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Nontrivial Solution of a Quasilinear Singular Elliptic Equation With Critical Sobolev-Hardy Exponent*

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Abstract: The nontrivial solution of a quasilinear singular elliptic equation is studied with critical Sobolev-Hardy exponent by virtue of Sobolev—Hardy inequality and the Mountain Pass Geometry.

Key words: quasilinear elliptic equation; critical exponent; nontrivial solution; Sobolev-Hardy inequality

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0 Introduction and the result of existence

In this paper we shall study the existence of the boundary value weighted problem:

$$-\operatorname{div}(|x|^{\beta}\nabla u) = |x|^{\alpha}|u|^{p-2}u + \lambda|x|^{\sigma}|u|^{q-2}u, \quad \text{in } \Omega;\ u = 0, \quad \text{on } \partial\Omega. \tag{1}$$

Here Ω is a bounded smooth domain containing the origin and we assume that the critical Sobolev—Hardy exponent $p = 2(N+\alpha)/(N+\beta-2)$, and suppose that $N \geqslant 3$, $2 < q < p$, $N+\alpha > 0, \alpha + 2 > \beta$, $\sigma + 2 > \beta$, $\frac{\beta}{2} \geqslant \frac{\alpha}{p}$ and $\beta \leqslant 0$. This type of problem has been studied in several papers. Let us mention some of them. The case when $\beta = 0, \alpha = 0$ and $q = 2$ was first treated in a famous paper by Brezis and Nirenberg[1]. Some of their results have been generalized by Escobar[2], including the case with variable nonsingular coefficients.

For the following generalization of problem:

$$-\operatorname{div}(|x|^{\beta}|\nabla u|^{m-2}\nabla u) = |x|^{\alpha}u^{p-1} + \lambda|x|^{\sigma}u^{m-1}, \text{ in } \Omega;\ u > 0, \text{in } \Omega;\ u = 0, \text{on } \partial\Omega.$$

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The restrictions on Ω are the same as before and Egnell [4] has given a brief discussion for a radial solution when $m+\alpha>\beta$, $N+\beta>m$, $m+\sigma>\beta$ and $p=m(N+\alpha)/(N+\beta-m)$.

We shall apply the variational methods to the energy functional associated with (1), namely

$$I(u)=\frac{1}{2}\int_\Omega |x|^\beta|\nabla u|^2-\frac{1}{p}\int_\Omega|x|^\alpha|u|^p-\frac{\lambda}{q}\int_\Omega|x|^\sigma|u|^q, \tag{2}$$

In order to deal with $I(u)$, let $H^1_{0,\beta}(\Omega)$ be the completion of $\{u\in C^1(\overline{\Omega})\,|\,u=0,\ x\in\partial\Omega\}$ with norm given by

$$\|u\|\left(\int_\Omega|x|^\beta|\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}, \tag{3}$$

We know when p, α and β satisfy that

$$p>2,\ N+\alpha>2,\ p=\frac{2(N+\alpha)}{N+\beta-2},\ \frac{\beta}{2}\geqslant\frac{\alpha}{p}, \tag{4}$$

the weighted Sobolev – Hardy inequality[5,9] asserts that there exists a positive constant c such that

$$\left(\int_{\mathbb{R}^N}|x|^\alpha|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\leqslant c\left(\int_{\mathbb{R}^N}|x|^\beta|\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}, \tag{5}$$

for all $u\in H^1_{0,\beta}(\mathbb{R}^N)$.

The result of this paper is:

Theorem If $\max\left(2,\frac{2(N+\sigma)}{N+\beta-2}-2\right)<q<p$ and $\beta\leqslant 0$. Then for all $\lambda>0$, problem (1) has a nontrivial solution.

1 The proof of some lemmas

Before proving the theorem, we prove several lemmas. Firstly we prove a compactness result.

Lemma 1 $H^1_{0,\beta}(\Omega)\hookrightarrow L^q(\Omega,|x|^\sigma\mathrm{d}x)$ is compact if $2<q<p=\dfrac{2(N+\sigma)}{N+\beta-2}$ and $\sigma>\left(\dfrac{N+\beta}{2}-1\right)q-N$.

Proof Let $\{u_n\}$ be a bounded sequence in $H^1_{0,\beta}(\Omega)$, then $u_n\rightharpoonup u$ weak in $H^1_{0,\beta}(\Omega)$, so $\{u_n\}$ is bounded in $H^1_{0,\beta}(\Omega\backslash B_\delta(0))$ for all $\delta>0$, where $B_\delta(0)\subset\Omega$. By Kondrahov compactness theorem guarantees the existence of a convergent subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $u_n\to u$ strongly in $L^q(\Omega\backslash B_\delta(0))$. We know that $u_n\to u$ a. e. in $\Omega\backslash B_\delta(0)$. Let $\delta\to 0$, by taking a diagonal sequence, $u_n\to u$ a. e. in Ω. By the Hölder inequality and the weighted Sobolev – Hardy inequality,

$$\int_{B_\delta(0)}|x|^\sigma|u_n|^q\leqslant\left(\int_{B_\delta(0)}|x|^\alpha|u_n|^p\right)^{\frac{q}{p}}\left(\int_{B_\delta(0)}|x|^{(\sigma-\frac{\alpha q}{p})\frac{p}{p-q}}\right)^{1-\frac{q}{p}}\leqslant C_1\int_{B_\delta(0)}|x|^{(\sigma-\frac{\alpha q}{p})\frac{p}{p-q}}$$

for some constant C_1. Therefore, for a given $\epsilon>0$, and $\sigma>\left(\dfrac{N+\beta}{2}-1\right)q-N$, we first fix δ such that

$$\int_{B_\delta(0)}|x|^\sigma|u_n|^q<\frac{\epsilon}{3}. \tag{6}$$

On the other hand, by the Egorov theorem, for all $\delta>0$, there exists some $\Omega_\delta\subset\Omega$, $|\Omega_\delta|<\delta$, such that $u_n\to u$ uniformly in $\Omega\backslash\Omega_\delta$, we have

$$\int_{(\Omega\backslash B_\delta(0))\backslash\Omega_\delta}|x|^\sigma|u_n-u|^q<\frac{\epsilon}{3}, \tag{7}$$

where δ is small enough.

$$\int_{(\Omega\backslash B_\delta(0))\cap\Omega_\delta}|x|^\sigma|u_n-u|^q\leqslant C_2\int_{\Omega_\delta}|u_n-u|^q\leqslant$$

$$C_2\Big(\int_{\Omega_\delta}|u_n-u|^q\Big)^{\frac{q}{p}}\Big(\int_{\Omega_\delta}1\Big)^{1-\frac{q}{p}}\leqslant C_3|\Omega_\delta|^{1-\frac{q}{p}}<\frac{\epsilon}{3}, \tag{8}$$

Then

$$\int|x|^\sigma|u_n-u|^q\leqslant\int_{B_\delta(0)}|x|^\sigma|u_n-u|^q+\int_{(\Omega\backslash B_\delta(0))\backslash\Omega_\delta}|x|^\sigma|u_n-u|^q+$$

$$\int_{(\Omega\backslash B_\delta(0))\cap\Omega_\delta}|x|^\sigma|u_n-u|^q<\epsilon. \tag{9}$$

Hence $H^1_{0,\beta}(\Omega)\hookrightarrow L^q(\Omega,|x|^\sigma dx)$ is compact.

We will apply the mountain pass theorem [1] without the (PS) condition:

Lemma 2 Let $I(u)$ be a C^1 functional on a Banach space E. Suppose there exist a neighborhood $B(0)$ of 0 in E and a constant α such that $I(u)\geqslant\alpha$ for every u in the boundary of $B(0)$,

$$I(0)<\alpha \text{ and } I(e)<\alpha \text{ for some } e\notin B(0).$$

Then there is a seqence $\{u_n\}$ in E such that

$$I(u_n)\to c \text{ and } I'(u_n)\to 0, \tag{10}$$

where $c=\inf\limits_{\gamma\in\Gamma}\max\limits_{0\leqslant t\leqslant 1}I(\gamma(t))\geqslant\alpha$, $\Gamma=\{\gamma\in C^1([0,1],E):\gamma(0)=0,\gamma(1)=e\}$.

Lemma 3 There are $\rho,\alpha>0$ suth that $I(u)\geqslant\alpha$ when $\|u\|=\rho$, there is $e\in H^1_{0,\beta}(\Omega)$, $\|e\|>\rho$ such that $I(e)\leqslant 0$.

Proof By the weight Sobolev - Hardy inequality, we have

$$I(u)=\frac{1}{2}\|u\|^2-\frac{1}{p}\int_\Omega|x|^\alpha|u|^p-\frac{\lambda}{q}\int_\Omega|x|^\sigma|u|^q\geqslant$$

$$\frac{1}{2}\|u\|^2-\frac{c}{p}\|u\|^p-\frac{\lambda}{q}\|u\|^q\Big(\int_\Omega|x|^{(\sigma-\frac{\alpha q}{p})\frac{p}{p-q}}\Big)^{\frac{q}{p}}=$$

$$\Big[\frac{1}{2}-\frac{c}{p}\rho^{p-2}-\frac{\lambda}{q}\rho^{q-2}\Big(\int_\Omega|x|^{(\sigma-\frac{\alpha q}{p})\frac{p}{p-q}}\Big)^{\frac{q}{p}}\Big]\rho^2\geqslant$$

$$\frac{1}{4}\rho^2>0.$$

for small ρ such that $\frac{1}{2}-\frac{c}{p}\rho^{p-2}-\frac{\lambda}{q}\rho^{q-2}\Big(\int_\Omega|x|^{(\sigma-\frac{\alpha q}{p})\frac{p}{p-q}}\Big)^{\frac{q}{p}}<\frac{1}{4}$.

On the other hand, take $\hat{u}\in C_0^\infty(\Omega)$ with $\hat{u}\geqslant 0,\hat{u}\not\equiv 0$. Hence for $t>0$,

$$I(t\hat{u})=\frac{t^2}{2}\|\hat{u}\|^2-\frac{t^p}{p}\int_\Omega|x|^\alpha|\hat{u}|^p-\frac{\lambda t^q}{q}\int_\Omega|x|^\sigma|\hat{u}|^q\leqslant$$

$$\frac{t^2}{2}\|\hat{u}\|^2 - \frac{t^p}{p}\int_\Omega |x|^\alpha |\hat{u}|^p$$

so that

$$I(t\hat{u}) \to -\infty \quad \text{as} \quad t \to \infty$$

Letting $e \equiv t\hat{u}$ with $t > 0$ large, we have $I(e) \leqslant 0$.

Applying the Mountain Pass Lemma without (PS) condition and lemma 3 there is a $(PS)_c$ sequence $\{u_n\} \in H^1_{0,\beta}(\Omega)$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$, then

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{p}\int_\Omega |x|^\alpha |u_n|^p - \frac{\lambda}{q}\int_\Omega |x|^\sigma |u_n|^q = c + o(1) \tag{11}$$

$$\langle I'(u_n),\varphi\rangle = \int_\Omega \nabla u_n \nabla \varphi - \int_\Omega |x|^\alpha |u_n|^{p-2}u_n\varphi - \lambda\int_\Omega |x|^\sigma |u_n|^{q-2}\varphi = o(1)\|\varphi\| \tag{12}$$

Taking $\varphi = u_n$,

$$\langle I'(u_n),u_n\rangle = \|u_n\|^2 - \int_\Omega |x|^\alpha |u_n|^p - \lambda\int_\Omega |x|^\sigma |u_n|^q = o(1)\|u_n\|, \tag{13}$$

we have

$$\frac{p-2}{2}\|u_n\|^2 - \lambda(1 - \frac{p}{q})\int_\Omega |x|^\alpha |u_n|^q \} = pc + o(1)\|u_n\| + o(1)$$

Noticing that $2 < q < p$ and showing that $\{u_n\}$ is bounded in $H^1_{0,\beta}(\Omega)$.

2 Proof of the theorem

Step 1 Indeed, $\{u_n\}$ is bounded in $H^1_{0,\beta}(\Omega)$ and by lemmas 2 and 3, we may assume that $(PS)_c$ sequence $\{u_n\}$ satisfies the set of conditions

$$u_n \rightharpoonup u \quad \text{weakly in} \quad H^1_{0,\beta}(\Omega), \quad u_n \to u \quad \text{strongly in} \quad L^q(\Omega, |x|^\alpha\, dx).$$

Therefore, for all $\varphi \in H^1_0(\Omega)$, we have

$$\langle I'(u_n),\varphi\rangle \to \langle I'(u),\varphi\rangle,$$

and from (12) consequently

$$\langle I'(u),\varphi\rangle = 0$$

for all $\varphi \in H^1_0(\Omega)$, which shows that u is a weak solution of equation (1).

Step 2 We shall now verify that the weak solution $u \not\equiv 0$. Indeed, assume that $u = 0$. We claim that

$$\int_\Omega |x|^\sigma |u_n|^q \to 0$$

by (11) and (13) become respectively

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{p}\int_\Omega |x|^\alpha |u_n|^p = c + o(1) \tag{14}$$

$$\|u_n\|^2 = \int_\Omega |x|^\alpha |u_n|^p + o(1) \tag{15}$$

Therefore

$$c = (\frac{1}{2} - \frac{1}{p})\|u_n\|^2 + o(1) \geqslant (\frac{1}{2} - \frac{1}{p})S\Big(\int_\Omega |x|^\alpha |u_n|^p\Big)^{\frac{2}{p}} + o(1) =$$

$$(\frac{1}{2}-\frac{1}{p})S\left[(\frac{1}{2}-\frac{1}{p})^{-1}c\right]^{\frac{2}{p}}+o(1)$$

So

$$c \geqslant (\frac{1}{2}-\frac{1}{p})S^{\frac{p}{p-2}}.$$

Thus we obtain a contradiction to $c < \left(\frac{1}{2}-\frac{1}{p}\right)S^{\frac{p}{p-2}}$, where c was given in (10). Thus $u \not\equiv 0$.

Step 3 We prove that

$$c < \left(\frac{1}{2}-\frac{1}{p}\right)S^{\frac{p}{p-2}} \tag{16}$$

where c was given in (10). Similar to [8], we only need to prove that for some $v_0 \in H^1_{0,\beta}(\Omega)$, $v_0 \not\equiv 0$ such that

$$\sup_{t\geqslant 0} I(tv_0) < \left(\frac{1}{2}-\frac{1}{p}\right)S^{\frac{p}{p-2}}. \tag{17}$$

According to [5], set

$$U_\epsilon(x) = (\epsilon + |x|^{\alpha-\beta+2})^{(2-N-\beta)/(\alpha-\beta+2)}, \quad \epsilon > 0 \tag{18}$$

We can get S. Similar to [1], let ϕ be a function $\phi \in C_0^\infty(\Omega)$, and $\phi(x) \equiv 1$ in a neighbourhood of the origin. Set

$$u_\epsilon(x) = \phi(x)U_\epsilon(x) \tag{19}$$

$$v_\epsilon(x) = u_\epsilon(x)\left(\int_\Omega |x|^\alpha |u_\epsilon(x)|^p\right)^{-\frac{1}{p}} \tag{20}$$

First we calculate directly

$$\int_\Omega |x|^\beta |\nabla u_\epsilon|^2 = \epsilon^{\frac{2-N-\beta}{\alpha-\beta+2}}\int_{\mathbb{R}^N} |x|^\beta |\nabla U_1|^2 + O(1) \tag{21}$$

$$\int_\Omega |x|^\alpha |u_\epsilon|^p = \epsilon^{-\frac{N+\alpha}{\alpha-\beta+2}}\int_{\mathbb{R}^N} |x|^\alpha |U_1|^p + O(1) \tag{22}$$

$$\int_\Omega |x|^\sigma |u_\epsilon|^q = \epsilon^{\frac{2N+2\sigma-(N+\beta-2)q}{2(\alpha-\beta+2)}}\int_{\mathbb{R}^N} |x|^\sigma |U_1|^q + O(1) \tag{23}$$

Therefore

$$\int_\Omega |x|^\beta |\nabla v_\epsilon|^2 = S + O(\epsilon^{\frac{N+\beta-2}{\alpha-\beta+2}}) \tag{24}$$

$$\int_\Omega |x|^\alpha |v_\epsilon|^p = 1 \tag{25}$$

$$\int_\Omega |x|^\sigma |v_\epsilon|^q = K\epsilon^{\frac{2N+2\sigma-(N+\beta-2)q}{2(\alpha-\beta+2)}} \tag{26}$$

Observe if $2 < q < p$, then

$$\int_\Omega |x|^\sigma |v_\epsilon|^q \to 0 \quad \text{as} \quad \epsilon \to 0 \tag{27}$$

By using these estiments we will show that there exists $\epsilon > 0$ small enough, such that

$$\sup_{t\geqslant 0} I(tv_\epsilon) < (\frac{1}{2}-\frac{1}{p})S^{\frac{p}{p-2}}$$

Let us consider the functions

$$g(t) = I(tv_\epsilon) = \frac{t^2}{2}\int |x|^\beta |\nabla v_\epsilon|^2 - \frac{t^p}{p} - \lambda \frac{t^q}{q}\int |x|^\sigma |v_\epsilon|^q$$

and

$$\bar{g}(t) = \frac{t^2}{2}\int |x|^\beta |\nabla v_\epsilon|^2 - \frac{t^p}{p}.$$

It is clear that $g(t) \to -\infty$ (as $t \to \infty$), then $\sup_{t\geqslant 0} I(tv_\epsilon)$ is attained for some $t_\epsilon > 0$ and

$$0 = g'(t_\epsilon) = t_\epsilon\left(\int |x|^\beta |\nabla v_\epsilon|^2 - t_\epsilon^{p-2} - \lambda t_\epsilon^{q-2}\int |x|^\sigma |v_\epsilon|^q\right)$$

therefore

$$\int |x|^\beta |\nabla v_\epsilon|^2 = t_\epsilon^{p-2} + \lambda t_\epsilon^{q-2}\int |x|^\sigma |v_\epsilon|^q > t_\epsilon^{p-2}$$

i. e. ,

$$t_\epsilon \leqslant \left(\int_\Omega |x|^\beta |\nabla v_\epsilon|^2\right)^{\frac{1}{p-2}} \tag{28}$$

This inequality implies

$$\int_\Omega |x|^\beta |\nabla v_\epsilon|^2 \leqslant t_\epsilon^{p-2} + \lambda\left(\int_\Omega |x|^\beta |\nabla v_\epsilon|^p\right)^{\frac{q-2}{p-2}} \cdot \int_\Omega |x|^\sigma |v_\epsilon|^q \tag{29}$$

Choosing ϵ small enough, by (24),(27) and (29), we get

$$t_\epsilon^{p-2} \geqslant \frac{S}{2} \tag{30}$$

That is, we have a lower bound for t_ϵ, independent of ϵ, Now we estimate $g(t_\epsilon)$. The function $\bar{g}$ attains its maximum at $t_0^* = \left(\int_\Omega |x|^\beta |\nabla v_\epsilon|^2\right)^{\frac{1}{p-2}}$, and is increasing at the interval $[0, t_0^*]$. Then, by (24),(28) and (29), we have

$$\begin{aligned} g(t_\epsilon) &= \bar{g}(t_\epsilon) - \frac{\lambda}{q} t_\epsilon^q \int |x|^\sigma |v_\epsilon|^q \leqslant \\ &\bar{g}\left(\left(\int_\Omega |x|^\beta |\nabla v_\epsilon|^2\right)^{\frac{1}{p-2}}\right) - \frac{\lambda}{q} t_\epsilon^q \int |x|^\sigma |v_\epsilon|^q \leqslant \\ &\left(\frac{1}{2} - \frac{1}{p}\right) S^{\frac{p}{p-2}} + C_4 \epsilon^{\frac{N+\beta-2}{\alpha-\beta+2}} - \frac{\lambda}{q}\left(\frac{S}{2}\right)^{\frac{q}{p-2}} \int |x|^\sigma |v_\epsilon|^q \end{aligned}$$

By (26) we have

$$g(t_\epsilon) \leqslant \left(\frac{1}{2} - \frac{1}{p}\right) S^{\frac{p}{p-2}} + C_4 \epsilon^{\frac{N+\beta-2}{\alpha-\beta+2}} - \lambda C_5 \epsilon^{\frac{2N+2\sigma-(N+\beta-2)q}{2(\alpha-\beta+2)}} \tag{31}$$

If

$$\frac{N+\beta-2}{\alpha-\beta+2} > \frac{2N+2\sigma-(N+\beta-2)q}{2(\alpha-\beta+2)},$$

That is, $q > \frac{2(N+\sigma)}{N+\beta-2} - 2$, then, for ϵ small enough, we get

$$g(t_\epsilon) < \left(\frac{1}{2} - \frac{1}{p}\right) S^{\frac{p}{p-2}},$$

and the proof is complete.

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临界 Sobolev-Hardy 指数的拟线性奇性椭圆型方程的非平凡解

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摘要: 利用 Sobolev-Hardy 不等式和山路几何研究了临界 Sobolev-Hardy 指数的拟线性奇性椭圆型方程的非平凡解.

关键词: 拟线性椭圆型方程; 临界指数; 非平凡解; Sobolev-Hardy 不等式