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A New Formulation of the Hamiltonian p_Median Problem

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Abstract

Location-Routing problems involve locating a number of facilities among candidate sites and establishing delivery routes to a set of users in such a way that the total system cost is minimized. A special case of these problems is Hamiltonian p_Median Problem (HpMP). In attempting to solve this problem, numerous mathematical formulations have been proposed. Most of them have in common that their descriptions as integer optimization problems are not polyhedral ones (ILP formulation). In this paper, an ILP formulation, based on the formulation of vehicle routing problem, is presented. The proposed formulation is simpler and more practicable than those have been proposed up to now.

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1 Introduction

In the last decade, the class of so called location-routing problems (LRPs) attracted a lot of research interest and many different problems variants have been developed in this field. This is due to its practical relevance in many real world situations. In such contexts, location and routing are intertwined decisions which must be modelled.

Research on LRPs is relatively new and the literature on it is much less developed than that associated with either pure location or pure routing problems. This paper deals with a special case of location-routing problems, the Hamiltonian p_Median Problem (HpMP), which has been introduced by Branco and Coelho [1]. Since one of the most promising approaches for an exact solution of a hard combinatorial optimization problem is the cutting plane method, we investigate mathematical formulations of the HpMP and then proposed an ILP formulation for it.

The Hamiltonian p_Median Problem can be formulated through a graph theory model as follows. Let G =(V,A) be a directed complete graph, where V=$\{1, \cdots, n\}$ is the vertex set and A the arc set. For each (i,j)∈A, let c_{ij} be the cost of arc (i,j). Then the Hp_MP consists of selecting p vertices (facilities) from V and assigning vertices in V (customers) to p pairwise disjoint cycles, each of which consisting exactly one of these p vertices, such that the total distribution cost is minimized. It is also clear that each vertex of a subtour can be chosen as a facility location. In summary, the HpMP is equivalent to determining p pairwise disjoint circuits covering the vertex set V, with respect to some objective function. In other words, let

$$C_p := \{\{C^1, C^2, \ldots, C^p\} : \; C^i = (V_i, A_i), \; V_i \cap V_j = \emptyset \;\; i \neq j, \; \bigcup_{i=1}^{p} V_i = V\}$$

denote the set of all Hamiltonian p_medians, where C^i is a circuit. As most interesting combinatorial optimization problems, the HpMP has been proven to be NP-Complete.

The remainder of this paper is organized as follows: Section 2 deals with the mathematical models for the HpMP. In section 3, a new ILP formulation for the HpMP is provided, which is simpler and more practicable than those have been proposed up to now, and section 4 gives our conclusive remarks.

2 Mathematical models for the HpMP

In the literature different formulations for the HpMP exist, two of which are given in [1]. But, both formulations have in common that their descriptions as integer optimization problems are not polyhedral ones (the set of feasible solutions is not described as the set of integral points of some polytope).

The problem of finding a general polyhedral ILP formulation lies in the description of V into p disjoint subtours. Therefore, Glaab and pott [2] introduced the term of m_partitions P_m of V as the set of all partitions of V into m subsets which are pairwise disjoint and which form a cover of V, i.e.

$$P_m := \{S = \{s_1, s_2, \ldots, s_m\} : \ s_i \subset V, \ s_i \cap s_j = \emptyset \ \ i \neq j, \ \bigcup_{i=1}^{m} s_i = V\}$$

moreover, for each element $S \in P_m$, let

$$A(S) := \{(i,j) \in A : \ i \in s_k \ \mathit{and} \ j \in s_l \ \ 1 \leq k < l \leq m\}$$

denote the directed m_cut associated with S. The existence of nonempty directed (p+1)_cut will guarantee the existence of at most p subtours.

Lemma Suppose F_n(m) be the cardinality of the set P_m of V, Then:

$$F_n(m) = \ F_{n-1}(m-1) \ + \ mF_{n-1}(m)$$

(Proof is available from the author on request)

Let $$x_{ijk} = \begin{cases} 1 & (i,j) \in C^k \\ 0 & Otherwise \end{cases}$$

then an ILP formulation for the HpMP is as follows (Glaab and Pott [2]):

$$A) \quad Min \quad \sum_{i,j} c_{ij} \Big(\sum_{k=1}^{p} x_{ijk} \Big)$$

$$s.t. \quad \sum_{i,k} x_{ijk} = 1 \qquad j = 1, \ldots, n \tag{1}$$

$$\sum_{j,k} x_{ijk} = 1 \qquad i = 1, \ldots, n \tag{2}$$

$$\sum_{i} x_{ijk} - \sum_{l} x_{jlk} = 0 \qquad j = 1, \ldots, n \quad k = 1, \ldots, p \tag{3}$$

$$\sum_{k} \sum_{(i,j) \in A(S)} x_{ijk} \geq 1 \qquad \forall S \in P_{p+1} \tag{4}$$

$$\sum_{i,j} x_{ijk} \geq 2 \qquad k = 1, \ldots, p \tag{5}$$

$$x_{ijk} \in \{0, 1\} \qquad i, j = 1, \ldots, n \quad k = 1, \ldots, p \tag{6}$$

The equations (1) and (2) ensure that each vertex has exactly one successor and one predecessor in $\bigcup_{k=1}^{p} C^k$; i.e. The C^ks are unions of circuits. Equation (3) guarantees that each vertex is assigned to exactly one of the sets C^k. Together with (1) and (2), the last condition also implies that any two subtours are vertex disjoint. The inequalities Subtour Number Constraints(4), exclude the existence of more than p different circuits. Finally, the inequalities (5) in connection with (4) ensure that each feasible solution consists of at least two arcs. Additionally, (5) in connection with (4) ensure that each feasible solution consists of exactly p circuits or subtours.

Clearly, in spite of definition a polyhedral ILP formulation, mode A is a complex and an ambiguous model due to the fact that formulating of subtour number constraints and (p+1)_partition of V are indeed complex operations.

3 A new ILP formulation for the HpMP

In this section a new ILP formulation for the HpMP is provided, which is based on the Vehicle routing problem (VRP) formulation. In the context of routing problems, VRP has been one of the widely studied topics, where goods must be picked up and/or delivered for a geographically dispersed set of customers. We don't focus on the role of modelling and implementation issues in VRP. The interested reader may consult a number of useful surveys of the field including Golden et al. [3] and Mole [4].

Suppose, 0, be a virtual depot and x_{ijk} is defined as mentioned earlier. Let c_{0i}= c_{i0}= L $\forall i$, where L is a nonnegative number. Then an ILP formulation for the HpMP is as follows:

$$B) \quad Min \quad \sum_{i,j} c_{ij} \Big(\sum_{k=1}^{p} (x_{ijk} + z_{ijk}) \Big)$$

$$s.t. \quad \sum_{i,k} x_{ijk} = 1 \qquad j = 1, \cdots, n \tag{1}$$

$$\sum_{j,k} x_{ijk} = 1 \qquad i = 1, \cdots, n \tag{2}$$

$$\sum_{j} x_{0jk} = 1 \qquad k = 1, \cdots, p \tag{3}$$

$$\sum_{i,j} x_{ijk} \geq 4 \qquad k = 1, , \cdots, p \tag{4}$$

$$\sum_{k,j} kx_{ijk} - \sum_{k,j} kx_{jik} = 0 \qquad i = 1, \cdots, n \tag{5}$$

$$u_i - u_j + (n+1)\sum_{k} x_{ijk} \leq n \quad i, j = 1, \cdots, n \tag{6}$$

$$x_{0jk} = \sum_{i} z_{ijk} \qquad j = 1, \ldots, n \quad k = 1, \cdots, p \tag{7}$$

$$x_{i0k} = \sum_{j} z_{ijk} \qquad i = 1, \ldots, n \quad k = 1, \cdots, p \tag{8}$$

$$\sum_{i,j} z_{ijk} = 1 \qquad k = 1, \cdots, p \tag{9}$$

$$z_{ijk} \in \{0, 1\} \qquad i, j = 0, \cdots, n \quad k = 1, \cdots, p \tag{10}$$

$$x_{ijk} \in \{0, 1\} \qquad i, j = 0, \cdots, n \quad k = 1, \cdots, p \tag{11}$$

$$u_i \geq 0 \qquad i = 0, 1, \ldots, n \tag{12}$$

The constraints (1-6) and (11-12) ensure the formation of a VRP feasible solution on V$\cup\{0\}$ such that each cycle of it has at least three nodes. The constraints (7-10) guarantees that if $x_{i0k} = x_{0jk} = 1$, then $z_{ijk} = 1$ (i.e. (i,j)$\in C^k$) and c$_{ij}$ will be added to the objective function. Furthermore, constant value 2pL is added to the HpMP optimal value. Hence, we can take L=0.

Theorem 1 The constraints (7-9) accompanied by (10'), satisfy (10).

$$z_{ijk} \geq 0 \qquad i, j = 0, 1, \ldots, n \qquad k = 1, 2, \ldots, p \tag{10'}$$

Proof Suppose k=l and $x_{a0l} = x_{0bl} = 1$.

$$\sum_{i=0}^{n} z_{ibl} = x_{0bl} = 1$$

$$\sum_{j=0}^{n} z_{ajl} = x_{a0l} = 1$$

$$\overline{\sum_{i=0}^{n} z_{ibl} + \sum_{j=0}^{n} z_{ajl}} = \sum_{i:\, i\neq a} z_{ibl} + z_{abl} + \sum_{j:\, j\neq b} z_{ajl} + z_{abl} = 2$$

Likewise, we have $\sum_{i:\, i\neq a} z_{ibl} + z_{abl} + \sum_{j:\, j\neq b} z_{ajl} \leq \sum_{i,j} z_{ijl} = 1.$

Therefore $z_{abl} = 1$ and due to constraint (9), $z_{ijl} = 0 \quad \forall i, j \ (i \neq a,\ j \neq b)$ □

Theorem 2 The constraint (9) is superfluous.

Proof By means of constraints (3) and (7), we have:

$$x_{0jk} = \sum_{i=0}^{n} z_{ijk}$$

$$\sum_{j=0}^{n} x_{0jk} = 1$$

$$\overline{\qquad\qquad\qquad}$$

$$1 = \sum_{j} x_{0jk} = \sum_{i,j} z_{ijk}$$

Hence the constraint (9) is superfluous □

In other words, model B could be simplified by adding constraint (10') and removing the constraints (9-10). Hence,

$$\begin{array}{lll} C) & Min & \sum_{i,j} c_{ij} \Big(\sum_{k=1}^{p} (x_{ijk} + z_{ijk}) \Big) \\ & s.t. & constraints \quad \{(1-8)\&(10')\&(11-12)\} \end{array}$$

4 Conclusions

In this paper we presented a new ILP formulation for the Hamiltonian p_median problem (HpMP) which is simpler and more practicable than those have been proposed up to now. This formulation is based on the Vehicle routing problem (VRP), which has been widely studied. But, in spite of the proposed ILP formulation, it is unlikely that it will be able to solve large-scale problems.

In many situations one has to settle for algorithm that run fast but may produce suboptimal solutions. Therefore, development of heuristic algorithms for this problem provides an interesting challenge for future research.

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