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The Dual of a Space of Cauchy Transforms

Y. Abu Muhanna and Z. Abdulhadi

Department of Mathematics
American University of Sharjah
P.O. Box 26666, Sharjah, United Arab Emirates
ymuhanna@aus.edu, zhadi@aus.edu

Abstract. Let F_α, $\alpha \geq 0$ be the class of Cauchy transforms of order α equipped with the bounded variation norm. The bounded linear functionals on F_α are characterized and an inclusion between duals is given.

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1. Introduction

Let $\mathbf{T}$ be the unit circle and $\mathbf{M}$ be the set of all complex–valued Borel measures on $\mathbf{T}$. For $\alpha > 0$ and $z \in \mathbf{D}$, we define the space of weighted Cauchy transforms F_α to be the family of all functions $f(z)$ such that

$$f(z) = \int_{\mathbf{T}} K_x^\alpha(z) d\mu(x) \tag{1.1}$$

where the Cauchy kernel $K_x(z)$ is given by

$$K_x(z) = \frac{1}{1 - \overline{x}z}$$

and where μ in (1.1) varies over all measures in $\mathbf{M}$. The class F_α is a Banach space with respect to the norm

$$\|f\|_{F_\alpha} = \inf \|\mu\|_{\mathbf{M}} \tag{1.2}$$

where the infimum is taken over all Borel measures μ satisfying (1.1). $\|\mu\|$ denotes the total variation norm of μ. For detailed information about the space F_α, see [2, 3, 4, 5].

Let W_α denote the space of all bounded linear functionals on F_α. Recall that L is called a bounded linear functional on F_α if L is linear and

$$|L(f)| \le A\|f\|_{F_\alpha}$$

for all $f \in F_\alpha$. Let W_α^* denote the subspace of W_α which consist of all bounded fuctional that preserves weak convergence. This means that if a sequence $\{f_n\}$ and f in $F_\alpha$with corresponding measures, as in (1.1), $\{\mu_n\}$ converges weakly to μ then $L(f_n) \to L(f)$ for each $L \in W_\alpha^*$.

It is known, [3], that $\|z^n\|_{F_\alpha} \le cn^{\alpha-1}$, for $0 \le \alpha < 1$ and that $\|z^n\|_{F_\alpha}$ is bounded when $\alpha \ge 1$.

Let

$$b_n = L(z^n).$$

Hence

$$\limsup_{n\to\infty} \left|b_n^{1/n}\right| \le 1.$$

So the functions

$$\begin{aligned} g(z) &= \sum_0^\infty b_n z^n, \\ g_\alpha(z) &= \sum_0^\infty A_n(\alpha) b_n z^n \end{aligned} \tag{1.3}$$

are analytic in D. Clearly, $L(\sum_0^k a_n z^n) = \sum_0^k a_k b_k$.

In Theorem 1, we characterize all bounded linear functionals on F_α and in Theorem 3, we show that $W_\alpha \subset W_\beta^* \subset W_\beta$, where $0 \le \beta < \alpha$.

2. Bounded linear functionals on F_α

In this section, we shall express a bounded linear functional L in terms of g_α. As a first step we have:

Lemma 1. *If* $K_x^\alpha(\rho z) = \sum_0^\infty A_n(\alpha)\rho^n \overline{x}^n z^n$, $\rho < 1$ *then*

$$L(K_x^\alpha(\rho z)) = \sum_0^\infty A_k(\alpha) b_k \rho^k \overline{x}^k = g_\alpha(\rho\overline{x}). \tag{2.1}$$

Proof. Let

$$
\begin{aligned}
(2.2) \qquad d\sigma_\rho(y) &= \operatorname{Re}\frac{1}{1-\overline{y\rho}}\frac{dt}{\pi},\\
d\sigma_{n\rho}(y) &= \operatorname{Re}\sum_0^k \overline{y}^k\rho^k\frac{dt}{\pi}\\
K^\alpha_{xn}(\rho z) &= \sum_0^n A_n(\alpha)\rho^k\overline{x}^k z^k
\end{aligned}
$$

where $y=e^{it}$. Then $K^\alpha_{xn}(\rho z)=\int_{\mathbf{T}} K^\alpha_{xy}(z)d\sigma_{n\rho}(y)$ and $K^\alpha_x(\rho z)=\int_{\mathbf{T}} K^\alpha_{xy}(z)d\sigma_\rho(y)$.

Hence $\|K^\alpha_x(\rho z)-K^\alpha_{xn}(\rho z)\|_{F_\alpha}\le\int\left|\operatorname{Re}\sum_{n+1}^\infty\overline{y}^k\rho^k\right|\frac{dt}{\pi}\to 0$, as $n\to\infty$. Hence, as L is continuous, the result follows. ∎

Lemma 2. *Let f be as in (1.1), then*

$$L(f)=\int_{\mathbf{T}} L(K^\alpha_x(z))d\mu(x).$$

Proof. Let $d\lambda_n=\sum_0^n\mu_j\chi_{x_j}\to d\mu$ in the total variation norm and let $f_n=\int_{\mathbf{T}} K^\alpha_x(z)d\lambda_n(x)$. Then $f_n\to f$ in F_α and, as $d\lambda_n$ is a finite sum,

$$L(f_n)=\int_{\mathbf{T}} L(K^\alpha_x(z))d\lambda_n(x)\to L(f).$$

Since $L(K^\alpha_x(z))$ is bounded,

$$\left|\int_{\mathbf{T}} L(K^\alpha_x(z))d\lambda_n(x)-\int_{\mathbf{T}} L(K^\alpha_x(z))d\mu(x)\right|\le C\|\mu-\lambda_n\|$$

and hence $L(f)=\int_{\mathbf{T}} L(K^\alpha_x(z))d\mu(x)$. ∎

Lemma 3. $g_\alpha(z)=\sum_0^\infty A_n(\alpha)b_nz^n$ *is a bounded function in D.*

Proof. By (2.1) and (2.2), $L(K^\alpha_x(\rho z))=\sum_0^\infty A_k(\alpha)b_k\rho^k\overline{x}^k=\int_{\mathbf{T}} L(K^\alpha_{xy}(z))d\sigma_\rho(y)$. Since $K^\alpha_x(z)$ is uniformly bounded by 1 in F_α, $L(K^\alpha_x(\rho z))$ is also uniformly bounded in C. Hence $g_\alpha(z)$ is a bounded function. ∎

The first three lemmas lead to the following proposition:

Proposition 4. *$g_\alpha(\overline{x}) = \lim\limits_{\rho\to 1} g_\alpha(\rho\overline{x}) = L(K_x^\alpha(z))$, for all x with $|x| = 1$.*

Proof. Lemma 1 imply that $g_\alpha(\rho\overline{x}) = \int\limits_{\mathbf{T}} L(K_{xy}^\alpha(z))d\sigma_\rho(y)$. This, the fact that $g_\alpha(z)$ is bounded and $d\sigma_\rho(y) = \operatorname{Re}\dfrac{1}{1-\overline{y\rho}}\frac{dt}{\pi}$ imply that $L(K_x^\alpha(z)) = g_\alpha(\overline{x})$. ∎

A consequence of Proposition 1 is:

Corollary 5. *Let f be as in (1.1), $f(z) = \sum\limits_0^\infty a_k z^k$ and L a bounded linear functional on F_α. Then*

$$L(f) = \int\limits_{\mathbf{T}} g_\alpha(\overline{x})d\mu(x) = \lim_{\rho\to 1}\sum_0^\infty A_k(\alpha)b_k a_k\rho^k, \tag{2.3}$$

where $g_\alpha \in H^\infty$, defined as in (1.3), has radial limits at all $x \in T$.

In the converse direction of Corollary 1, we have

Lemma 6. *Let g and g_α be related as in (1.3). If $g_\alpha \in H^\infty$ with radial limits at all $x \in T$ then g generates a bounded linear functional on F_α.*

Proof. Define $L(K_x^\alpha(z)) = g_\alpha(\overline{x})$ and if f as in (1), $L(f) = \int\limits_{\mathbf{T}} g_\alpha(\overline{x})d\mu(x)$. Clearly L is linear and as g_α is bounded $|L(f)| \leq \|g_\alpha\|_{H^\infty}\|f\|_{F_\alpha}$. ∎

In conclusion, Corollary 1 and Lemma 4 lead to the following theorem which characterizes the dual of F_α :

Theorem 7. *The dual of $F_\alpha = W_\alpha = \{g : g_\alpha \in H^\infty$ with radial limits at all $x \in T\}$ and*
$W_\alpha^* = \{g : g_\alpha$ is continuous on $\overline{D}\}$.

Remark 1. *The dual of $F_1 = W_1 = \{g : g \in H^\infty$ with radial limits at all $x \in T\}$.*

Remark 2. *The dual of $F_2 = W_2 = \{g : g$ is analytic on $\overline{D}\} =$The dual of F_α, for all $\alpha > 2$.This is the same as the dual of the space of all analytic functions equipped with the topology of uniform convergence on compact subsets of D.*

3. Some Properties of W_α

We start this section with the statement of the well known Abel's Theorem:

Theorem 8. *(Abel's Theorem) If* $\sum_1^\infty c_n$ *is convergent, with* $\left|\sum_1^\infty c_n\right| \leq M$ *and* $\{d_n\}$ *is a decresing positive sequence converging to* 0 *then* $\left|\sum_{s+1}^u c_n d_n\right| \leq 2Md_{s+1}$ *and hence* $\sum_1^\infty c_n d_n$ *converges.*

As a consequence of Abel's Theorem we have:

Lemma 9. *Let* g *and* g_α *be as in (1.3) with* $\alpha > 1$*. If* $g_\alpha \in H^\infty$ *then* g *is continuous on* $\overline{D}$*.*

Proof. Let $d_n = \frac{1}{A_n(\alpha)}$, where $\alpha > 1$. Then d_n is decreasing to 0. Let $c_n = A_n(\alpha)b_n z^n$. Then apply Abel's Theorem to get

$$\left|\sum_{s+1}^{u} c_n d_n\right| = \left|\sum_{s+1}^{u} b_n z^n\right| \leq 2\left\|g_\alpha\right\|_\infty d_{s+1} \to 0.$$

and consequently

$$\left|\sum_{s+1}^{u} b_n e^{int}\right| \leq 2\left\|g_\alpha\right\|_\infty d_{s+1} \to 0.$$

Hence $\sum_1^\infty b_n z^n$ is uniformly convergent on $\overline{D}$. Therefore, it is continuous. ∎

Here is another important consequence of Abel's Theorem.

Lemma 10. *If* $g_\alpha \in H^\infty$ *for any* $\alpha \geq 0$ *then* g_β *is continuous for any* $\beta < \alpha$*.*

Proof. Let $d_n = \dfrac{A_n(\beta)}{A_n(\alpha)}$ and $c_n = A_n(\alpha)b_n z^n$. Then it can be shown that d_n decreases to 0. Hence, by Abel's Theorem,

$$\left|\sum_{s+1}^{u} c_n d_n\right| = \left|\sum_{s+1}^{u} A_n(\beta) b_n z^n\right| \leq 2\left\|g_\alpha\right\|_\infty d_{s+1} \to 0$$

and that

$$\left|\sum_{s+1}^{u} A_n(\beta) b_n e^{int}\right| \leq 2\left\|g_\alpha\right\|_\infty d_{s+1} \to 0$$

Hence $\sum_1^\infty A_n(\beta) b_n z^n$ is uniformly convrgent on $\overline{D}$. Hence, it is continuous. ∎

As a consequence of the last lemma, we have:

Theorem 11. *$W_\alpha \subset W_\beta^* \subset W_\beta$, where $0 \leq \beta < \alpha$.*

Finally we have the following property of W_α, $\alpha > 0$.

Theorem 12. *If $g \in W_\alpha$ and $g_\alpha \in F_\alpha$ then $\sum_0^\infty A_n(\alpha) |b_n|^2 < \infty$.*

Proof. Let g, g_α be as in (1.3) and that L is the corresponding linear functional. In specific, $L(z^n) = b_n$ and $g_\alpha = \sum_1^\infty A_n(\alpha) b_n r e^{int} \in F_\alpha$ is bounded. Hence $\overline{g_\alpha(\overline{z})} = \sum_1^\infty \frac{1}{n} \overline{b_n} z^n \in F_\alpha$ is bounded. Apply L to $\overline{g_\alpha(\overline{z})}$ to conclude the result. ∎

Remark 3. *The result implies, for $\alpha = 0$, that the area of the image of g_0 is finite.*

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