_j^2 \rho_i(y_i)\rho_j(y_j) < \psi_i(x), \psi_j(x) >_{W_2} \\ = & \sum_{i=0}^{\infty} B_i^2 \rho_i^2(y_i), \end{array} \tag{3.17}$$

thus

$$c = \sum_{i=0}^{\infty} B_i^2 \rho_i^2(y_i) - \sum_{i=0}^{\infty} B_i^2 \rho_i^2(y_i) = 0,$$

namely
$\| r_k \|_{W_2} \to 0$, as $k \to \infty$. Consequently, $r_k(x) \to 0$, as $k \to \infty$, in the sense of $\| \cdot \|_{W_2}$.

□

4 solution of Eqs.(1.4) and a numerical example

According to (3.16) we will present a numerical example for solving Eqs.(1.4) in the reproducing kernel space $W_2[0,1]$. All computations are performed by the Mathematica 5.0 software package.

Example

Considering equation

$$\begin{cases} 4x(x+1)u''(x) + (3+11x)u'(x) + u(x) = w(x) \\ u(0) = u(1) = 0 \end{cases}$$

where $x \in [0,1]$. The true solutions are $u(x) = 9x(e^{x-1} - 1), w(x) = 9(-3e(1+4x) + e^x(3 + 23x + 23x^2 + 4x^3))/e$. Using our method, we obtain approximate solution $u_n(x)(n = 100)$ on $[0,1]$. The numerical results are given in following Table 1.

Table 1:

Node	True solution $u(x)$	Approximate solution $u_n(x)$	Absolute error	Relative error
0.02	-0.112444	-0.112467	2.26149E-5	2.01122E-4
0.10	-0.534087	-0.533464	6.23436E-4	1.16729E-3
0.18	-0.906501	-0.905599	9.0147E-4	9.9445E-4
0.26	-1.22355	-1.22282	7.29015E-4	5.95818E-4
0.34	-1.47843	-1.47794	4.98556E-4	3.37219E-4
0.42	-1.66358	-1.66332	2.65589E-4	1.59649E-4
0.50	-1.77061	-1.77056	4.85874E-5	2.7441E-5
0.58	-1.79022	-1.79033	1.17293E-4	6.5519E-5
0.66	-1.71208	-1.71231	2.23113E-4	1.30317E-4
0.74	-1.5248	-1.52506	2.62453E-4	1.72123E-4
0.82	-1.21571	-1.21594	2.3552E-4	1.93731E-4
0.90	-0.770817	-0.770985	1.67664E-4	2.17514E-4
0.98	-0.174648	-0.17469	4.20038E-5	2.40506E-4

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Received: January 12, 2007