Applied Mathematical Sciences, Vol. 1, 2007, no. 38, 1851 - 1862

Convolution and Actuarial Risk in a Pension Fund

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Abstract

The aim of the paper is to propose a new method to evaluate the actuarial risks in a pension fund.

We recall:

a) the utilisation of discrete Fourier transforms in the random variables (with integer realisations), to calculate convolutions in an exact and fast way. [par 1A];

b) the definition of the contributions and benefits according to the "Projected Unit Method" Guidance Note 9;[par.1B].

The idea is to apply the distribution of the sum of the random variables in order to calculate directly the risk. That is done multiplying each probability (of sum of random variables) for the corresponding negative realisation (each realisation of any employee, for each year, is the difference between the accrued contributions and the corresponding benefits).

The first application is the calculus of guarantee in a defined benefit pension fund; particularly the guarantee that -in every year- the succession of assets is not minor than the succession of liabilities.

Keywords: Exact and fast sum of discrete random variables to evaluate the actuarial risk; Applications

1) Introduction

It is useful to give three references (the first two, cmp[2][3]) while the third (Ornstein-Uhlenbeck process) cmp e.g.[10].

1A) An exact and fast method to calculate the sum of random variables with integer realisations

Let (cmp [8][11]) $X_1, X_2, ..., X_N$ be discrete random variables (r.vs.), not necessarily with the same distribution, having null covariance and integer realisations.

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The sum of r.vs. is

$$X^{(n)} = \sum_{s=1}^{n} X_s \qquad\qquad n = 1, ..., N \tag{1}$$

For each r.v. X_s , $AR_s(\max)$ is the greatest realisation and $AR_s(\min)$ the smallest realisation.

Let us denote with an integer positive $peso_s$ the number of r.vs.with the same chacteristics. Then the maximum value of realisations is

$$M^{(n)} = \sum_{s=1}^{n} (AR_s(\max) - AR_s(\min) + 1) * peso_s \tag{2}$$

We write now only the conclusion of this procedure (in order to have a complete demonstration see ref. [11])

The final formula [11] of the difference between two cumulative distributions calculated by integer number (for $k > m$) is

$F_{X^{(N)}}(k) - F_{X^{(N)}}(m) =$

$$\begin{aligned} = \ & \frac{k-m}{M^{(n)}} + \frac{2}{M^{(n)}} * \\ & * \sum_{\tau=1}^{(M^{(n)}-1)/2} \rho(\frac{2\pi\tau}{M^{(n)}}) \frac{\sin \frac{\pi\tau}{M^{(n)}}(k-m)}{\sin \frac{\pi\tau}{M^{(n)}}} * \\ & * \cos\left(\alpha(\frac{2\pi\tau}{M^{(n)}}) - \frac{\pi\tau(m+k-1)}{M^{(n)}} \right) \end{aligned} \tag{3}$$

Besides is $\rho(\frac{2\pi\tau}{M^{(n)}}) =$

$$= \left(\prod_{s=1}^{n} \left[\sum_{v=0}^{M_s-1} p_v^{(s)} \cos(f_{s,\nu}) \right]^2 + \left[\sum_{v=0}^{M_s-1} p_v^{(s)} \sin(f_{s,\nu}) \right]^2 \right)^{0.5} \tag{4}$$

where $f_{s,\nu} = \frac{2\pi\tau * v}{M_s}$.

Let the sum of the arguments be

$$\alpha(\frac{2\pi\tau}{M^{(n)}}) = \sum_{s=1}^{n} \alpha_s(\frac{2\pi\tau}{M_s}) \tag{5}$$

In order to evaluate this sum we must calculate the argument of the s-th r v and then

$$\tan \alpha_s(\frac{2\pi\tau}{M_s}) = \frac{\sum_{r=0}^{M_s-1} p_v^{(s)} * \sin(\frac{2\pi\tau * r}{M^{(s)}})}{\sum_{r=0}^{M_s-1} p_v^{(s)} * \cos(\frac{2\pi\tau * r}{M^{(s)}})} \tag{6}$$

where r =AR$_s$(min), ...AR$_s$(max), are the realisations of the s-th r.v. .
Then we calculate $\alpha_s(\frac{2\pi\tau}{M_s})$ where

$$-\frac{\pi}{2} \leq \alpha_s(\frac{2\pi\tau}{M_s}) \leq \frac{\pi}{2}. \quad (7)$$

1B) Actuarial features of a defined benefit pension fund [1]

In order to realise an easy check, we write the same symbols of [1] with very few variants.

In the pages 37-38 of [2] Blake illustrates the scheme of the "defined benefit" pension fund to which we refer in this paper.

Let us consider a new entry employee with age x and without any contribution year.

Let $W(x, 0)$ denote the starting salary, T the maximum number of annual contributions and g the projected annual growth in salary.

The salary after t years is $(0 \leq t \leq T)$

$$W(x, t) = W(x, 0) * (1 + g)^t \quad (8)$$

Let $\frac{t}{60}$ be the annual factor for service for t years of contribution,

$P(x + t, \zeta)$ the probability of remaining in the scheme between age $x + t$ and ζ the old age retirement,

$\bar{a}_\zeta$ the present value of a life annuity of 1 Euro for annum,

$D(x + t, \zeta)$ the discount factor between age x+t and the retirement old age ζ , that is$(1 + i)^{-(T-t)}$ where i is the constant discount rate.

Besides γ is the contribution in percentage of the pensionable earning; this contribution is prefixed cmp [2], 117 (then γ is not necessarily an actuarial equilibrium rate).

After t *contributions, (for an new entry of age x) the present value of benefits, function of W(x,t) is*

$$\frac{t}{60} * W(x, t) * P(x + t, \zeta) * \ddot{a}_\zeta * (1 + i)^{-(T-t)} \quad (9)$$

The old age pension is equal to

$$W(x, 0) * (1 + g)^t * \frac{t}{60} \quad (10)$$

where t represents the contribution years.

2) Definition of random variables

We want to define the r.v. for any type of new entry in the pension Fund.

2A) Introduction

We assume that the r.v. $X_s^{(r)}$ (that is the s-th employee, at the end of the r-th year) is based on two mutually exclusive events:

- the first one is the expected value of the realisation , that is the difference between the expected value of the accrued contributions and the corresponding liabilities ; this event has the probability that the s-th employee lives until old age;

- as the only benefit is the old age pension, the second event is a null benefit with probability equal to complement to 1 of the probability of the first event.

All the evaluations are referred to the a initial moment (r=0) by the stochastic factor $e^{-Y(r)}$ (Ornstein-Uhlenbeck process).

So we write $Z_s^{(r)} = X_s^{(r)} * e^{-Y(r)}$.

We make the following assumptions:

a) for a fixed r, the r.vs. $X_s^{(r)}$ (demographic factor) are independent from one another ;

b) the r.v. $e^{-Y(r)}$ (financial factor) are dependent;

c) the r.v. $X_s^{(r)}$ and $e^{-Y(r)}$ are mutually independent (that is the demographic factors are independent on the financial ones).

2B) The basic idea.

The projected events of a contract are[3]

- at the end of the first year the asset is the accrued capital of the contributions paid at the beginning of the year , while the only liability of the fund is the old age benefit which is equal to $\frac{1}{60}$ of the last annual earnings. The pension is evaluated at the end of the year and will be paid only if the assured person reaches the old age;

- at the end of the second year the asset will be formed by accruing the contribution paid in the first year (accrued for two years) plus the contribution of the second year (accrued for one year), while the benefit is the old age pension which is equal to $\frac{2}{60}$ of the last salary evaluated at the end of the second year (the pension is paid only if the employee is alive at the old age), and so on.

It is useful to underline that the problem *is a succession of mutually exclusive events, that is the employee can have (at old age) the pension calculated at the end of the first year - if only one contribution has been paid - or can have the pension calculated at the end of the second year - if two contributions have been paid, etc.*

2C) Analysis of events

[3]Pension Fund with defined benefits (cmp Guidance 9 ; Retirement Benefits- Actuarial Reports of The Faculty and Institute of Actuaries) (cmp [1] , 37)

The r.v. $X_s^{(r)}$ (for the s-th employee, aged x at the entry, alive at the end of age $x+r$, $r = 1, 2, ...T.$) is formed by two mutually exclusive events:

- the first one, is the difference ($d_s^{(r)}$) between the expected value of the accrued contributions(${}^{(c)}d_s^{(r)}$) and the actuarial present value of the old age pension calculated as $\frac{r}{60}$ of the last salary(${}^{(0)}d_s^{(r)}$); to this difference corresponds the probability to live until the old age, that is ${}_{65-x-r}p_{x+r}$;

- the second event has probability $(1- {}_{65-x-r}p_{x+r})$ with zero benefit.

We will study the two alternatives:

I) the first event

The first realisation $d_s^{(r)}$ referred at this event is formed by the difference between asset and liability, that is,

IA*) Asset* ${}^{(c)}d_s^{(r)}$ *(referred at the end of r-th year (if employee is alive)*

The asset is the succession of contributions paid at the beginning of the v-th year (v=0,1,2,...r-1) and accrued up to the end of the r-th year(r=1,2,...T)

$$\left(\frac{l_{x+r+1}}{l_x} * R(0) * \gamma\right)\left[\sum_{v=0}^{r-1}(1+g)^{\nu} * (1+i)^{r-\nu}\right]; \qquad ((A))$$

IB) Liability ${}^{(o)}d_s^{(r)}$ *(referred at the end of the r -th year. if the employee is alive)*

It is necessary to write many specifications about (cmp also [2], 37-38) :

a) x is the age of a new entry in the fund,

b) $R(0) * (1+g)^r$ the projected earning after r contributions,

c) $\frac{r}{60}$ the accrual factor for service by r contributions,

d) $(1+g)^{65-r}$ the revaluation factor for earnings between age x+r and retirement age 65,

e) $\bar{a}_{65}$ the expected annuity factor (the present value of a life annuity of Euro 1 per annum) at retirement age 65,

f) $(1+i)^{-(65-r)}$ the discount factor between age $x+r$ and retiremement age 65(the discount rate is constant).

The liabilities of the pension Fund after r contributions are

$$\frac{l_{x+r+1}}{l_x} * \left\{R(0)\frac{r}{60}(1+g)^r * (\frac{1+g}{1+i})^{65-x-r}\tilde{a}_{65}\right\} \qquad ((B))$$

Then we can write that, if the assured person is alive until the end of the r-th year, an actuarial equivalence between the accrued contributions (A) and the actuarial present value of the corresponding liabilities (B) exists.

It is important to underline that the difference (realisation) (A)-(B) will exist if and only if the assured person lives until the old age (with probability

${}_{65-x-r}p_{x+r}$).

II) the second event.

The assets and liabilities will be always null if the assured person does not live until the old age (probability $1 -_{65-x-r} p_{x+r}$).

A generical r.v is the following

Tab1 Random variable at the end of year ($X_s^{(r)}$) for the s-th employee (aged x)

realisations	probability
$\frac{l_{x+r+1}}{l_x} * (R(0) * \gamma) [(**) - (*)]$	${}_{65-x-r}p_{x+r+1}$
0	$1 -_{65-x-r} p_{x+r+1}$

(**) $\sum_{v=0}^{r-1}(1+g)^{\nu} * (1+i)^{r-\nu}$

(*) $\left(\frac{r}{60}(1+g)^r * (\frac{1+g}{1+i})^{65-x-r}\tilde{a}_{65}\right)$

If we have many r.v.s with the same realisation and the same probabilities it is sufficient to indicate only one r.v. with a repetion factor.

2D) Conclusion of definition of random variables

As actuarial equilibrium of the pension Fund, one can write about the r.v. :

- if the sum of r.v. has all positive realisations, then the guarantee is zero;
- if the sum of r,v. has both positive and negative realisations, then a cost of the guarantee can exist ; this cost is the sum of the product between negative realisations and the corresponding probabilities.

2E) Evaluation of the costs of the guarantee

According to the collective based on S employees the mean of r.v. $Z_s^{(r)}$ is

$$E[Z^{(r)}] = \sum_{s=1}^{S} E[Z_s^{(r)}].$$

We underline (for the r-th year) :

- if $E[Z^{(r)}] = 0$ then the pension fund is in actuarial equilibrium,
- if $E[Z^{(r)}] < 0$ the fund has a deficit (and then the guarantee is useful),
- if $E[Z^{(r)}] > 0$ the fund has an active (no guarantee is needed).

3) Analysis of the actuarial risks (demographic and financial risk)

On the basis of the r.vs. (defined before) , it is easy to calculate the distribution of the sum according to [4]

Let us denote, for the r-th year ,

$$X^{(r)} = \sum_{s=1}^{S} X_s^{(r)} \qquad \text{and}$$

$$Z^{(r)} = \sum_{s=1}^{S} Z_s^{(r)} = X^{(r)} * e^{-Y(r)}$$

From [4] we can write [4] $(r = 1, 2, ...T.)$

Demographic risk $V\ [X^{(r)}] * E[e^{-2Y(r)}]$

Financial risk $\left(E[X^{(r)}]\right)^2 * V\ [e^{-Y(r)}]$

4) Applications

Let us consider a defined benefit pension fund, that provides only old age pension at 65

The collective is based on 1000 employees (males and female with 4 different ages, different initial salaries and distribution of the employees as in table A1)

4A) The scheme of the pension fund

The pension of old age at the moment of old age is =last salary$*(1+g)^t*$ $\frac{t}{60}$ where t are the contribution year $(0 \le t \le 40 = T)$

g the annual rate of variation of initial salary (1.50%),

g_Z the annual rate of variation of initial pension (1.00%),

r_B the annual rate of investiment in the period of benefits (2.5%),

i the annual deterministic rate (2.80%).

About the Ornstein-Uhlenbeck process(cmp [4]), let us set $\alpha = 0.10$ and $\sigma = 0.01$,while the initial rate $\delta_0 = 0.028$ and the long period rate $\delta = 0.06$

If we multiply the present value of a life annuity of 1 Euro by the annual benefit , we have the actuarial present value of one pension (cmp example [2], 37-38).

The mortality table is ISTAT[5] 2001 in which all the death probabilities from age 66 until 104 have been multiplied by 0.93.

Table A.1 Distribution, by age and gender, of employees and earnings

[4]We work without suscripts in X and Y in mean and variance because in this paper they are not necessary.

[5]ISTAT is the italian Organisation for statistics

Males			
age	insured	salary	term
25	100	1000	40
35	260	1100	30
45	120	1300	20
55	120	1150	10
Females			
age	insured	salary	term
25	140	900	40
35	100	1000	30
45	120	1200	20
55	40	1050	10

Tax rate $\gamma = 0.20$ is prefixed for the collective [6].

4B) Three cases about the guarantee [7]

It is useful to write the following three cases; we have diminuished the final classes of distribution from the initial fifty-one to a smaller number.

Probability distribution of the sum of the r.vs. $Z^{(1)}$

Table I Year = 1 Probability Distribution of the sum of the random variables $Z^{(1)}$ $(m = 562;\ \sigma = 279)$

Interv.Realis	Probab
(-∞, −1111]	0.000 000 001 6
[-1110, -275]	0.001 392 056 6
[-274, 0]	0.004 879 249 0
[+1, +2512]*	0.993 728 692.8

Interv.(-∞ , +2512) probability=1 000 000 000 0

* We remind that the cost is formed by only negative realisations

Table IA Year = 5 Probability Distribution of the sum of the random variables $Z^{(5)}$ $(m = -11844;\ \sigma = 1198)$

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[6] Cmp [2] , 114

[7] With the program trig21se.cpp one can have the distributions for the years 1,2,3,4,5,6,7,8,9,10,20,30,40

Interv.Realis.	Probab.
(-∞, −19035]	0.000 000 001 0
[-19034, -15440]	0.001 288 388 1
[-15439, -13642]	0.064 915 186 9
[13641, -13043]	0.092 085 872 1
[-13042, -11844]	0.343 064 312 6
[-11843, -11245]	0.191 276 595 4
[-11244, -10646]	0.149 001 327 1
[-10645, -8249]	0.156 849 813 7
[-8248, -3455]	0.001 518 504 0

Interv.(-∞,-3455) probability=1 000 000 000 0

Tab. IB Year =30 Probability Distribution of the sum of the random variables $Z^{(30)}$ **(**$m = -28812$**;** $\sigma = 1179$**)**

Interv Realis	Probab
(-∞, −12449]	10^{-10}
[-12448, -6555]	10^{-10}
[-6555,+20559	0.0000000005
[+20560, +35884](*)	0.9999999995(*)

Interv.(-∞ ,34884 probability=1 000 000 000 0

** For the costs, the positive realisation are not useful

5C) All the year

Tab A. Annual cost of guarantee in percentage of total annual earnings . Sum of the random variables $Z^{(r)}$ **(r=1,2.....10; 20,30,40)**

r	cost	earnings	cost /earnings
1	1.	1106530	< 0.0005
2	437	1121083	< 0.0005
3	2798	1135646	0.002 5
4	6729	1150174	0.005 9
5	11725	1164652	0.010 1
6	17336	1178984	.0.014 7
7	24046	1193220	0.020 2
8	30759	1207321	0.025 5
9	37911	1221288	0.031 0
10	44950	1235076	0.0364
20	50867	1165353	0.043 6
30	< 1	873237	< 0.0005
40	0(*n.s.*)	368547	*n.s.*

n.s. Years with the realisations are all positive

From Table A) we can observe in the first and second year the cost/earnings ratio is smaller than 0.0005; from the third year till the twentieth, the ratio rises from 0.25% up to 4.36%), then from the 21-st till the 40th year the ratio decreases.

We do not show an application about the division of the actuarial into demographic and financial risk, because the definition of the random variables (cmp par 2) produces a prevalent financial risk (for applications cmp [2]applied to a life insurance portfolio)

6) Conclusions

This paper has the aim to present a new procedure to calculate the actuarial risks and the costs to guarantee that the accrued contributions are sufficient to face (at the various times) the benefits deriving from the contributions.

If this is not realized, it means that the accrued contributions do not even the benefits, and this causes an actuarial deficit (accrued capital not sufficient) and the subject who guarantee the fund must pay this difference.

We remind that this study considers only a defined benefit pension fund, according Guidance 9 ; Retirement Benefits (Actuarial Reports of The Faculty and Institute of Actuaries- United Kingdom)[8].

I) Methodology

The new idea is the utilisation of the distribution of the sum of the random variables to calculate directly the cost of this guarantee.

These evaluations are made by multiplying each probability (of the sum of the random variables) for the corresponding realisation (i.e.the only negative difference between accrued contributions and the corresponding benefits).

It is important to underline that - for the evaluation of the cost- the differences must be negative.

II) Results about the cost of the guarantee

The collective (cmp A1) is classified into 8 homogeneous groups that distinguish themselves by the number of employees, gender, salary, age, term of the contract (only two groups can reach 40 years of contributions - all the others leave the fund after 10,20,30 years of contributions)

The actuarial risk of deficit is negligible for the first two years:

- then the cost (in percentage of the earnings of the whole collective) increases from the 3-rd year (0,25%) until the 20-th (4.36%)
- from the 21-st to the 40th year the cost diminuishes until to reach negligible values.

If we want to analyse the division of the actuarial risk, we must consider a homogeneous group (and not a collective classified into eight different groups).

[8](cmp [2] D.Blake, 2003, 37)

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Appendix nr 1

The software for the sum of discrete random variables with integer realisations (positive, negative or null) is trig20ge.cpp.

If one would like to have this programme in C++ language, one can write an email (the first edition of this program is twelve years old - Bibl [11], 1995)

Received: January 23, 2007