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On the Improvement of the Performances of an Observer: A Discrete Linear System

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Abstract

We consider the discrete system $x_{i+1} = Ax_i + Bu_i$ with the output equation $y_i = Cx_i$, A, B and C are appropriate matrices and the initial state x_0 is supposed to be unknown.
One of the tools most famous for the estimate of the unknown state x_i or Tx_i (T being a matrix of an adequate order) is the use of the observer $z_{i+1} = Fz_i + Dy_i + Px_i$ where F, D and P are suitable matrices. Although this observer constitutes an asymptotic estimator of x_i (or of Tx_i), its reliability is narrowly related to the speed of the convergence $\lim_{i\to\infty} \|z_i - x_i\| = 0$ (or $\lim_{i\to\infty} \|z_i - Tx_i\| = 0$).
In this paper and to contribute to this context, we propose a class $\mathcal{M}$ of observer initial states such as the corresponding observer checks $\|z_i - x_i\| \le \alpha_i;\ \forall i \ge 0$ (or $\|z_i - Tx_i\| \le \alpha_i;\ \forall i \ge 0$) with $\lim_{i\to+\infty} \alpha_i = 0$, $\alpha = (\alpha_i)_{i\ge 0}$ means a desired mode of convergence. The problem for delayed discrete systems is also considered.

Keywords: Discrete systems, observers, estimation error, discrete delayed systems.

1 Introduction

The development of better mathematical model was always a priority for engineers, physicists, biologists,....Toward this end, the scientists developed a

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mathematical arsenal as sophisticated as diversified, we quote by way of examples identifiability ([21], [41], [31], [32], [6], [43]), robustness ([7], [10]), sentinels ([23], [24]), filtering ([3], [2] , [16]),.... One of the components of this arsenal is the theory of observers. The observer was first proposed and developed by D.G.Luenberger in [25], and further developed in [26]. Since these early papers which concentrated on observers for purely deterministic continuous-time linear time-invariant systems, observer theory has been extended by several researchers to include time-varying systems([4], [18], [5]), discrete systems ([30], [13] , [1], [27]), delayed systems([28], [29]) and nonlinear systems ([19], [11] , [20], [12], [9]).

The use of state observer proves to be useful in not only system monitoring and regulation but also detecting as well as identifying failures in dynamical systems. The presence of disturbances, dynamical uncertainties, and nonlinearities pose a great challenges in practical applications. Toward this end, the high-performance robust observer design problem has been topic of considerable interest recently, and several advanced observer designs have been proposed (see [8], [14], [15], [35], [37], [38], [39], [42]).

In addition to their practical utility, observers offer a unique theoretical fascination. The associated theory is intimately related to the fundamental linear system concepts of controllability, observability dynamic response, and stability, and provides a simple setting in which all of these concepts interact.

In this paper, we consider the discrete linear system governed by

$$\begin{cases} x_{i+1} = Ax_i + Bu_i\,, \qquad i \geq 0 \\ x_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

where x_0 is supposed to be unknown.

the corresponding output function is given by

$$y_i = Cx_i\,, \qquad i \geq 0 \tag{2}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^q$ are, respectively, the state variable, the control variable and the output variable, while A, B and C are constant matrices of appropriate dimensions.

An observer for the system above is a dynamic system which has as its inputs the inputs u_i and available outputs y_i, and whose state is an asymptotic estimation of Tx_i, where T is an appropriate matrix. More precisely, the state observer is described by

$$\begin{cases} z_{i+1} = Fz_i + Dy_i + Pu_i\,, \qquad i \geq 0 \\ z_0 \in \mathbb{R}^p \end{cases} \tag{3}$$

where $z_i \in \mathbb{R}^p$ and F, P and D are constant matrices of suitable dimensions and verifying

$$\lim_{i\to\infty}(z_i - Tx_i) = 0 \tag{4}$$

Admittedly, the convergence (4) constitutes the fundamental goal of the observer (3). Unfortunately, for certain systems, it is not sufficient that the error $e_i = z_i - Tx_i$ converges to 0, but the speed of this convergence is also a paramount factor. For example, if the system (1) represents a compartment model describing the evolution of the quantity of a substance in a living organism ([17], [22], [36]), the observer can become without interest if we must wait a long time to have $z_i \simeq Tx_i$. As another example, one can quote the kinematics of an engine moving in space according to the equation (1), the slowness of convergence (4) can have as a consequence the loss forever of the engine.
Our contribution in the solution of this problem consists in supposing that the unknown initial state x_0 is localized in a convex and compact polyhedron $\mathcal{P}$ and to design a set $\mathcal{M}$ such that the corresponding observer defined by

$$\begin{cases} z_{i+1} = Fz_i + Dy_i + Pu_i \ , & i \geq 0 \\ z_0 \in \mathcal{M} \end{cases}$$

checks the performances

$$\|z_i - Tx_i\| \leq \alpha_i \ , \quad \forall i \geq 0$$

where $(\alpha_i)_{i\geq 0}$ is a real positive sequence decreasing to 0 and representing a predefined speed (for examples $\alpha_i = \frac{1}{i},\ \frac{1}{i^2},\ e^{-i}, ...$). For the characterization of the set $\mathcal{M}$, we propose simple algorithms based on mathematical programming techniques, simplex method made it possible to lead to numerical simulations. Finally, we show in section 6 that the adopted approach can be extended to discrete systems with delays on the state.

2 Problem statement

We consider the linear discrete-time systems described by the difference equation

$$\begin{cases} x_{i+1} = Ax_i + Bu_i \ , & i \geq 0 \\ x_0 \ \ is \ \ unknown \end{cases} \tag{5}$$

the output function is given by

$$y_i = Cx_i \ , \quad i \geq 0 \tag{6}$$

where $x_i \in \mathbb{R}^n,\ u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^q$ are, respectively, the state vector, the control vector and the output vector, while A, B and C are constant matrices of respective dimensions $(n \times n),\ (n \times m),\ and\ (n \times q)$.
We also consider the corresponding observer whose state is represented by

$$\begin{cases} z_{i+1} = Fz_i + Dy_i + Pu_i\ , & i \geq 0 \\ z_0 \in \mathbb{R}^p \end{cases} \tag{7}$$

F, P and D are real matrices of suitable dimensions. For $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, let $e_i(x_0)$ be defined as the error between the observer state z_i and its estimate Tx_i where x_i is the solution of (5) corresponding to the initial state x_0, i.e

$$e_i(x_0) = z_i - Tx_i \tag{8}$$

the system (7) is an observer for the system defined by (5) and (6) if

$$\lim_{i \to +\infty} e_i(x_0) = 0 \tag{9}$$

Sufficient conditions for the existence of an observer are given by the following proposition

Proposition 2.1 *Equation (7) specifies an observer of the system given by (5) and (6) if the following hold*

1. $P = TB$
2. TA - $FT = DC$
3. *The operator F is stable*

Moreover, we have

$$e_i(x_0) = F^i(z_0 - Tx_0)\ ,\ \forall i \geq 0\ . \tag{10}$$

Proof
For all $i \geq 0$, we have

$$\begin{aligned} e_{i+1}(x_0) &= z_{i+1} - Tx_{i+1} \\ &= Fz_i + Pu_i + Dy_i - TAx_i - TBu_i \\ &= F(z_i - Tx_i) + (FT - TA + DC)x_i + (P - TB)u_i \end{aligned}$$

Therefore, the constraints 1. and 2. yields

$$e_{i+1}(x_0) = Fe_i(x_0)$$

which implies

$$e_i(x_0) = F^i e_0(x_0) = F^i(z_0 - Tx_0)\ , \qquad \forall i \geq 0.$$

We deduce, since F is stable that $\lim\limits_{i \to \infty} e_i(x_0) = 0$.

■

The problem being addressed in this paper can be formulated as follows : Given $\mathcal{P}$ a convex and compact polyhedron of $\mathbb{R}^n$ containing the unknown initial state x_0 and a positive decreasing sequence $(\alpha_i)_{i\geq 0}$ which verifies

$$\frac{\alpha_i}{\alpha_{i+1}} \leq \frac{\alpha_{i-1}}{\alpha_i}, \qquad \forall i \geq 1 \tag{11}$$

($\alpha_i = \frac{1}{i}$; $\alpha_i = \zeta^{-i}$, $\zeta < 1$; $\alpha_i = \frac{1}{(i+1)^r}$, $r \in [1, +\infty[$; ...), and suppose that the conditions of proposition 2.1 are checked, we investigate all the observer initial states z_0 for whose the resulting error (8) satisfies the pointwise-in-time conditions

$$\|e_i(x_0)\| \leq \alpha_i \ , \quad \forall i \geq 0 \ , \quad \forall x \in \mathcal{P}.$$

More precisely we aim to determine $\mathcal{M}$ the set of α-admissible observer initial states given by

$$\mathcal{M} = \{z_0 \in \mathbb{R}^p / \ \|F^i(z_0 - Tx)\| \leq \alpha_i \ , \ \forall i \geq 0 \ , \ \forall x \in \mathcal{P}\}.$$

3 On the properties of the set $\mathcal{M}$

3.1 Preliminary results

In the following, for $x \in \mathbb{R}^n$, $\mathcal{M}_x$ will denote the set defined by

$$\mathcal{M}_x = \{z_0 \in \mathbb{R}^p / \ \|F^i(z_0 - Tx)\| \leq \alpha_i \ , \ \forall i \geq 0\} \tag{12}$$

The following theorem holds

Theorem 3.1 *Let $\mathcal{P}$ be a convex and compact polyhedron of $\mathbb{R}^n$ containing x_0 and whose vertices are v_1 , v_2 , ... , v_r.*
Then

$$\mathcal{M} = \bigcap_{j=1}^{r} \mathcal{M}_{v_j} \ .$$

Proof

It is clear that $\mathcal{M} \subset \bigcap_{j=1}^{r} \mathcal{M}_{v_j}$.

reciprocally, let $z \in \bigcap_{j=1}^{r} M_{v_j}$, and $x \in \mathcal{P}$ expressed as a convex combination of the vertices of $\mathcal{P}$, then

$$\|F^i(z - Tv_j)\| \leq \alpha_i \ , \ \forall i \geq 0 \ , \ \forall \ j = 1, ..., r \quad \textit{and}$$
$$x = \sum_{j=1}^{r} \lambda_j v_j \ , \ 0 \leq \lambda_j \leq 1 \ , \ \sum_{j=1}^{r} \lambda_j = 1.$$

Therefore, for $i \geq 0$

$$\begin{aligned}\|F^i(z-Tx)\| &= \|F^i(z-\sum_{j=1}^{r}\lambda_j T v_j)\| \\ &= \|\sum_{j=1}^{r}\lambda_j F^i(z-Tv_j)\| \\ &\leq \sum_{j=1}^{r}\lambda_j\|F^i(z-Tv_j)\| \\ &\leq \sum_{j=1}^{r}\lambda_j\alpha_i \ = \ \alpha_i\end{aligned}$$

Hence $z \in M$.

■

In the following, int(V) will indicate the interior of V, $\mathcal{B}(x,\epsilon)$ will denote the ball with center x and radius ϵ, and S is the set defined by

$$S = \{\xi \in \mathbb{R}^p / \ \|F^i\xi\| \leq \alpha_i \ , \ \forall i \geq 0\}. \tag{13}$$

It is obvious that

$$\mathcal{M}_x = S + Tx, \ \forall x \in \mathbb{R}^n \tag{14}$$

moreover, we have the following results.

Proposition 3.1 *i) S and $\mathcal{M}$ are convex compact sets and S is symmetric. ii) Suppose that F verifies $\lim\limits_{i\to+\infty}\frac{\|F^i\|}{\alpha_i} = 0$, then $0 \in int(S)$ and $int(\mathcal{M}_x) \neq \emptyset, \ \forall x \in \mathbb{R}^n$.*

Proof
A simple application of the definitions of S and $\mathcal{M}$ leads to properties i).
The assumption in ii) implies that there exists $\gamma > 0$ such that for all $\xi \in \mathbb{R}^p$, $i \in \mathbb{N}$, $\|F^i\xi\| \leq \gamma\alpha_i\|\xi\|$. Then $\mathcal{B}(0,\frac{1}{\gamma}) \subset S$, i.e. $0 \in int(S)$ and consequently from relation (14), $int(\mathcal{M}_x) \neq \emptyset, \ \forall x \in \mathbb{R}^n$.

■

3.2 On the accessibility of the set $\mathcal{M}$

It is clear that the characterization of the set S is practically impossible because of the infinite number of the inequations from which it derives, thus and with an aim of curing this handicap, let us define the sets

$$S_k = \{\xi \in \mathbb{R}^p / \ \|F^i(\xi)\| \ \leq \ \alpha_i \ , \ \forall \ i \in \{0,1,...,k\}\} \ ; \ k \in \mathbb{N} \ .$$

Definition 3.1 *The set S is said to be finitely accessible if there is an integer k such that $S = S_k$.*
The smallest integer k^* verifying the condition above is called the access-index of S.

The following proposition summarize relations between the sets defined above

Proposition 3.2 .
i) $S = \bigcap_{k\geq 0} S_k$ *and* $S_{k+1} \subset S_k$; $\forall k \in \mathbb{N}$.
ii) $\xi \in S_{k+1} \Leftrightarrow \xi \in S_k$ *and* $\|F^{k+1}\xi\| \leq \alpha_{k+1}$.
iii) $\xi \in S_{k+1} \Rightarrow \frac{\alpha_k}{\alpha_{k+1}} F\xi \in S_k$.

Proof
i) and ii) are immediate from the definitions of the sets S and S_k.
To prove iii), suppose that $\xi \in S_{k+1}$ then for every $i \in \{0, 1, ..., k\}$, we have

$$\begin{aligned} \| F^i(\tfrac{\alpha_k}{\alpha_{k+1}} F\xi) \| &= \tfrac{\alpha_k}{\alpha_{k+1}} \| F^{i+1}\xi \| \\ &\leq \tfrac{\alpha_k}{\alpha_{k+1}} \alpha_{i+1} \end{aligned}$$

since $(\frac{\alpha_j}{\alpha_{j+1}})_{j\geq 0}$ is decreasing, then

$$\frac{\alpha_k}{\alpha_{k+1}} \leq \frac{\alpha_i}{\alpha_{i+1}} \quad , \quad \forall\, i \in \{0, 1, ..., k\}$$

which implies that

$$\|F^i(\frac{\alpha_k}{\alpha_{k+1}} F\xi)\| \leq \alpha_i \quad , \quad \forall\, i \in \{0, 1, ..., k\}$$

we deduce that

$$\frac{\alpha_k}{\alpha_{k+1}} F\xi \in S_k \, .$$

■

An equivalent assertion for S to be finitely accessible is given by the following proposition

Proposition 3.3 *S is finitely accessible if and only if there is $k \in \mathbb{N}$ such that $S_{k+1} = S_k$.*

Proof
suppose that S is finitely accessible, then there is $k \in \mathbb{N}$ such that $S = S_k$, which implies that $S_k \subset S_{k+1}$, thus we can deduce from proposition 3.2 the

equality $S_k = S_{k+1}$.
Conversely, if $S_k = S_{k+1}$ for some integer $k \in \mathbb{N}$, i.e. $S_k \subset S_{k+1}$, it follows from proposition 3.2 that for $\xi \in S_k$ we have

$$\frac{\alpha_k}{\alpha_{k+1}} F\xi \ \in \ S_k$$

and by iteration

$$(\frac{\alpha_k}{\alpha_{k+1}})^j F^j \xi \ \in \ S_k \ , \quad \forall j \geq 0$$

then

$$\|(\frac{\alpha_k}{\alpha_{k+1}})^j \ F^{i+j} \ \xi\| \ \leq \ \alpha_i \ , \quad \forall \ i \in \{0, 1, ..., k\} \ , \ \ \forall j \geq 0$$

in particular, for i = k, we have

$$\|F^{k+j} \ \xi\| \ \leq \ \frac{\alpha_{k+1}^j}{\alpha_k^{j-1}} \ \ , \quad \forall j \geq 1.$$

Using the properties of $(\alpha_i)_{i \geq 0}$, we establish by recurrence that for all $j \geq 1$, $\frac{\alpha_{k+1}^j}{\alpha_k^{j-1}} \ \leq \ \alpha_{k+j}$ thus

$$\|F^{k+j} \ \xi\| \ \leq \ \alpha_{k+j}$$

Then $\xi \in S$, hence $S_k \subset S$, we deduce from proposition 3.2 i) that $S = S_k$.

■

The following theorem gives sufficient condition for S to be finitely accessible.

Theorem 3.2 *We suppose that* $\lim\limits_{i \to +\infty} \frac{\|F^i\|}{\alpha_i} = 0$, *then* S *is finitely accessible.*

Proof
The fact that $\lim\limits_{i \to +\infty} \frac{\|F^i\|}{\alpha_i} = 0$ implies the existence of an integer $k_0 \geq 1$ such that

$$\frac{\|F^{k_0+1}\|}{\alpha_{k_0+1}} \ \leq \ \frac{1}{\alpha_0}.$$

For $\xi \in S_{k_0}$, we have $\|\xi\| \leq \alpha_0$ then

$$\begin{aligned} \|F^{k_0+1}\xi\| \ &\leq \ \|F^{k_0+1}\|\|\xi\| \\ &\leq \ \tfrac{\alpha_{k_0+1}}{\alpha_0} \, \alpha_0 \ = \ \alpha_{k_0+1} \end{aligned}$$

thus $\xi \in S_{k_0+1}$, and from propositions 3.2 and 3.3, we deduce that $S_{k_0} = S_{k_0+1} = S$.

■

3.3 An algorithmic approach

The proposition 3.3 inspires the following theoretical algorithm

Step 1 k:=0

Step 2 Repeat
k:= k+1
Determination of S_k , S_{k+1}
Until $S_k \ = \ S_{k+1}$

Step 3 $k^* = k$
$S \ = \ S_{k^*}$

The inconvenient of the above algorithm is the difficulty of testing $S_k = S_{k+1}$, therefore the following approach is proposed.

Consider the norm $\|.\|_\infty$ on $\mathbb{R}^p$ defined for every $\xi = (\xi_1, \xi_2, ..., \xi_p) \in \mathbb{R}^p$ by

$$\|\xi\|_\infty = \max_{1\leq i\leq p} |\xi_i|$$

and the function $f_j : \mathbb{R}^p \to \mathbb{R}$, $j = 1, 2, ..., 2p$ defined for every $\xi = (\xi_1, \xi_2, ..., \xi_p) \in \mathbb{R}^p$ by

$$f_{2l}(\xi) = -\xi_l - 1 \ , \quad l = 1, 2, ..., p$$
$$f_{2l-1}(x) = \xi_l - 1 \ , \quad l = 1, 2, ..., p$$

then the set S_k is described as follows

$$S_k \ = \ \{\xi \in \mathbb{R}^p \ / \ f_j(\frac{1}{\alpha_i} F^i \xi) \ \leq \ 0 \ ; \ j = 1, 2, ..., 2p \ ; \ i = 0, 1, ..., k\}$$

We deduce that

$$\begin{aligned} S_k \ = \ S_{k+1} \ &\Leftrightarrow \ S_k \ \subset \ S_{k+1} \\ &\Leftrightarrow \ \forall \xi \in S_k \ , \ \forall i \in \{1, 2, ..., 2p\} \ , \ \ f_i(\tfrac{1}{\alpha_{k+1}} F^{k+1} \xi) \ \leq \ 0 \\ &\Leftrightarrow \ \sup_{\xi \in S_k} f_i(\tfrac{1}{\alpha_{k+1}} F^{k+1} \xi) \ \leq \ 0 \ , \ \ \forall i \in \{1, 2, ..., 2p\}. \end{aligned}$$

This encourages us to propose the following algorithmic implementation

Algorithm

Step 1 k:=0

Step 2 For $i = 1, 2, ..., 2p$, do:

$$\begin{cases} maximize \;\; J_i(\xi) \; = \; f_i(\frac{1}{\alpha_{k+1}} F^{k+1}\xi) \\ f_j(\frac{1}{\alpha_l} F^l \xi) \; \leq \; 0 \\ j = 1, 2, ..., 2p \;\; , \;\; l = 0, 1, ..., k \end{cases}$$

Let J_i^* be the maximum value of J_i.
If $(J_1^* \leq 0, \; J_2^* \leq 0, \; ... \; , J_{2p}^* \leq 0)$ then set $k^* := k$ and stop.
Else continue

Step 3 Replace k by k+1 and return to step 2.

The optimization problem cited in step 2 is a mathematical programming problem and can be solved by standard methods.

4 observer initial state design

In this section, we shall assume $\mathcal{P}$ to be a convex and compact polyhedron of $\mathbb{R}^n$ containing x_0 and whose vertices are $v_1 \; , \; v_2 \; , \; ... \; , \; v_r$.
Using the results of section 3, we can easily establish the following proposition

Proposition 4.1 *Suppose that F verifies*

$$\lim_{i \to +\infty} \frac{\|F^i\|}{\alpha_i} = 0 \tag{15}$$

then $\mathcal{M}$ is the set of all $z_0 \in \mathbb{R}^p$ satisfying the constraints

$$\begin{cases} \|F^i(z_0 - Tv_j)\|_\infty \; \leq \; \alpha_i \\ 0 \; \leq \; i \; \leq \; k^* \\ 1 \; \leq \; j \; \leq \; r \end{cases} \tag{16}$$

where k^* is the index-access of S .

Remark 4.1 *It follows from the previous proposition that the set $\mathcal{M}$ of the α-admissible observer initial states is entirely determined if the system (16) described by a finite number of linear inequalities in the unknown z_0 has a feasible solution.*

However, and while basing itself on more precise information on the localization of the unknown initial state x_0, we will give, in the following, conditions on the width of the polyhedron $\mathcal{P}$ and this, with an aim of solving the system of inequations (16).

Proposition 4.2 *Suppose that there is $\rho > 0$ such that $\mathcal{B}(0, \rho) \subset S$ and $\mathcal{P}$ is such that $diam\mathcal{P} \leq \frac{\rho}{\|T\|}$.*
Then

$$T\mathcal{P} \subset \mathcal{M}.$$

Moreover, if $diam\mathcal{P} < \frac{\rho}{\|T\|}$ then

$$T\mathcal{P} \subset int(\mathcal{M}).$$

Proof
Suppose that $diam\mathcal{P} \leq \frac{\rho}{\|T\|}$, for $j\ ,\ l\ = 1, ..., r$, we have

$$\begin{aligned} \|Tv_j - Tv_l\| &\leq \|T\|\|v_j - v_l\| \\ &\leq \|T\|diam(\mathcal{P}) \\ &\leq \rho \end{aligned}$$

then

$$Tv_j \in \bigcap_{l=1}^{r} (S + Tv_l)$$

we deduce from relation (14) and theorem 3.1 that

$$Tv_j \in \mathcal{M}$$

therefore since $\mathcal{M}$ is convex,

$$T\mathcal{P} \subset \mathcal{M}.$$

Suppose now that $diam\mathcal{P} < \frac{\rho}{\|T\|}$ and consider $\beta = \rho - \|T\|diam(\mathcal{P})$.
For $j = 1, ..., r$ and $z \in \mathcal{B}(Tv_j\ ,\ \beta)$, we have

$$\|z - Tv_j\| \leq \beta \leq \rho$$

then

$$z - Tv_j \in \mathcal{B}(0\ ,\ \rho) \subset S$$

i.e,

$$z \in S + Tv_j.$$

On the other hand, for $i \neq j$, we have

$$\begin{aligned} \|z - Tv_i\| &\leq \|Tv_i - Tv_j\| + \|Tv_j - z\| \\ &\leq \|T\|\|v_i - v_j\| + \beta \\ &\leq \|T\|diam(\mathcal{P}) + \beta\ =\ \rho \end{aligned}$$

then

$$z \in S + Tv_i$$

therefore

$$z \in \bigcap_{k=1}^{r} (S + Tv_k)$$

we deduce that $z \in \mathcal{M}$, hence

$$\mathcal{B}(Tv_j , \beta) \subset \mathcal{M}$$

i.e, $Tv_j \in int(\mathcal{M})$ therefore, since $int(\mathcal{M})$ is convex,

$$T\mathcal{P} \subset int(\mathcal{M}).$$

■

With an aim of improving the preceding result and to give more concrete conditions on the diameter of $\mathcal{P}$, we propose the following result.

Proposition 4.3 *If we suppose that the sequence* $(\frac{\|F^i\|}{\alpha_i})_{i\geq 0}$ *is bounded and* $\mathcal{P}$ *is such that* $diam\mathcal{P} \leq \frac{1}{\gamma\|T\|}$ *(respectively* $diam\mathcal{P} < \frac{1}{\gamma\|T\|}$*), where* $\gamma = \sup_{i\geq 0} \frac{\|F^i\|}{\alpha_i}$*, then*

$$T\mathcal{P} \subset \mathcal{M} \; (respectively \;\; T\mathcal{P} \subset int(\mathcal{M})).$$

Proof
We show that the hypothesis of proposition 4.2 is verified. Indeed, since $(\frac{\|F^i\|}{\alpha_i})_{i\geq 0}$ is bounded, we consider

$$\gamma = \sup_{i\geq 0} \frac{\|F^i\|}{\alpha_i} \in \mathbb{R}_+^*$$

then

$$\|F^i\| \leq \gamma\alpha_i \quad \forall\, i \geq 0.$$

For $\rho = \frac{1}{\gamma}$ and $\xi \in \mathcal{B}(0, \rho)$, we have

$$\begin{aligned} \|F^i\xi\| &\leq \|F^i\|\|\xi\| \qquad \forall\, i \geq 0 \\ &\leq \gamma\alpha_i.\rho = \alpha_i \end{aligned}$$

thus $\xi \in S$, therefore

$$\mathcal{B}(0, \rho) \subset S.$$

■

It comes from proposition 4.1 that though the condition (15) guarantees the equality

$$\mathcal{M} = \{z_0 \in \mathbb{R}^p \ / \ \|F^i(z_0 - Tv_j)\|_\infty \leq \alpha_i \ , \ 0 \ \leq \ i \ \leq \ k^* \ , \ 1 \leq j \leq r\}$$

it is not sure that $\mathcal{M} = \bigcap_{j=1}^{r} (S + Tv_j)$ is nonempty, therefore the only condition (15) is insufficient for the description of at least an α-admissible observer initial state z_0. To cure this handicap and on the basis of a more precise site of the initial state x_0, we establish in the following proposition, sufficient conditions which ensure on the one hand the feasibility of the system of inequations (16)(i.e. $\mathcal{M} \neq \emptyset$) and on the other hand, the design of a part of the set $\mathcal{M}$.

Proposition 4.4 *If* $\lim_{i \to +\infty} \frac{\|F^i\|}{\alpha_i} = 0$ *and* $diam \mathcal{P} \leq \frac{1}{\mu \|T\|}$ *, where* $\mu = \max_{0 \leq i \leq k^*} \frac{\|F^i\|}{\alpha_i}$*, then*

$$\mathcal{M} = \{z_0 \in \mathbb{R}^p \ / \ \|F^i(z_0 - Tv_j)\|_\infty \leq \alpha_i \ , \ 0 \ \leq \ i \ \leq \ k^* \ , \ 1 \leq j \leq r\} \ \neq \ \emptyset \ .$$

Moreover, we have

$$T\mathcal{P} \ \subset \ \mathcal{M} \quad \text{and} \quad \mu \ = \ \sup_{i \geq 0} \frac{\|F^i\|}{\alpha_i}.$$

If we have the strict inequality $diam \mathcal{P} < \frac{1}{\mu \|T\|}$*, then* $int(\mathcal{M}) \neq \emptyset$ *and we have* $T\mathcal{P} \subset int(\mathcal{M})$*.*

Proof

We recall that from proposition 3.2, S is finitely accessible and $S = S_{k^*}$.

Let us consider $\gamma = \sup_{i \geq 0} \frac{\|F^i\|}{\alpha_i}$, we have $\gamma \geq \mu$.

If $\gamma > \mu$, then there is $i_0 > k^*$ and $z_0 \in \mathcal{B}(0,1)$ such that

$$\|F^{i_0} z_0\| \ > \ \mu \alpha_{i_0}$$

which implies

$$\|F^{i_0}(\frac{1}{\mu} z_0)\| \ > \ \alpha_{i_0}$$

thus

$$\frac{1}{\mu} z_0 \ \notin \ S. \tag{17}$$

On the other hand, for $0 \leq i \leq k^*$

$$\begin{array}{rcl} \|F^i(\frac{1}{\mu} z_0) & \leq & \frac{1}{\mu} \|F^i\| \|z_0\| \\ & \leq & \frac{1}{\mu} \alpha_i \mu \ = \ \alpha_i \end{array}$$

then $\frac{1}{\mu} z_0 \in S_{k^*} = S$, which is in contradiction with (17).

■

5 Numerical simulation

Consider the discrete linear system governed by

$$\begin{cases} x_{i+1} = Ax_i + Bu_i \ , & i \geq 0 \\ x_0 \ is \ unknown \end{cases}$$

with the observation

$$y_i = Cx_i \ , \quad i \geq 0$$

where $A = \begin{pmatrix} -1.7 & 3.2 \\ -3 & 3.42 \end{pmatrix}$; $B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $C = \begin{pmatrix} -1 & 1 \end{pmatrix}$ and consider the observer whose state is defined by

$$\begin{cases} z_{i+1} = Fz_i + Dy_i + Pu_i \ , & i \geq 0 \\ z_0 \in \mathbb{R}^2 \end{cases}$$

where $F = \begin{pmatrix} 0.3 & 1.2 \\ 0 & 0.42 \end{pmatrix}$; $D = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; $P = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
and let T be the identity matrix

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

It is obvious that

1. P = TB
2. TA - FT = DC
3. The eigenvalues of F are 0.3 and 0.42, then F is stable.

In order to improve the performances of our observer (z_i), we consider the sequence $(\alpha_i)_{i\geq 0}$ defined by $\alpha_i = \frac{1}{2^i}$, then $\lim\limits_{i\to+\infty} \frac{\|F^i\|}{\alpha_i} = 0$. Using the algorithm defined in subsection 3.3, the simplex method gives $k^* = 4$. We have by proposition 4.4, $\sup\limits_{i\geq 0} \frac{\|F^i\|}{\alpha_i} = \max\limits_{0\leq i\leq 4} \frac{\|F^i\|}{\alpha_i} = \frac{1}{0.2511}$.

For $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the polyhedron $\mathcal{P}$ with vertices

$$v_1 = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix} \ , \ v_2 = \begin{pmatrix} -0.1 \\ 0 \end{pmatrix} \ and \ v_3 = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}$$

A simple calculation gives $diam\mathcal{P} \simeq 0.14$ then $diam\mathcal{P} < \frac{1}{\mu\|T\|} = 0.2511$, thus proposition 4.4 insures that the set $\mathcal{M}$ of α-admissible observer initial states corresponding to the polyhedron $\mathcal{P}$ is nonempty and is entirely determined by proposition 4.1, and we have the following scheme:

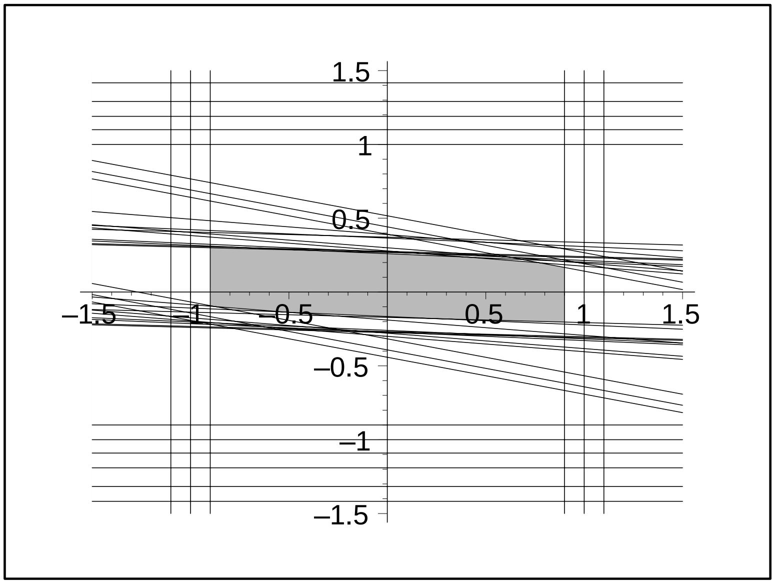

Figure 1: The colored region gives the graphic representation of $\mathcal{M}$ the set of α-admissible observer initial states corresponding to the polyhedron $\mathcal{P}$.

Remark 5.1 .
1) Like it was established in proposition 4.4, we notice that the triangle $T\mathcal{P}$ *whose three vertices are* v_1, v_2 *and* v_3 *is well inside the set of* α*-admissible observer initial states corresponding to the polyhedron* $\mathcal{P}$*.*
2) For $z_0 \in \mathcal{M}$*, we know that the estimation error (10) verifies* $\|e_i(x_0)\| \leq \alpha_i \ , \ \ \forall i \geq 0$ *from any initial state* $x_0 \in \mathcal{P}$*. By proposition 4.1, it is sufficient to verify it for* $i = 0, ..., 4$ *and from vertices* $v_j, \ j = 1, 2, 3$ *as initial states. To illustrate that, we take* $z_0 = \begin{pmatrix} 0.8 \\ -0.1 \end{pmatrix} \in \mathcal{M}$*, and we represent the* l^∞ *norm of the estimation errors (10) which are plotted in Fig.2 from initial states* v_1, v_2, v_3*, and their comparison with curve 1 representing* $\alpha_i, \ i \geq 0$*.*
3) If $z_0 \notin \mathcal{M}$*, we cannot know starting from which rank* i_0 *one will have* $\|e_i(x_0)\| \leq \alpha_i \ , \ \ \forall i \geq i_0$ *from some initial state* $x_0 \in \mathcal{P}$*. Fig.3 represents the estimation error (10) from initial state* v_1 *for* $z_0 = \begin{pmatrix} -1.5 \\ 0.5 \end{pmatrix}$ *(resp.*$z_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$*) which are not in* $\mathcal{M}$*, and their comparison with curve 1 representing* $\alpha_i, \ i \geq 0$*. Note that* $i_0 = 9$*(resp.* $i_0 = 14$*).*

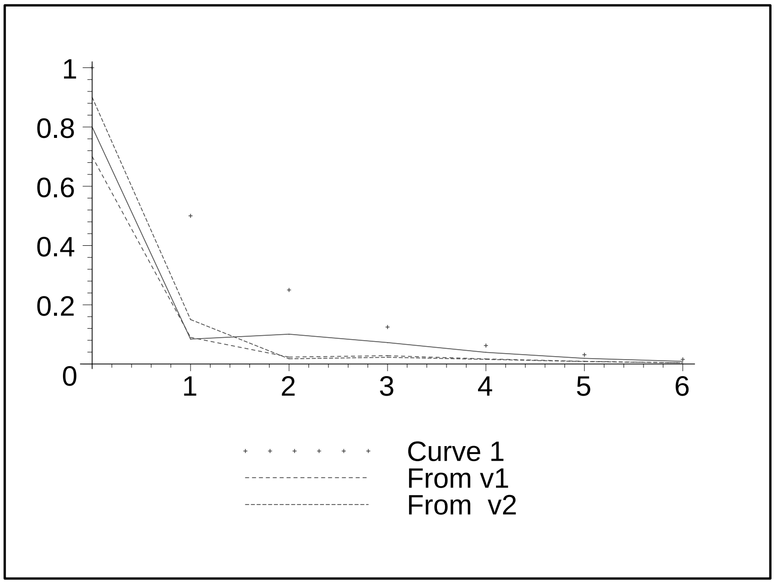

Figure 2: The simulation results for l^∞ norm of the estimation error for $z_0 = (0.8, -0.1)$

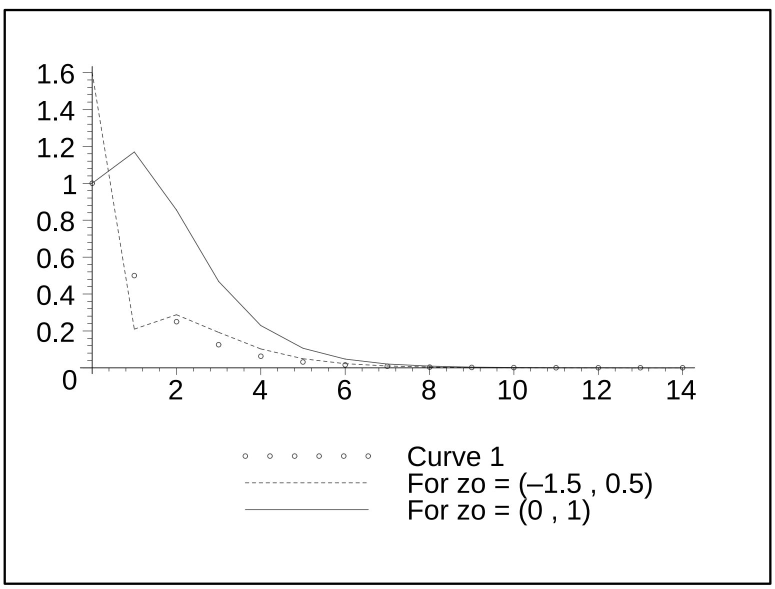

Figure 3: The simulation results for l^∞ norm of the estimation error for $z_0 \notin \mathcal{M}$

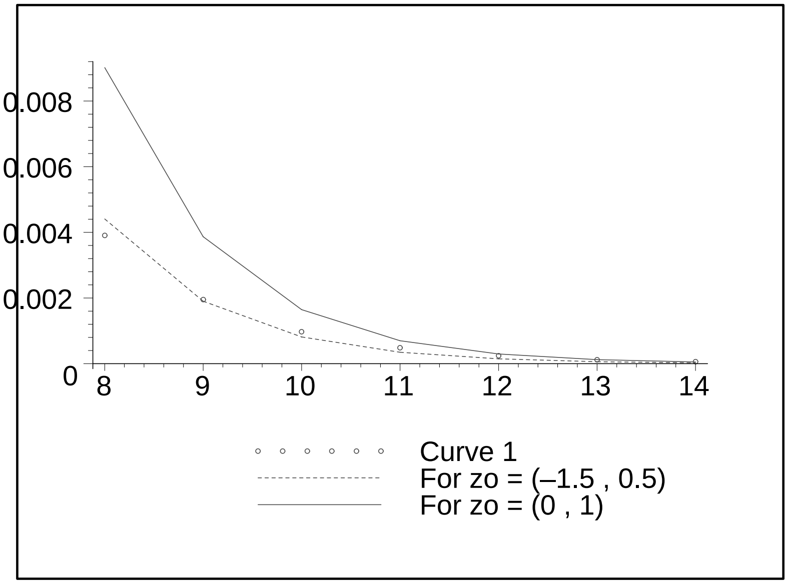

Figure 4:

6 Discrete-time delayed system

In this section, we consider the discrete delayed system given by

$$\begin{cases} x_{i+1} = \sum_{j=0}^{N} A_j x_{i-j} + Bu_i \,, \qquad i \geq 0 \\ x_k \in \mathbb{R}^n \;\; for \;\; k \;\in\; \{-N, \; -N+1, \; ... \;, \; 0\} \end{cases} \tag{18}$$

the corresponding delayed output function is

$$y_i = \sum_{k=0}^{R} C_k x_{i-k} \,, \qquad i \geq 0 \tag{19}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^q$ are, respectively, the state variable, the control variable and the output variable, while A_j, B and C_j are constant matrices of respective dimensions $(n \times n),\ (n \times m),\ and\ (q \times n)$. R and N are positive integers such that $R \leq N$.
Without loss of generality, we assume that R = N, if not $(R < N)$ we can get $C_k = 0$ for $k = R+1, \; ... \;, N$.
In the following, we suppose that the initial state $(x_{-N}, x_{-N+1}, ..., x_0)$ is unknown, then we have to solve the states estimation problem of system (18) in basis of the output (19).
To solve this problem, we propose in the following to design an observer of the

form

$$\begin{cases} z_{i+1} = \sum_{j=0}^{N} F_j z_{i-j} + Pu_i + Dy_i \ , \qquad i \geq 0 \\ z_k \in \mathbb{R}^p \ \ for \ \ k \ \in \ \{-N, \ -N+1, \ ... \ , \ 0\} \end{cases} \tag{20}$$

where $z_i \in \mathbb{R}^p$ is the observer state, F_j, P and D are constant matrices of respective dimensions $(p \times p), \ (p \times m), \ and \ (p \times q)$.
For $\widetilde{x_0} = (x_0, x_{-1}, ..., x_{-N})$ an initial state of system (18), and T a matrix of suitable dimension, let us introduce, for $i \geq 0$, the vectors

$$\widetilde{e_i}(\widetilde{x_0}) = (e_i(\widetilde{x_0}), \ e_{i-1}(\widetilde{x_0}), \ ..., \ e_{i-N}(\widetilde{x_0}))^T \ \in \ \mathbb{R}^{(N+1)p}$$

where

$$e_i(\widetilde{x_0}) = z_i - Tx_i \tag{21}$$

is the estimation error from the initial state $\widetilde{x}_0$, and consider the new matrix $\widetilde{F}$ of dimension $(N+1)p \times (N+1)p$ defined by

$$\widetilde{F} = \begin{pmatrix} F_0 & F_1 & \cdots & \cdots & F_N \\ I_p & 0_p & \cdots & \cdots & 0_p \\ 0_p & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_p & \cdots & 0_p & I_p & 0_p \end{pmatrix}$$

where I_p and 0_p are respectively the identity and the zero matrices of order p. In order to lighten the notations, and when there is no confusion , we will denote $\widetilde{e_i}(\widetilde{x_0})$ by $\widetilde{e_i}$ and $e_i(\widetilde{x_0})$ by e_i.
The following propositions give sufficient conditions for the existence of an observer.

Proposition 6.1 *For $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, the equation (20) specifies an observer of the system (18, 19) if the following hold*

1. $F_jT - TA_j = -DC_j \ , \ \ for \ \ 0 \leq \ j \ \leq \ N$

2. $P \ = \ TB$

3. *The matrix $\widetilde{F}$ is stable.*

Moreover, we have $\widetilde{e_i} = \widetilde{F}^i.\widetilde{e_0}$, $\forall i \geq 0$.

Proof

Using (18, 19) and (20) yield

$$\begin{array}{rcl} e_{i+1} & = & z_{i+1} - Tx_{i+1} \\ & = & \sum_{j=0}^{N} F_j z_{i-j} \ + \ D\sum_{j=0}^{N} C_j x_{i-j} \ + \ Pu_i \ - \ \sum_{j=0}^{N} TA_j x_{i-j} \ - \ TBu_i \\ & = & \sum_{j=0}^{N} F_j e_{i-j} \ + \ \sum_{j=0}^{N} (F_jT + DC_j - TA)x_{i-j} \ + \ (P - TB)u_i \end{array} .$$

If the conditions (1) and (2) hold then the observer error becomes

$$e_{i+1} = \sum_{j=0}^{N} F_j e_{i-j} \tag{22}$$

which is is equivalent to

$$\widetilde{e}_{i+1} = \widetilde{F}\widetilde{e}_i$$

hence

$$\widetilde{e}_{i+1} = \widetilde{F}^i \widetilde{e}_0 \ .$$

Thus

$$\begin{array}{rcl} z_i \ is \ an \ asymptotic \ state \ estimator \ of \ Tx_i & \Leftrightarrow & \lim_{j \to +\infty} (z_i - Tx_i) \ = \ 0 \\ & \Leftrightarrow & \lim_{i \to +\infty} e_i = 0 \\ & \Leftrightarrow & \lim_{i \to +\infty} \widetilde{e}_i = 0 \\ & \Leftrightarrow & The \ matrix \ \widetilde{F} \ is \ stable \end{array} .$$

■

Proposition 6.2 *For $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, the equation (20) specifies an observer of the system (18, 19) if the following hold*

1. $F_jT - TA_j = -DC_j \ , \ \ for \ \ 0 \leq \ j \ \leq \ N$
2. $P \ = \ TB$
3. $\sum_{j=0}^{N} \|F_j\| \ < \ 1.$

Proof

It is established in ([40]) that the condition (3) is sufficient to insure the stability of $\widetilde{F}$, then we use proposition 6.1 to conclude.

■

Under the conditions of proposition 6.1 and given $\widetilde{\mathcal{P}}$ a convex and compact polyhedron of $\mathbb{R}^{(N+1)n}$ containing the unknown initial state $\widetilde{x}_0 = (x_0, x_{-1}, ..., x_{-N})$ of system (18), we are interested to determine all observer initial state conditions $\widetilde{z}_0 = (z_0, z_{-1}, ..., z_{-N})$ of system (20) such that the error (21) verifies

$$\|e_i(\widetilde{x})\| \ \leq \ \alpha_i \quad \forall\, i \geq -N \ , \ \ \forall \widetilde{x} \in \widetilde{\mathcal{P}}$$

where $(\alpha_i)_{i\geq -N}$ is a positive decreasing sequence which verifies condition (11). In other words, we aim to characterize the set $\widetilde{\mathcal{M}}$ of α-admissible initial states given by

$$\widetilde{\mathcal{M}} = \{(z_0,\ z_{-1},\ ...,\ z_{-N}) \in \mathbb{R}^{(N+1)p} \ / \ \|e_i(\widetilde{x})\| \ \leq \ \alpha_i \ , \ \ \forall\, i \geq -N \ , \ \forall \widetilde{x} \in \widetilde{\mathcal{P}}\}.$$

Toward this end, let us define

$$\widetilde{\mathcal{M}}_{\widetilde{x}_0} = \{(z_0,\ z_{-1},\ ...,\ z_{-N}) \in \mathbb{R}^{(N+1)p} \ / \ \|e_i(\widetilde{x}_0)\| \ \leq \ \alpha_i \ , \ \ \forall\, i \geq -N\}$$

and $\widetilde{T}$ the matrix of dimension $(N+1)p \times (N+1)n$

$$\widetilde{T} = \begin{pmatrix} T & 0_{p\times n} & \cdots & 0_{p\times n} \\ 0_{p\times n} & T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{p\times n} \\ 0_{p\times n} & \cdots & 0_{p\times n} & T \end{pmatrix}$$

where $0_{p\times n}$ is and the $p \times n$ -zero matrix.
Thus the following proposition holds

Proposition 6.3 *For* $\widetilde{x}_0 = (x_0,\ x_{-1},\ ...,\ x_{-N}) \ \in \mathbb{R}^{(N+1)n}$*, we have*

$$\widetilde{\mathcal{M}}_{\widetilde{x}_0} = \{\widetilde{z}_0 = (z_0,\ z_{-1},\ ...,\ z_{-N}) \in \mathbb{R}^{(N+1)p} \ / \ \|\widetilde{F}^i(\widetilde{z}_0 \ - \ \widetilde{T}\widetilde{x}_0)\| \ \leq \ \beta_i \quad \forall\, i \geq 0\}$$

where $\beta_i = \alpha_{i-N}$*,* $i \geq 0$.

Proof
For $i \geq 0$, let us define the vectors

$$\widetilde{x}_i = (x_i,\ x_{i-1},\ ...,\ x_{i-N})$$

$$\widetilde{z}_i = (z_i,\ z_{i-1},\ ...,\ z_{i-N})$$

then we have

$$\widetilde{e}_i = \widetilde{z}_i - \widetilde{T}\widetilde{x}_i \ .$$

From proposition 6.1, it follows that

$$\begin{array}{rcl} \widetilde{e}_i & = & \widetilde{F}^i \widetilde{e}_0 \\ & = & \widetilde{F}^i(\widetilde{z}_0 \ - \ \widetilde{T}\widetilde{x}_0) \end{array} .$$

If $\widetilde{z}_0 \in \widetilde{\mathcal{M}}_{\widetilde{x}_0}$ then

$$\|e_i\| \ \leq \ \alpha_i \quad \forall\, i \geq -N$$

or

$$\|e_i\| \ \leq \ \alpha_i \ , \ \|e_{i-1}\| \ \leq \ \alpha_{i-1} \ , \ ... \ , \ \|e_{i-N}\| \ \leq \ \alpha_{i-N} \quad \forall\, i \geq -N$$

since

$$\|\widetilde{e}_i\| = max(\|e_i\| \ , \ ... \ , \ \|e_{i-N}\|)$$

we deduce

$$\|\widetilde{e}_i\| \leq max(\|\alpha_i\| \ , \ ... \ , \ \|\alpha_{i-N}\|) \ = \ \alpha_{i-N}$$

i.e,

$$\|\widetilde{e}_i\| \leq \beta_i \ , \ \ \forall\, i \geq 0.$$

Conversely if $\widetilde{z}_0 \in \mathbb{R}^{(N+1)p}$ is such that $\|\widetilde{e}_i\| \leq \alpha_{i-N} \ , \ \ \forall\, i \geq 0$
then

$$\|e_i\| \ \leq \ \|\widetilde{e}_{i+N}\| \ \leq \ \alpha_i \ , \ \ \forall\, i \geq -N \ .$$

Hence $\widetilde{z}_0 \in \widetilde{\mathcal{M}}_{\widetilde{x}_0}$.

■

From proposition 6.3, it follows that $\widetilde{\mathcal{M}}_{\widetilde{x}_0}$ is of the same form as the set M_{x_0} defined by (12), so $\widetilde{\mathcal{M}}_{\widetilde{x}_0}$ can be expressed as in (14)

$$\widetilde{\mathcal{M}}_{\widetilde{x}_0} \ = \ \widetilde{S} \ + \ \widetilde{T}\widetilde{x}_0$$

where

$$\widetilde{S} = \{\xi \in \mathbb{R}^{(N+1)p} / \ \|\widetilde{F}^i\xi\| \leq \beta_i \ , \ \forall i \geq 0\}.$$

Therefore, it is obvious that theorem 3.2 gives sufficient conditions to characterize the set $\widetilde{S}$ by a finite number of inequalities, and the results on the characterization of the set of α-admissible observer initial states of section 4 can be translated to the set $\widetilde{\mathcal{M}}$.

In the following proposition, we give other sufficient conditions to characterize the set $\widetilde{\mathcal{M}}_{\widetilde{x}_0}$ by a finite number of inequalities

Proposition 6.4 *Suppose that*

$$\sum_{j=0}^{N} \|F_j\|^2 \ \leq \ \frac{\alpha_{N+1}^2}{\displaystyle\sum_{i=0}^{N} \alpha_i^2}$$

then

$$\widetilde{\mathcal{M}}_{\widetilde{x}_0} = \{(z_0, \ z_{-1}, \ ..., \ z_{-N}) \in \mathbb{R}^{(N+1)p} \ / \ \|e_i\| \ \leq \ \alpha_i \quad \forall\, i \in \{-N, ..., 0, ..., N\}\}$$

The following lemma will help us to prove proposition 6.4.

Lemma 6.1 *Suppose that*

$$\|\sum_{j=0}^{N} F_j z_j\| \;\leq\; \alpha_{N+1} \quad , \quad \forall\, z_j \;\in\; \mathcal{B}(0, \alpha_{N-j})$$

Then

$$\widetilde{\mathcal{M}}_{\widetilde{x}_0} = \{(z_0,\ z_{-1},\ ...,\ z_{-N}) \in \mathbb{R}^{(N+1)p} \ / \ \|e_i\| \;\leq\; \alpha_i \quad \forall\, i \in \{-N, ..., 0, ..., N\}\}$$

where N is the number of delays in the state variable of system (18).

Proof
From relation (22) , we have

$$e_i = \sum_{j=0}^{N} F_j e_{i-j-1} \quad , \quad \forall\, i \geq N-1$$

If $\widetilde{z}_0 \in \widetilde{\mathcal{M}}^N_{\widetilde{x}_0} = \{(z_0,\ z_{-1},\ ...,\ z_{-N}) \in \mathbb{R}^{(N+1)p} \ / \ \|e_i\| \leq \alpha_i \quad \forall\, i \in \{-N, ..., 0, ..., N\}\}$,
from the hypothesis of lemma 6.1, we have

$$\|e_{N+1}\| \;=\; \|\sum_{j=0}^{N} F_j e_{N-j}\| \;\leq\; \alpha_{N+1}$$

then

$$\widetilde{z}_0 \;\in\; \widetilde{\mathcal{M}}^{N+1}_{\widetilde{x}_0} = \{(z_0,\ z_{-1},\ ...,\ z_{-N}) \in \mathbb{R}^{(N+1)p} \ / \ \|e_i\| \;\leq\; \alpha_i$$
$$\forall\, i \in \{-N, ..., 0, ..., N, N+1\}\}$$

therefore

$$\widetilde{\mathcal{M}}^N_{\widetilde{x}_0} \;\subset\; \widetilde{\mathcal{M}}^{N+1}_{\widetilde{x}_0}$$

we deduce from propositions 3.2 and 3.3 that

$$\widetilde{\mathcal{M}}^N_{\widetilde{x}_0} \;=\; \widetilde{\mathcal{M}}^{N+1}_{\widetilde{x}_0} \;=\; \widetilde{\mathcal{M}}_{\widetilde{x}_0}\ .$$

■

Proof of proposition 6.4
We prove that the condition of lemma 6.1 is verified. Indeed for every

$z_j \in \mathcal{B}(0, \alpha_{N-j})$, we have

$$\begin{aligned}
\|\sum_{j=0}^{N} F_j z_j\| \le \alpha_{N+1} &\le (\sum_{j=0}^{N} \|F_j\|^2)^{\frac{1}{2}} (\sum_{j=0}^{N} \|z_j\|^2)^{\frac{1}{2}} \\
&\le (\sum_{j=0}^{N} \|F_j\|^2)^{\frac{1}{2}} (\sum_{j=0}^{N} \alpha_{N-j}^2)^{\frac{1}{2}} \\
&\le (\sum_{j=0}^{N} \|F_j\|^2)^{\frac{1}{2}} (\sum_{j=0}^{N} \alpha_i^2)^{\frac{1}{2}} \\
&\le \alpha_{N+1} .
\end{aligned}$$

■

7 Conclusion

In this paper, we are interested to estimate a discrete system state, we suppose that the initial state is unknown but localized in a convex and compact polyhedron. We determine, under certain hypothesis, a class $\mathcal{M}$ such that the Luenberger observer $(z_i)_{i\geq 0}$ initialized with $z_0 \in \mathcal{M}$ allows to realize the performance

$$\|z_i - Tx_i\| \le \alpha_i \; ; \; \forall i \geq 0$$

where $\alpha = (\alpha_i)_i$ is a predefined mode of convergence. After giving a theoretical and algorithmic characterization of the set $\mathcal{M}$, we showed that the used approach is easily extended to discrete delayed systems .
As a natural continuation of this work and inspired by what was done in [33], [34], we investigate the same problem in the presence of perturbations. It will be also interesting to study the continuous case.

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