Applied Mathematical Sciences, Vol. 1, 2007, no. 38, 1863 - 1877

On Certain Class of Univalent Functions with Negative Coefficients

K. Al-Shaqsi and *M. Darus [1]

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
ommath@hotmail.com
*maslina@pkrisc.cc.ukm.my

Abstract

In [2], we introduced the generalization of Ruscheweyh derivatives operator D_λ^n. In the present paper, we study a new class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$, defined by D_λ^n for $0 \le \alpha < 1, 0 < \beta \le 1, -1 \le a < b \le 1, 0 < b \le 1, 0 < \gamma \le \frac{b}{b-a}$, $n \in \mathbb{N}_0$ and $\lambda \ge 0$,

Mathematics Subject Classification: 30C45

Keywords: Univalent functions, starlike functions, Ruscheweyh derivative operator

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the unit disc $\mathbf{U} = \{z : |z| < 1\}$.

Definition 1.1 *[2] We define the operator D_λ^n by:*

$$D_\lambda^0 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \ge 0,$$
$$D_\lambda^1 f(z) = (1-\lambda)zf'(z) + \lambda z(zf'(z))',$$
$$D_\lambda^n f(z) = D_\lambda\left(\frac{z(z^{n-1}f(z))^n}{n!}\right), \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

[1]Corresponding author

If f is given by (1.1), then we write

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \big[1 + \lambda(k-1)\big] C(n,k) a_k z^k,$$

where

$$C(n,k) = \binom{k+n-1}{n} = \frac{\prod_{j=1}^{k-1}(j+n)}{(k-1)!}, \qquad k \geq 2.$$

Definition 1.2 *Let $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq a < b \leq 1$, $b > 0$, $0 < \gamma \leq b/(b-a)$, $n \in \mathbb{N}_0$, and $\lambda \geq 0$ we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{T}_\lambda^n(a,b,\alpha,\beta,\gamma)$ if and only if*

$$\left| \frac{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1}{(b-a)\gamma\left[\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - \alpha\right] - b\left[\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1\right]} \right| < \beta, \quad z \in \mathbb{U}. \tag{1.2}$$

Further, let $\mathcal{M}$ denote the subclass of $\mathcal{A}$ consisting functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0. \tag{1.3}$$

Let us define

$$\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma) = \mathcal{T}_\lambda^n(a,b,\alpha,\beta,\gamma) \cap \mathcal{M}$$

It is interesting to note the various classes of functions studied by other authors including $\mathcal{M}_0^0(-1,1,0,1,1/2) \equiv S^*$ by Silverman [1], $\mathcal{M}_0^0(-1,1,\alpha,1,\gamma) \equiv S_0^*(\alpha,\gamma)$ by Owa [3], and $\mathcal{M}_0^0(-1,1,\alpha,\beta,\gamma) \equiv S_0^*(\alpha,\beta,\gamma)$ by Owa [4],[5].

2 Characterization Theorem

Theorem 2.1 *Let the function f be defined by (1.3). Then $f \in \mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty} \Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma) a_k \leq \beta\gamma(b-a)(1-\alpha), \tag{2.4}$$

where

$$\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma) = \Big\{k - 1 + \beta[b(k+1) - (b-a)\gamma(k+\alpha)]\Big\}[1 + \lambda(k-1)]C(n,k),$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq a < b \leq 1$, $b > 0$, $0 < \gamma \leq b/(b-a)$, $n \in \mathbb{N}_0$, $\lambda \geq 0$.

Proof. Assume that the inequality (2.1) holds. Then for $|z| = 1$ we have

$$\begin{aligned}
&\left|z(D_\lambda^n f(z))' - D_\lambda^n f(z)\right| - \beta\Big|(b-a)\gamma[z(D_\lambda^n f(z))' - \alpha D_\lambda^n f(z)] \\
&\qquad\qquad -b[z(D_\lambda^n f(z))' - D_\lambda^n f(z)]\Big| \\
= \;& \left|z(D_\lambda^n f(z))' - D_\lambda^n f(z)\right| - \beta\Big|[(b-a)\gamma - b]z(D_\lambda^n f(z))' \\
&\qquad\qquad +[b-(b-a)\gamma\alpha]D_\lambda^n f(z)\Big| \\
= \;& \Big|\sum_{k=2}^{\infty}(1-k)[1+\lambda(k-1)]C(n,k)a_k z^k\Big| \\
&-\beta\Big|[(b-a)\gamma - b]z - [(b-a)\gamma - b]\sum_{k=2}^{\infty}k[1+\lambda(k-1)]C(n,k)a_k z^k \\
&+[b-(b-a)\gamma\alpha]z - [b-(b-a)\gamma\alpha]\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n,k)a_k z^k\Big| \\
= \;& \Big|\sum_{k=2}^{\infty}(1-k)[1+\lambda(k-1)]C(n,k)a_k z^k\Big| \\
&-\beta\Big|(b-a)\gamma(1-\alpha)z - [(b-a)\gamma - b]\sum_{k=2}^{\infty}k[1+\lambda(k-1)]C(n,k)a_k z^k \\
&-[b-(b-a)\gamma\alpha]\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n,k)a_k z^k\Big| \\
\le \;& \sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]C(n,k)a_k|z|^k - \beta(b-a)\gamma(1-\alpha)|z| \\
&+\beta|(b-a)\gamma - b|\sum_{k=2}^{\infty}k[1+\lambda(k-1)]C(n,k)a_k|z|^k \\
&+\beta[b-(b-a)\gamma\alpha]\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n,k)a_k|z|^k \\
\le \;& \sum_{k=2}^{\infty}\Big\{(k-1)+\beta[b-(b-a)\gamma]k + \beta[b-(b-a)\gamma\alpha]\Big\}[1+\lambda(k-1)]C(n,k)a_k|z|^k \\
&-\beta\gamma(b-a)(1-\alpha) \le 0.
\end{aligned}$$

Consequently, by the maximum modulus theorem, the function f is in the class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$.

Conversely, assume that the function f defined by (1.3) is in the class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$. Then

$$\left|\frac{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-1}{(b-a)\gamma\left[\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-\alpha\right]-b\left[\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-1\right]}\right|<\beta, \quad \Leftrightarrow$$

$$\left|\sum_{k=2}^{\infty}(1-k)[1+\lambda(k-1)]C(n,k)a_k z^k\right|$$
$$\leq \beta\Big|(b-a)\gamma(1-\alpha)z-[(b-a)\gamma-b]\sum_{k=2}^{\infty}k[1+\lambda(k-1)]C(n,k)a_k z^k$$
$$-[b-(b-a)\gamma\alpha]\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n,k)a_k z^k\Big|.$$

Since $|\text{Re}(z)|\leq|z|$ for any z, we have

$$\text{Re}\left\{\frac{\sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]C(n,k)a_k z^k}{(b-a)\gamma(1-\alpha)z-\sum_{k=2}^{\infty}[b(k+1)-(b-a)\gamma(k+\alpha)][1+\lambda(k-1)]C(n,k)a_k z^k}\right\}<\beta.$$

Upon clearing the denominator in the last inequality and letting $|z|\to 1$ through real values, we obtain

$$\sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]C(n,k)a_k$$
$$\leq \beta\gamma(b-a)(1-\alpha)-\sum_{k=2}^{\infty}\beta[b(k+1)-(b-a)\gamma(k+\alpha)][1+\lambda(k-1)]C(n,k)a_k,$$

and this inequality gives the required condition.

Finally, the result is sharp with the extremal function f given by

$$f(z)=z-\frac{\beta\gamma(b-a)(1-\alpha)}{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)}z^k.\square$$

3 Closure Theorems

First, let the function f_j be defined for $j=1,2,...,m$, by

$$f_j(z)=z-\sum_{k=2}^{\infty}a_{k,j}a_k, \qquad z\in\mathbb{U}, \qquad (a_{k,j}\geq 0). \tag{3.5}$$

We shall prove the following results for the closure of functions in the class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$.

Theorem 3.1 *Let the functions f_j defined by (4.1) be in the class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$ for every $j=1,2,...,m$. Then the function g defined by*

$$g(z)=z-\sum_{k=2}^{m} b_k z^k, \quad ; \quad b_k\ge 0, \quad with \quad b_k=\frac{1}{m}\sum_{j=1}^{m} a_{k,j},$$

also belongs to the class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$.

Proof. As $f_j(z)\in\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$ it follows from Theorem 2.1 that

$$\sum_{k=2}^{\infty}\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)a_{k,j}\le\beta\gamma(b-a)(1-\alpha), \qquad j=1,2,...,m.$$

Therefore

$$\begin{aligned}\sum_{k=2}^{\infty}\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)b_k &= \sum_{k=2}^{\infty}\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)\frac{1}{m}\sum_{j=1}^{m}a_{k.j}\\ &\le \beta\gamma(b-a)(1-\alpha),\end{aligned}$$

hence, by Theorem 2.1, $g(z)\in\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$.

Theorem 3.2 *Let the functions f_j defined by (4.1) be in the class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$ for every $j=1,2,...,m$. Then the functions h defined by*

$$h(z)=\sum_{j=1}^{m}c_j f_j(z), \quad (c_j\ge 0)$$

is also in the class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$, where $\sum_{j=1}^{m}c_j=1$.

Proof. According to the definition of h, we can write

$$h(z)=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}c_j a_{k,j}\right)z^k.$$

Further, since functions $f_j(z)$ are in $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$ for every $j=1,2,...,m$ we get

$$\sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)a_{k,j} \leq \beta\gamma(b-a)(1-\alpha)$$

for every $j = 1, 2, ..., m$. We can see that

$$\begin{aligned}
&\Rightarrow \sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)\left(\sum_{j=1}^{m} c_j a_{k,j}\right) \\
&\Rightarrow \sum_{j=1}^{m} c_j \ \left(\sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)a_{k,j}\right) \\
&\leq \left(\sum_{j=1}^{\infty} c_j\right)\beta\gamma(b-a)(1-\alpha) = \beta\gamma(b-a)(1-\alpha).
\end{aligned}$$

Hence the theorem follows.

Theorem 3.3 *Let the functions* $f_1(z) = z - \sum\limits_{k=2}^{\infty} a_{k,1}z^k$, $a_{k,1} \geq 0$, *and* $f_2(z) = z - \sum\limits_{k=2}^{\infty} a_{k,2}z^k$, $a_{k,2} \geq 0$, *be in the class* $\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$, *respectively* $\mathcal{M}_{\lambda}^{n+1}(a,b,\alpha,\beta,\gamma)$. *Then the function* $p(z)$ *defined by*

$$p(z) = z - \frac{2+n}{4+n}\sum_{k=2}^{\infty}(a_{k,1}+a_{k,2})z^k \in \mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma).$$

Proof. Let $f_1(z) \in \mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$ and $f_2(z) \in \mathcal{M}_{\lambda}^{n+1}(a,b,\alpha,\beta,\gamma)$, by using Theorem 2.1 we get, respectively,

$$\sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)a_{k,1} \leq \beta\gamma(b-a)(1-\alpha),$$

and

$$\sum_{k=2}^{\infty} \Phi_{\lambda}^{n+1}(k,a,b,\alpha,\beta,\gamma)a_{k,2} \leq \beta\gamma(b-a)(1-\alpha).$$

We have

$$\begin{aligned}
\frac{2+n}{2}\sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)a_{k,2} &\leq \sum_{k=2}^{\infty} \Phi_{\lambda}^{n+1}(k,a,b,\alpha,\beta,\gamma)a_{k,2} \\
&\leq \beta\gamma(b-a)(1-\alpha).
\end{aligned}$$

Then,

$$\frac{2+n}{2}\sum_{k=2}^{\infty}\Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)a_{k,1}\leq\frac{2+n}{2}\beta\gamma(b-a)(1-\alpha),$$

and

$$\frac{2+n}{2}\sum_{k=2}^{\infty}\Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)a_{k,2}\leq\beta\gamma(b-a)(1-\alpha).$$

imply

$$\frac{2+n}{4+n}\sum_{k=2}^{\infty}\Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)(a_{k,1}+a_{k,2})\leq\beta\gamma(b-a)(1-\alpha),$$

and from this we deduce that

$$p(z)=z-\frac{2+n}{4+n}\sum_{k=2}^{\infty}(a_{k,1}+a_{k,2})z^{k}\in\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma).\square$$

4 Integral Operators

Theorem 4.1 *Let the function f defined by (1.3), be in the class $\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$, and let c be a real number such that $c>-1$. Then the function F defined by*

$$F(z)=\frac{c+1}{z^{c}}\int_{0}^{z}t^{c-1}f(t)dt, \tag{4.6}$$

also belongs to the class $\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z)=z-\sum_{k=2}^{\infty}b_{k}z^{k},\qquad \textit{where}\qquad b_{k}=\frac{c+1}{c+k}a_{k}.$$

Therefore

$$\begin{aligned}\sum_{k=2}^{\infty}\Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)b_{k} &= \sum_{k=2}^{\infty}\Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)\frac{c+1}{c+k}a_{k}\\ &< \sum_{k=2}^{\infty}\Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)a_{k}\\ &\leq \beta\gamma(b-a)(1-\alpha).\end{aligned}$$

Since $f\in\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$ and hence by Theorem 2.1, $F\in\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$.

Theorem 4.2 *Let c be a real number such that $c > -1$. If $F \in \mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$, then the function f defined by (4.1) is univalent in $|z| < R$, where*

$$R = \inf_k \left[\frac{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)(c+1)}{\beta\gamma(b-a)(1-\alpha)(c+k)k}\right]^{\frac{1}{k-1}}, \qquad k \geq 2.$$

The result is sharp with the extremal function f given by

$$f(z) = z - \frac{\beta\gamma(b-a)(1-\alpha)(c+k)}{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)(c+1)k} z^k, \qquad k \geq 2. \tag{4.7}$$

Proof. Let $F(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$, it follows from (4.1) that

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{c+1} = z - \sum_{k=2}^{\infty} \frac{c+k}{c+1} a_k z^k.$$

$$F(z) \in \mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma) \Rightarrow \sum_{k=2}^{\infty} \frac{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)a_k}{\beta\gamma(b-a)(1-\alpha)} \leq 1.$$

If

$$\frac{k(c+k)|z|^{k-1}}{c+1} < \frac{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)}{\beta\gamma(b-a)(1-\alpha)},$$

or if

$$|z| < \left[\frac{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)(c+1)}{\beta\gamma(b-a)(1-\alpha)(c+k)k}\right]^{\frac{1}{k-1}},$$

then

$$\begin{aligned} |f'(z) - 1| &= \left| - \sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k z^{k-1} \\ &< \sum_{k=2}^{\infty} \frac{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)a_k}{\beta\gamma(b-a)(1-\alpha)} \leq 1. \end{aligned}$$

But from $|f'(z) - 1| < 1$, $|z| < R$, we deduce that f is univalent in the disc $|z| < R$.
The result is sharp and the extremal function is given by (4.2). □

Theorem 4.3 *Let $c \in \mathbb{R}$, $c > -1$. If*

$$F(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$$

then, the function f given by (4.1) is starlike of order δ $(0 \le \delta < 1)$ in $|z| < R^*$, where

$$R^* = \inf_k \left[\frac{(1-\delta)(c+1)\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)}{(k-\delta)(c+k)\beta\gamma(b-a)(1-\alpha)} \right]^{\frac{1}{k-1}}, \qquad k \ge 2.$$

The result is sharp.

Proof. We must show that $\left|\frac{zf'(z)}{f(z)} - 1\right| < (1-\delta)$, for $|z| < R^*$. We have

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)} - 1\right| &= \left| \frac{-\sum\limits_{k=2}^{\infty} (k-1)\frac{c+k}{c+1} a_k z^{k-1}}{1 - \sum\limits_{k=2}^{\infty} \frac{c+k}{c+1} a_k z^{k-1}} \right| \\ &\le \frac{\sum\limits_{k=2}^{\infty} (k-1)\frac{c+k}{c+1} a_k |z|^{k-1}}{1 - \sum\limits_{k=2}^{\infty} \frac{c+k}{c+1} a_k |z|^{k-1}} < 1-\delta. \end{aligned} \tag{4.8}$$

Hence (4.3) holds true if

$$\sum_{k=2}^{\infty} (k-1)\frac{c+k}{c+1} a_k |z|^{k-1} < (1-\delta)\left\{ 1 - \sum_{k=2}^{\infty} \frac{c+k}{c+1} a_k |z|^{k-1} \right\},$$

or, equivalently,

$$\sum_{k=2}^{\infty} \left(\frac{k-\delta}{1-\delta}\right)\left(\frac{c+k}{c+1}\right) a_k |z|^{k-1} < 1. \tag{4.9}$$

By using

$$\sum_{k=2}^{\infty} \frac{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma) a_k}{\beta\gamma(b-a)(1-\alpha)} \le 1.$$

Then (4.4) will be true if

$$\left(\frac{k-\delta}{1-\delta}\right)\left(\frac{c+k}{c+1}\right) a_k |z|^{k-1} < \frac{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)}{\beta\gamma(b-a)(1-\alpha)},$$

or if

$$|z| < \left[\frac{(1-\delta)(c+1)\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)}{(k-\delta)(c+k)\beta\gamma(b-a)(1-\alpha)} \right]^{\frac{1}{k-1}}$$

Hence, $f(z) \in S^*(\delta)$ for $|z| < R^*$. The sharpness follows if we take the function F, given by

$$F(z) = z - \frac{\beta\gamma(b-a)(1-\alpha)}{\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)} z^k, \qquad k \ge 2. \quad \square$$

5 The Hadamard products

Let $f, g \in \mathcal{M}$ where f defined by (1.3) and g given by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \qquad b_k \geq 0. \tag{5.10}$$

Then the Hadamard product or convolution of f and g denoted by

$$f(z) * g(z) = (f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k.$$

Theorem 5.1 *If the functions f, g defined by (1.3) and (5.1), belong to the same class $\mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$, then the Hadamard product $f * g$ belongs to the class $\mathcal{M}_\lambda^n(a, b, \alpha, \beta^2, \gamma)$.*

Proof. Since $f(z) \in \mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$ by using Theorem 2.1 we have

$$a_k \leq \frac{\beta\gamma(b-a)(1-\alpha)}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)}, \qquad k \geq 2. \tag{5.11}$$

If $g(z) \in \mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$, then

$$\sum_{k=2}^{\infty} \Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma) b_k \leq \beta\gamma(b-a)(1-\alpha). \tag{5.12}$$

Since, $0 < \beta^2 \leq \beta \leq 1$, we have

$$\sum_{k=2}^{\infty} \Phi_\lambda^n(k, a, b, \alpha, \beta^2, \gamma) b_k \leq \sum_{k=2}^{\infty} \Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma) b_k$$

and then

$$\begin{aligned} \sum_{k=2}^{\infty} \Phi_\lambda^n(k, a, b, \alpha, \beta^2, \gamma) a_k b_k &\leq \sum_{k=2}^{\infty} \Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma) a_k b_k \\ &\leq \frac{\beta^2\gamma^2(b-a)^2(1-\alpha)^2}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)} \\ &\leq \beta^2\gamma(b-a)(1-\alpha). \end{aligned}$$

Because

$$\begin{aligned} &\frac{\beta^2\gamma^2(b-a)^2(1-\alpha)^2}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)} \leq \beta^2\gamma(b-a)(1-\alpha) \Leftrightarrow \\ &\beta^2\gamma(b-a)(1-\alpha)\{\gamma(b-a)(1-\alpha) - \Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)\} \leq 0 \Leftrightarrow \\ &\gamma(b-a)(1-\alpha) - (n+1)(1+\lambda) + (n+1)(1+\lambda)\beta[\gamma(b-a)\alpha - b] \\ &\qquad +(n+1)(1+\lambda)2\beta[\gamma(b-a) - b] \leq 0. \end{aligned}$$

According to Theorem 2.1 we obtain $f * g \in \mathcal{M}_\lambda^n(a, b, \alpha, \beta^2, \gamma)$. $\square$

Theorem 5.2 *Let $p > 0$ and $\frac{p+2-\sqrt{p^2+4p}}{2} \leq \alpha \leq \frac{1}{1+p}$. If the functions f and g defined by (1.3) and (5.1) belong to the same class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta^2,\gamma)$, then the Hadamard product $f*g$ belongs to the class $\mathcal{M}_\lambda^n(a,b,1-p\alpha,\beta^2,\gamma)$.*

Proof. By using (5.2) and (5.3), we obtain

$$\begin{aligned}\sum_{k=2}^{\infty}\Phi_\lambda^n(k,a,b,1-p\alpha,\beta^2,\gamma)a_kb_k &\leq \sum_{k=2}^{\infty}\Phi_\lambda^n(k,a,b,\alpha,\beta^2,\gamma)a_kb_k\\ &\leq \sum_{k=2}^{\infty}\Phi_\lambda^n(k,a,b,\alpha,\beta,\gamma)a_kb_k\\ &\leq \frac{\beta^2\gamma^2(b-a)^2(1-\alpha)^2}{\Phi_\lambda^n(2,a,b,\alpha,\beta,\gamma)}\\ &\leq \beta^2\gamma(b-a)(1-\alpha)^2 \leq \beta^2\gamma(b-a)p\alpha.\end{aligned}$$

which implies that $f*g \in \mathcal{M}_\lambda^n(a,b,1-p\alpha,\beta^2,\gamma)$. □

When $p=1$, Theorem 5.2 gives the following

Corollary 5.3 *Let $\frac{3-\sqrt{5}}{2} \leq \alpha \leq \frac{1}{2}$ and let the functions f and g defined by (1.3) and (5.1) belong to the same class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$. Then the Hadamard product $f*g$ belongs to the class $\mathcal{M}_\lambda^n(a,b,1-\alpha,\beta^2,\gamma)$.*

As for $p=2$, Theorem 5.2 gives the following

Corollary 5.4 *Let $2-\sqrt{3} \leq \alpha \leq \frac{1}{3}$ and let the functions f and g defined by (1.3) and (5.1) belong to the same class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$. Then the Hadamard product $f*g$ belongs to the class $\mathcal{M}_\lambda^n(a,b,1-2\alpha,\beta^2,\gamma)$.*

Remark 5.5 *From definition of the class $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$ it is easy to see that if $0<\beta_1\leq\beta_2\leq 1$, then $\mathcal{M}_\lambda^n(a,b,\alpha,\beta_1,\gamma) \subset \mathcal{M}_\lambda^n(a,b,\alpha,\beta_2,\gamma)$*

Remark 5.6 *Since $0<\beta^2\leq\beta\leq 1$, we have $\mathcal{M}_\lambda^n(a,b,\alpha,\beta^2,\gamma) \subset \mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$ and $\mathcal{M}_\lambda^n(a,b,1-p\alpha,\beta^2,\gamma) \subset \mathcal{M}_\lambda^n(a,b,1-p\alpha,\beta,\gamma)$*

6 Inclusion properties of the classes $\mathcal{M}_\lambda^n(a,b,\alpha,\beta,\gamma)$

Theorem 6.1 *Let $0\leq\alpha_2\leq\alpha_1<1$, $0<\beta_1\leq\beta_2\leq 1$ and $0<\gamma_1\leq\gamma_2\leq\frac{b}{b-a}$. Then we have $\mathcal{M}_\lambda^n(a,b,\alpha_1,\beta_1,\gamma_1) \subset \mathcal{M}_\lambda^n(a,b,\alpha_2,\beta_2,\gamma_2)$.*

Proof. Let $f \in \mathcal{M}_\lambda^n(a,b,\alpha_1,\beta_1,\gamma_1)$. Then by using Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\Phi_\lambda^n(k,a,b,\alpha_1,\beta_1,\gamma_1)a_k \leq \beta_1\gamma_1(b-a)(1-\alpha_1). \tag{6.13}$$

From this we deduce that

$$\sum_{k=2}^{\infty}[b(k+1)-(b-a)\gamma_1(k+\alpha_1)][1+\lambda(k-1)]C(n,k)a_k \leq \gamma_1(b-a)(1-\alpha_1),$$

$$\sum_{k=2}^{\infty}[-(k+\alpha_1)][1+\lambda(k-1)]C(n,k)a_k \leq 1-\alpha_1,$$

and

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n,k)a_k \leq 1.$$

Let $\alpha_1=\alpha_2+\nu$, $\beta_1=\beta_2-\varepsilon$, $\gamma_1=\gamma_1-\theta$ where $\nu,\varepsilon,\theta\geq 0$, then from (6.1) we have

$$\sum_{k=2}^{\infty}\Phi_{\lambda}^{n}(k,a,b,\alpha_2+\nu,\beta_2-\varepsilon,\gamma_2-\theta)a_k \leq \beta_1\gamma_1(b-a)(1-\alpha_1).$$

and

$$\begin{aligned} & \Phi_{\lambda}^{n}(k,a,b,\alpha_2+\nu,\beta_2-\varepsilon,\gamma_2-\theta) \\ = \; & \Phi_{\lambda}^{n}(k,a,b,\alpha_2,\beta_2,\gamma_2)-\varepsilon[b(k+1)-\gamma_1(b-a)(k+\alpha_1)][1+\lambda(k-1)]C(n,k) \\ & -\theta\beta_2(b-a)[-(k+\alpha_1)][1+\lambda(k-1)]C(n,k)-\gamma_2\beta_2(b-a)\nu[1+\lambda(k-1)]C(n,k) \end{aligned}$$

we have

$$\begin{aligned} & \sum_{k=2}^{\infty}\Phi_{\lambda}^{n}(k,a,b,\alpha_2,\beta_2,\gamma_2)a_k \\ \leq \; & \beta_1\gamma_1(b-a)(1-\alpha_1)+\varepsilon\sum_{k=2}^{\infty}[b(k+1)-(b-a)\gamma_1(k+\alpha_1)][1+\lambda(k-1)]C(n,k)a_k \\ & +\beta_2\theta(b-a)\sum_{k=2}^{\infty}[-(k+\alpha_1)][1+\lambda(k-1)]C(n,k)a_k \\ & +\gamma_2\beta_2(b-a)\nu\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n,k)a_k \\ \leq \; & \beta_1\gamma_1(b-a)(1-\alpha_1)+\varepsilon\gamma_1(b-a)(1-\alpha_1)+\beta_2\theta(b-a)(1-\alpha_1)+\gamma_2\beta_2(b-a)\nu \\ = \; & \beta_2\gamma_2(b-a)(1-\alpha_2). \end{aligned}$$

According to Theorem 2.1 we obtain $f\in\mathcal{M}_{\lambda}^{n}(a,b,\alpha_2,\beta_2,\gamma_2)$ and $\mathcal{M}_{\lambda}^{n}(a,b,\alpha_1,\beta_1,\gamma_1)\subset\mathcal{M}_{\lambda}^{n}(a,b,\alpha_2,\beta_2,\gamma_2)$. □

Corollary 6.2 *Let $0\leq\alpha_1\leq\alpha_2<1$. Then we have $\mathcal{M}_{\lambda}^{n}(a,b,\alpha_1,\beta,\gamma_1)\subset\mathcal{M}_{\lambda}^{n}(a,b,\alpha_2,\beta,\gamma)$.*

Corollary 6.3 *Let* $0 < \gamma_1 \le \gamma_2 \le \frac{b}{b-a}$*. Then* $\mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma_1) \subset \mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma_2)$*.*

Theorem 6.4 *Let* $0 \le \alpha_2 \le \alpha_1 < 1$*,* $0 < \beta_1 \le \beta_2 \le 1$ *and* $0 < \gamma_1 \le \gamma_2 \le \frac{b}{b-a}$*. If the functions* f *defined by (1.3) and* g *defined by (5.1) be in the class* $\mathcal{M}_\lambda^n(a, b, \alpha_1, \beta_1, \gamma_1)$ *and* $\mathcal{M}_\lambda^n(a, b, \alpha_2, \beta_2, \gamma_2)$*, respectively, then the Hadamard product* $f * g$ *belongs to the class* $\mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$ *where* $\alpha = \min(\alpha_1, \alpha_2)$*,* $\beta = \max(\beta_1, \beta_2)$ *and* $\gamma = \max(\gamma_1, \gamma_2)$*.*

Proof. Since

$$\begin{aligned}
\alpha = \min(\alpha_1, \alpha_2) &\Rightarrow \alpha \le \alpha_1 \text{ and } \alpha \le \alpha_2 \\
\beta = \max(\beta_1, \beta_2) &\Rightarrow \beta \ge \beta_1 \text{ and } \beta \ge \beta_1 \\
\gamma = \max(\gamma_1, \gamma_2) &\Rightarrow \gamma \ge \gamma_1 \text{ and } \gamma \ge \gamma_2
\end{aligned}$$

from Theorem 6.1 we have $f \in \mathcal{M}_\lambda^n(a, b, \alpha_1, \beta_1, \gamma_1) \Rightarrow f \in \mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$ and $g \in \mathcal{M}_\lambda^n(a, b, \alpha_2, \beta_2, \gamma_2) \Rightarrow g \in \mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$. From Theorem 5.1 and Remark 5.6 we have $f * g \in \mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$. □

Theorem 6.5 *Let* $-1 \le a_2 \le a_1 < b_1 \le b_2 \le 1$*,* $0 < b_1$*. Then we have* $\mathcal{M}_\lambda^n(a_1, b_1, \alpha, \beta, \gamma) \subset \mathcal{M}_\lambda^n(a_2, b_2, \alpha, \beta, \gamma)$*.*

Proof. Let $f \in \mathcal{M}_\lambda^n(a_1, b_1, \alpha, \beta, \gamma)$, then

$$\sum_{k=2}^{\infty} \Phi_\lambda^n(k, a_1, b_1, \alpha, \beta, \gamma) a_k \le \beta\gamma(b_1 - a_1)(1 - \alpha).$$

Since $a_1 \ge a_2$, $b_1 \le b_2 \Rightarrow b_1 - a_1 \le b_2 - a_2$ and because $0 \le \alpha < 1$, $0 < \beta \le 1$, $0 < \gamma \le \frac{b}{b-a}$ from defined of $\mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$ we deduce that $\mathcal{M}_\lambda^n(a_1, b_1, \alpha, \beta, \gamma) \ge \mathcal{M}_\lambda^n(a_2, b_2, \alpha, \beta, \gamma)$. We have

$$\begin{aligned}
\sum_{k=2}^{\infty} \Phi_\lambda^n(k, a_2, b_2, \alpha, \beta, \gamma) a_k &\le \sum_{k=2}^{\infty} \Phi_\lambda^n(k, a_1, b_1, \alpha, \beta, \gamma) a_k \\
&\le \beta\gamma(b_1 - a_1)(1 - \alpha) \\
&\le \beta\gamma(b_2 - a_2)(1 - \alpha),
\end{aligned}$$

and according to Theorem 2.1 we obtain $f \in \mathcal{M}_\lambda^n(a_2, b_2, \alpha, \beta, \gamma)$ which imply that $\mathcal{M}_\lambda^n(a_1, b_1, \alpha, \beta, \gamma) \subset \mathcal{M}_\lambda^n(a_2, b_2, \alpha, \beta, \gamma)$. □

Corollary 6.6 $-1 \le a_2 \le a_1 < b \le 1$, $0 < b \le 1$*. Then we have* $\mathcal{M}_\lambda^n(a_1, b, \alpha, \beta, \gamma) \subset \mathcal{M}_\lambda^n(a_2, b, \alpha, \beta, \gamma)$*.*

Corollary 6.7 $-1 \le a < b_1 \le b_2 \le 1$, $0 < b_1 \le b_2 \le 1$*. Then we have* $\mathcal{M}_\lambda^n(a, b_1, \alpha, \beta, \gamma) \subset \mathcal{M}_\lambda^n(a, b_2, \alpha, \beta, \gamma)$*.*

Theorem 6.8 $\mathcal{M}_{\lambda}^{n+1}(a,b,\alpha,\beta,\gamma) \subset \mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$.

Proof. Since $f \in \mathcal{M}_{\lambda}^{n+1}(a,b,\alpha,\beta,\gamma)$, by using Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \Phi_{\lambda}^{n+1}(k,a,b,\alpha,\beta,\gamma)a_k \leq \beta\gamma(b-a)(1-\alpha),$$

because

$$C(n,k) \leq C(n+1,k);\ k \geq 2 \text{ and } n \geq 0.$$

and

$$\Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma) > 0,$$

then

$$\Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma) \leq \Phi_{\lambda}^{n+1}(k,a,b,\alpha,\beta,\gamma)$$

and

$$\sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k,a,b,\alpha,\beta,\gamma)a_k \leq \sum_{k=2}^{\infty} \Phi_{\lambda}^{n+1}(k,a,b,\alpha,\beta,\gamma)a_k \leq \beta\gamma(b-a)(1-\alpha).$$

According to Theorem 2.1 we obtain

$$f \in \mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma) \Rightarrow \mathcal{M}_{\lambda}^{n+1}(a,b,\alpha,\beta,\gamma) \subset \mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma).\Box$$

Remark 6.9 *From definition of the class $\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$ or from Theorem 2.1 it is easy to see that if $0 \leq \lambda_1 \leq \lambda_2$, then $\mathcal{M}_{\lambda_2}^{n}(a,b,\alpha,\beta,\gamma) \subset \mathcal{M}_{\lambda_1}^{n}(a,b,\alpha,\beta,\gamma)$.*

Remark 6.10 *From Theorem 6.8 we have*

$$\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma) \subset \mathcal{M}_{\lambda}^{0}(a,b,\alpha,\beta,\gamma)$$

from Remark 6.9, we have

$$\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma) \subset \mathcal{M}_{0}^{0}(a,b,\alpha,\beta,\gamma)$$

and from Theorem 6.1 and Theorem 6.5 we have

$$\mathcal{M}_{0}^{0}(a,b,\alpha,\beta,\gamma) \subset \mathcal{M}_{0}^{0}(-1,1,0,1,\frac{1}{2})$$

and

$$f \in \mathcal{M}_{0}^{0}(-1,1,0,1,\frac{1}{2}) \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 1,$$

the class of starlike functions with negative coefficients. Since these functions are univalent, then all functions in the class $\mathcal{M}_{\lambda}^{n}(a,b,\alpha,\beta,\gamma)$ are also univalent.

ACKNOWLEDFEMENT. The work presented here was partially supported by SAGA(Scientific Advancement Grant Allocation) STGL-012-2006, Academy of Sciences, Malaysia.

References

[1] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51**,(1975),109-116.

[2] K. Al Shaqsi and M. Darus, On univalent functions with respect to k-symmetric points defined by a generalized Ruscheweyh derivatives operator.(Submitted)

[3] S. Owa, On the starlike functions of order α and type β, *Math. Japonica* **27**,(6) (1982),723-735.

[4] S. Owa, On a class of starlike functions, *J. Korean Math. Soc.* **19**, (1) (1982/83), 29-38.

[5] S. Owa, A remark on the Hadamard products of starlike functions II, *Math. Japonica* **27**, (6) (1982),747-752.

Received: February 16, 2007