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A Linearized Oscillation Result for Even-order Neutral Differential Equations[1]

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Abstract

Consider the even-order nonlinear neutral delay differential equation $\frac{d^n}{dt^n}[(x(t)-p(t)g(x(t-\tau))]-Q(t)h(x(t-\sigma))=0, \quad t \geq t_0$, where $p, Q \in C([t_0,\infty),R), \tau > 0, \sigma \geq 0$. We obtain a linearized oscillation result by an associate linear equation in the case when the coefficient $p(t)$ takes values in the interval $(-1,0)$, and thereby establish new criteria as proposed in an earlier open problem.

Keywords: Even-order; neutral differential equation; linearization; oscillation

1. Introduction

During last ten years, the linearized oscillation theory for nonlinear neutral delay differential equations has been extensively developed, for example see [1−9]. Roughly speaking, it has been proved that, under appropriate hypotheses, certain nonlinear neutral delay differential equations have the same oscillatory character as an associated linear equation.

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Consider the even-order nonlinear neutral differential equation

$$\frac{d^n}{dt^n}[x(t) - p(t)g(x(t-\tau))] - Q(t)h(x(t-\sigma)) = 0, \quad t \geq t_0, \tag{1}$$

where n is an even number and

$$p, Q \in C([t_0, \infty), R), \quad g, h \in C(R, R), \quad \tau > 0, \quad \sigma \geq 0. \tag{2}$$

The first linearized oscillation result of Eq.(1) was established by Chuanxi and Ladas [3], where the coefficient $p(t)$ takes values in the interval $(0, 1)$. The question naturally arises as to how one may establish the corresponding linearized oscillation results of (1) for the case when $p(t)$ takes values outside the interval $(0, 1)$. Also see the open problem 6.12.7 in [4]. About the study of the above problem, to the present time, the cases when $-\infty < p(t) \leq -1$ and $p(t) \geq 1$ have been considered in [9] . However, the case $-1 < p(t) < 0$ has not yet been handled. Our aim in this paper is to answer the above problem for the case when$-1 < p(t) < 0$. Our main result is the following theorem.

Theorem A *Assume that (2) holds and that*

$$\limsup_{t\to\infty} p(t) = -P_0 \in (-1, 0), \quad \liminf_{t\to\infty} p(t) = -p_0 \in (-1, 0), \tag{3}$$

$$\lim_{t\to\infty} Q(t) = q \in (0, \infty), \tag{4}$$

$$0 \leq \frac{g(u)}{u} \leq 1 \quad for \quad u \neq 0 \quad and \quad \lim_{u\to 0} \frac{g(u)}{u} = 1, \tag{5}$$

$$uh(u) > 0 \quad for \quad u \neq 0 \quad and \quad \lim_{u\to 0} \frac{h(u)}{u} = 1. \tag{6}$$

If every bounded solution of the linear equation

$$\frac{d^n}{dt^n}[y(t) + p_0^{-1} y(t-\tau)] - qy(t-\sigma) = 0 \tag{7}$$

oscillates, then every bounded solution of Eq.(1) also oscillates.

The proof of the above Theorem will be given in section 2.

Let $\rho = \max\{\tau, \sigma\}$. By a solution of Eq.(1) we mean a function $x \in C([t_1 - \rho, \infty), R)$ for some $t_1 \geq t_0$, such that $x(t) - p(t)g(x(t-\tau))$ is n times continuously differentiable on $[t_1, \infty)$ and (1) is satisfied for $t \geq t_1$.

Let $t_1 \geq t_0$ and let $\varphi \in C([t_1 - \rho, t_1], R)$ be a given initial function, and let $z_k, k = 0, 1, ..., n-1$, be given initial constants. Using the method of steps one can see that Eq. (1) has a unique solution $x \in C([t_1 - \rho, \infty), R)$ such that

$$x(t) = \varphi(t) \quad \text{for} \quad t \in [t_1 - \rho, t_1]$$

and
$$\frac{d^k}{dt^k}[\varphi(t)-p(t)g(\varphi(t-\tau))]_{t=t_1}=z_k \quad \text{for} \quad k=0,1,2,...,n-1.$$

As usual, a solution of Eq.(1) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

In the sequel, for convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2. Proof of Theorem A

The following lemmas will be useful in the proof of Theorem A.

Lemma 1 *Let n be even and assume that*

$$p\in(0,1), \quad \tau,q\in(0,\infty) \quad \textit{and} \quad \sigma\in[0,\infty). \tag{8}$$

If every bounded solution of the linear equation

$$\frac{d^n}{dt^n}[x(t)+p^{-1}x(t-\tau)]-qx(t-\sigma)=0 \tag{9}$$

oscillates, then there exists an $\varepsilon\in(0,q)$ such that every bounded solution of the equation

$$\frac{d^n}{dt^n}[(x(t)+(p^{-1}+\varepsilon)x(t-\tau)]-(q-\varepsilon)x(t-\sigma)=0 \tag{10}$$

also oscillates.

Proof. By lemma 4 in [3], the hypothesis that every bounded solution of Eq.(9) oscillates implies that the characteristic equation of Eq.(9)

$$f(\lambda)=\lambda^n+p^{-1}\lambda^n e^{-\lambda\tau}-qe^{-\lambda\sigma}=0$$

has no real roots $\in(-\infty,0)$. This and $f(0)=-q<0$ imply that

$$f(\lambda)<0 \quad \text{for} \quad \text{all} \quad \lambda\in(-\infty,0]$$

and hence $\tau<\sigma$. Clearly, $f(-\infty)=-\infty$ and so

$$f(\lambda)\le\sup_{\xi\in(-\infty,0]}f(\xi):=m<0 \quad \text{for} \quad \text{all} \quad \lambda\in(-\infty,0].$$

Next we set
$$\delta=\frac{1}{3}q \quad \text{and} \quad g(\lambda)=\delta(-\lambda^n e^{-\lambda t}-e^{-\lambda\sigma}).$$

Then it is easy to see that

$$f(\lambda) - g(\lambda) = \lambda^n(1 + (p^{-1} + \delta)e^{-\lambda t}) - (q - \delta)e^{-\lambda\sigma} \to -\infty \text{ as } \lambda \to -\infty,$$

which implies that there exists a $\lambda_0 < 0$ such that

$$f(\lambda) - g(\lambda) \le \frac{1}{2}m \text{ for } \lambda \le \lambda_0.$$

Let

$$\mu = \sup_{\lambda\in[\lambda_0,0]} (\lambda^n e^{-\lambda\tau} + e^{-\lambda\sigma})$$

and set

$$\varepsilon = \min\{\delta, -\frac{1}{2}m\mu\}.$$

To complete the proof, by lemma 4 in [3] it suffices to show that the characteristic equation

$$\lambda^n + (p^{-1} + \varepsilon)\lambda^n e^{-\lambda\tau} - (q - \varepsilon)e^{-\lambda\sigma} = 0 \tag{11}$$

has no real roots in $(-\infty, 0]$. In fact, because n is even, we have for $\lambda \le \lambda_0$

$$\begin{aligned} \lambda^n + (p^{-1} + \varepsilon)\lambda^n e^{-\lambda\tau} - (q - \varepsilon)e^{-\lambda\sigma} &= f(\lambda) + \varepsilon(\lambda^n e^{-\lambda\tau} + e^{-\lambda\sigma}) \\ &\le f(\lambda) + \delta(\lambda^n e^{-\lambda\tau} + e^{-\lambda\sigma}) \\ &= f(\lambda) - g(\lambda) \le \frac{1}{2}m < 0. \end{aligned}$$

and for $\lambda_0 \le \lambda \le 0$

$$\begin{aligned} \lambda^n + (p^{-1} + \varepsilon)\lambda^n e^{-\lambda\tau} - (q - \varepsilon)e^{-\lambda\sigma} &= f(\lambda) + \varepsilon(\lambda^n e^{-\lambda\tau} + e^{-\lambda\sigma}) \\ &\le m + \mu\varepsilon \le m - \frac{1}{2}m = \frac{1}{2}m < 0. \end{aligned}$$

The proof is complete.

Lemma 2 [3] *Consider the equation*

$$\frac{d^n}{dt^n}[x(t) - P(t)x(t - \tau)] - Q(t)x(t - \sigma) = 0, \tag{12}$$

where n is even,and

$$P, Q \in C((t_0, \infty), R), \quad Q(t) \ge 0 \quad for \quad t \ge t_0 \quad and \quad \tau > 0, \quad \sigma \ge 0. \tag{13}$$

Assume that there are numbers p_1 and p_2 such that

$$p_1 \le P(t) \le p_2 < -1 \tag{14}$$

and that

$$\int_{t_0}^{\infty} Q(s)ds = \infty. \tag{15}$$

Let $x(t)$ be an eventually bounded positive solution of Eq.(12) and set

$$y(t) = x(t) - P(t)x(t-\tau).$$

Then eventually

$$y^{(n)}(t) \geq 0, \quad (-1)^i y^{(n-i)}(t) > 0 \quad for \quad i = 1, 2, ..., n. \tag{16}$$

$$\lim_{t\to\infty} y^{(i)}(t) = 0 \quad for \quad i = 0, 1, ...n-1. \tag{17}$$

Now we are ready to prove Theorem A by using the Banach Contraction Principle.

Proof of Theorem A . Assume that Eq.(1) has a bounded nonoscillatory solution $x(t)$. We will assume that $x(t)$ is eventually positive. The case when $x(t)$ is eventually negative is similar and will be omitted. Choose $t_1 \geq t_0$ to be such that

$$x(t-\tau) > 0, \quad x(t-\sigma) > 0 \quad \text{for} \quad t \geq t_1.$$

Set

$$Z(t) = x(t) - p(t)g(x(t-\tau)). \tag{18}$$

Then $Z(t) > 0$ and

$$Z^{(n)}(t) = Q(t)h\left(x(t-\sigma)\right) \geq 0 \quad for \quad t \geq t_1. \tag{19}$$

So, $Z^{(i)}(t)(i = 0, 1, ..., n-1)$ are eventually positive or eventually negative and so either

$$Z^{(n-1)}(t) < 0, \tag{20}$$

or

$$Z^{(n-1)}(t) > 0. \tag{21}$$

We claim that (20) holds. Otherwise (21) holds which implies that there exists $\beta > 0$ such that eventually

$$Z^{(n-1)}(t) \geq \beta.$$

This yields $Z(t) \to \infty$, which is a contradiction because of the bounded nature of $x(t)$ and $p(t)$. Hence (20) holds. Let

$$\lim_{t\to\infty} Z^{(n-1)}(t) = \alpha \in (-\infty, 0].$$

Integrating (19) from $t \geq t_1$ to ∞, we have

$$\alpha - Z^{(n-1)}(t) = \int_t^\infty Q(s)h(x(s-\sigma))ds,$$

which, together with (4) and (6), yields

$$\liminf_{t\to\infty} x(t) = 0. \tag{22}$$

Now we claim that

$$\limsup_{t\to\infty} x(t) = 0. \tag{23}$$

Indeed, let $\lim_{t\to\infty} Z(t) = L$, then $L \in [0, \infty)$, and from (22), there exists a sequence λ_n, such that

$$\lim_{n\to\infty} \lambda_n = \infty, \quad \lim_{n\to\infty} x(\lambda_n) = 0. \tag{24}$$

Set

$$Z(t) - Z(t-\tau) = x(t) - p(t)g(x(t-\tau)) - x(t-\tau) + p(t-\tau)g(x(t-2\tau)). \tag{25}$$

By replacing t with λ_n in (25), we have

$$\begin{aligned} Z(\lambda_n) - Z(\lambda_n - \tau) &= x(\lambda_n) - p(\lambda_n)g(x(\lambda_n - \tau)) \\ &\quad -x(\lambda_n - \tau) + p(\lambda_n - \tau)g(x(\lambda_n - 2\tau)) \\ &\leq x(\lambda_n) - [p(\lambda_n) + 1]x(\lambda_n - \tau), \end{aligned}$$

that is

$$x(\lambda_n) + Z(\lambda_n - \tau) - Z(\lambda_n) \geq [p(\lambda_n) + 1]x(\lambda_n - \tau)$$

and so

$$\begin{aligned} 0 &= \lim_{n\to\infty} [x(\lambda_n) + Z(\lambda_n - \tau) - Z(\lambda_n)] \\ &\geq \liminf_{n\to\infty} [p(\lambda_n) + 1]x(\lambda_n - \tau) \\ &\geq (1 - p_0) \liminf_{n\to\infty} x(\lambda_n - \tau). \end{aligned}$$

Since $-p_0 \in (-1, 0), x(t)$ is eventually positive, there exists a sequence λ_{n_k} such that

$$\lim_{t\to\infty} x(\lambda_{n_k} - \tau) = 0. \tag{26}$$

By replacing t with λ_{n_k} in (18), from (24),(26), we have

$$L = \lim_{k\to\infty} Z(\lambda_{n_k}) = \lim_{k\to\infty} [x(\lambda_{n_k}) - p(\lambda_{n_k})g(x(\lambda_{n_k} - \tau))] = 0.$$

From the definition of $Z(t)$, we have

$$0 = \lim_{t\to\infty} Z(t) \geq \limsup_{t\to\infty} x(t).$$

Since $x(t)$ is eventually positive, it follows that

$$\limsup_{t\to\infty} x(t) = 0.$$

Which, together with (22), yields

$$\lim_{t\to\infty} x(t) = L = 0. \tag{27}$$

Next we rewrite Eq. (1) in the form

$$\frac{d^n}{dt^n}(x(t) + P^*(t)x(t-\tau)) - Q^*(t)x(t-\sigma) = 0, \tag{28}$$

where

$$P^*(t) = -p(t)g(x(t-\tau))/x(t-\tau), \quad Q^*(t) = Q(t)h(x(t-\sigma))/x(t-\sigma).$$

From (3)–(6) and (27) we have

$$\limsup_{t\to\infty} P^* \leq p_0, \quad \lim_{t\to\infty} Q^*(t) = q. \tag{29}$$

According to the definition of $Z(t)$, we can rewrite Eq. (28) in the form

$$Z^{(n)}(t) + P^*(t-\sigma)\frac{Q^*(t)}{Q^*(t-\tau)}Z^{(n)}(t-\tau) = Q^*(t)Z(t-\sigma). \tag{30}$$

Since every bounded solution of Eq. (7) oscillates, by Lemma 1 it follows that there is an $\varepsilon \in (0, q)$ such that

$$\lambda^n + (p^{-1} + \varepsilon)\lambda^n e^{-\lambda\tau} - (q-\varepsilon)e^{-\lambda\sigma} < 0 \text{ for all } \lambda \in (-\infty, 0]. \tag{31}$$

For this $\varepsilon > 0$, let $\alpha \in (0,1)$ be such that $\alpha q > q - \varepsilon$, and let $\beta > 1$ be such that

$$\alpha q > \beta(q-\varepsilon) \text{ or } \frac{q}{\beta} > \frac{(q-\varepsilon)}{\alpha}.$$

From (29) we see that there exists $t_2 > t_1 + \sigma$ such that

$$P^*(t-\sigma) \cdot \frac{Q^*(t)}{Q^*(t-\tau)} < p_0 + \varepsilon < p_0^{-1} + \varepsilon, \quad Q^*(t) > \frac{q}{\beta} \text{ for } t \geq t_2. \tag{32}$$

Substituting this into (30),we get

$$Z^{(n)}(t)+(p_0^{-1}+\varepsilon)Z^{(n)}(t-\tau)>\frac{q}{\beta}Z(t-\sigma),t\geq t_2. \tag{33}$$

Set

$$G(t)=\left(Z^{(n)}(t)+(p_0^{-1}+\varepsilon)Z^{(n)}(t-\tau)\right)/Z(t-\sigma), \tag{34}$$

then we have by (33)

$$G(t)>\frac{q}{\beta}\quad\text{for}\quad t\geq t_2. \tag{35}$$

From (34) we see that

$$Z^{(n)}(t)+(p_0^{-1}+\varepsilon)Z^{(n)}(t-\tau)=G(t)Z(t-\sigma). \tag{36}$$

Integrating both sides of (36) from $t\geq t_2$ to ∞ $n-1$ times and using Lemma 2, we get

$$Z'(t)+(p_0^{-1}+\varepsilon)Z'(t-\tau)+\frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}G(s)Z(s-\sigma)ds=0.$$

In what follows, for the sake of convenience, we set

$$a=p_0^{-1}+\varepsilon,\quad H(t)=\frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}G(s)Z(s-\sigma)ds.$$

Then we have

$$Z'(t)+aZ'(t-\tau)+H(t)=0.$$

Integrating this from t to ∞, we get

$$Z(t)+aZ(t-\tau)=\int_t^\infty H(u)du,$$

or equivalently

$$Z(t)=-\frac{1}{a}Z(t+\tau)+\frac{1}{a}\int_{t+\tau}^\infty H(u)du.$$

Integrating it, we obtain

$$Z(t)=\sum_{i=1}^{k}(-1)^{i+1}a^{-i}\int_{t+i\tau}^\infty H(u)du+(-1)^k a^{-k}Z(t+k\tau).$$

Since $a>1$ and $Z(t)\to 0$ as $t\to\infty$, we let $k\to\infty$ to obtain

$$Z(t) \;=\; \sum_{i=1}^{\infty}(-1)^{i+1}a^{-i}\int_{t+i\tau}^\infty H(u)du$$

$$\begin{aligned} &= \sum_{i=1}^{\infty}\sum_{j=1}^{i}(-1)^{j+1}a^{-j}\int_{t+i\tau}^{t+(i+1)\tau} H(u)du \\ &= \sum_{i=1}^{\infty}\int_{t+i\tau}^{t+(i+1)\tau} \frac{1-(-a)^{-i}}{1+a}H(u)du \\ &= \sum_{i=1}^{\infty}\int_{t+i\tau}^{t+(i+1)\tau} \frac{1}{1+a}\{1-(-a)^{-[(u-t)/\tau]}\}H(u)du \\ &= \frac{1}{1+a}\int_{t+\tau}^{\infty}\{1-(-a)^{-[(u-t)/\tau]}\}H(u)du. \end{aligned}$$

That means

$$\begin{aligned} Z(t) &= \frac{1}{(1+p_0^{-1}+\varepsilon)(n-2)!}\int_{t+\tau}^{\infty}\{1-(-p_0^{-1}-\varepsilon)^{-[(u-t)/\tau]}\} \\ &\quad \times\int_u^{\infty}(s-u)^{n-2}G(s)Z(s-\sigma)dsdu, \end{aligned}$$

where $[\cdot]$ denotes the greatest integer function. This together with (35) and (32) yields

$$\begin{aligned} Z(t) &\geq \frac{q-\varepsilon}{\alpha(1+p_0^{-1}+\varepsilon)(n-2)!}\int_{t+\tau}^{\infty}\{1-(-p_0^{-1}-\varepsilon)^{-[(u-t)/\tau]}\} \\ &\quad \times\int_u^{\infty}(s-u)^{n-2}Z(s-\sigma)dsdu, \quad t\geq t_2. \end{aligned} \tag{37}$$

From (31) we know that $\tau<\sigma$. Now, let X be the set of all continuous and bounded functions on $[t_2+\tau-\sigma,\infty)$ with the sup-norm. Then X is a Banach space. Set

$$A=\{w\in X: 0\leq w(t)\leq 1, \quad \text{for} \quad t\geq t_2+\tau-\sigma\}.$$

Clearly, A is bounded, closed and convex subset of X. Define a mapping $S: A\to X$ as follow:

$$(Sw)(t)=\begin{cases} \dfrac{q-\varepsilon}{(1+p_0^{-1}+\varepsilon)(n-2)!Z(t)}\displaystyle\int_{t+\tau}^{\infty}\{1-(-p_0^{-1}-\varepsilon)^{-[(u-t)/\tau]}\} \\ \times\displaystyle\int_u^{\infty}(s-u)^{n-2}Z(s-\sigma)w(s-\sigma)dsdu, \quad t\geq t_2, \\ (Sw)(t_2)+e^{r(t_2-t)}-1, \quad t_2+\tau-\sigma\leq t\leq t_2, \end{cases}$$

where $r=(\ln(2-\alpha))/(\sigma-\tau)>0$.
Since for any $w\in A$ and $t\geq t_2$ we have by (37)

$$\begin{aligned} 0\leq (Sw)(t) &\leq \frac{q-\varepsilon}{(1+p_0^{-1}+\varepsilon)(n-2)!Z(t)}\int_{t+\tau}^{\infty}\{1-(-p_0^{-1}-\varepsilon)^{-[(u-t)/\tau]}\} \\ &\quad \times\int_u^{\infty}(s-u)^{n-2}Z(s-\sigma)dsdu\leq\alpha\leq 1, \end{aligned}$$

it follows that $0 \le (Sw)(t) \le 1$ for all $t \ge t_2+\tau-\sigma$ and so S maps A into itself. Next we claim that S is a contradiction on A. In fact, for any $w_1, w_2 \in A$ and $t \ge t_2$ we have

$$\begin{aligned} |(Sw_1)(t) \;&-\; (Sw_2)(t)| \\ &\le \frac{q-\varepsilon}{(1+p_0^{-1}+\varepsilon)(n-2)!Z(t)} \int_{t+\tau}^{\infty} \{1-(-p_0^{-1}-\varepsilon)^{-[(u-t)/\tau]}\} \\ &\quad \times \int_u^{\infty} (s-u)^{n-2} Z(s-\sigma)|w_1(s-\sigma)-w_2(s-\sigma)|dsdu \\ &\le \alpha||w_1-w_2||, \end{aligned}$$

and for $t_2+\tau-\sigma \le t \le t_2$ we have

$$|(Sw_1)(t)-(Sw_2)(t)| = |(Sw_1)(t_2)-(Sw_2)(t_2)| \le \alpha||w_1-w_2||.$$

Hence

$$||Sw_1 - Sw_2|| = \sup_{t \ge t_2+\tau-\sigma} |(Sw_1)(t)-(Sw_2)(t)| \le \alpha||w_1-w_2||.$$

Since $0 < \alpha < 1$, it follows that S is a contradiction on A. Therefore, by the Banach Contradiction Principle S has a fixed point $w \in A$, i.e.

$$\begin{aligned} w(t) &= \frac{q-\varepsilon}{(1+p_0^{-1}+\varepsilon)(n-2)!Z(t)} \int_{t+\tau}^{\infty} \{1-(-p_0^{-1}-\varepsilon)^{-[(u-t)/\tau]}\} \\ &\quad \times \int_u^{\infty} (s-u)^{n-2} Z(s-\sigma) w(s-\sigma) dsdu, \quad t \ge t_2, \end{aligned} \tag{38}$$

and for $t_2+\tau-\sigma \le t < t_2$ we have

$$w(t) = w(t_2) + e^{r(t_2-t)} - 1 > 0.$$

Now, we set

$$y(t) = Z(t)w(t).$$

Then $y(t)$ is a positive continuous function on $[t_2+\tau-\sigma, \infty)$ and satisfies for $t \ge t_2$

$$\begin{aligned} y(t) &= \frac{q-\varepsilon}{(1+p_0^{-1}+\varepsilon)(n-2)!} \int_{t+\tau}^{\infty} \{1-(-p_0^{-1}-\varepsilon)^{-[(u-t)/\tau]}\} \\ &\quad \times \int_u^{\infty} (s-u)^{n-2} y(s-\sigma) dsdu. \end{aligned}$$

This implies that for $t \ge t_2+\tau$

$$y(t) + (p_0^{-1}+\varepsilon)y(t-\tau) = \frac{q-\varepsilon}{(n-2)!} \int_t^{\infty} \int_u^{\infty} (s-u)^{n-2} y(s-\sigma) dsdu.$$

Differentiating it n times, we get

$$\frac{d^n}{dt^n}(y(t) + (p_0^{-1} + \varepsilon)y(t - \tau)) = (q - \varepsilon)y(t - \sigma), \quad t \geq t_2 + \tau,$$

which contradicts (31) and so the proof is complete.

REFERENCES

1. G. Ladas and C. Qian, Linearized oscillations for odd-order neutral differential equations. *J. Differential Equations*, **88**(1990), 238-247.

2. Q. Chuanxi and G. Ladas, Linearized oscillations for equations with positive and negative coefficients. *Hiroshima Math. J.*, **20** (1990), 331-340.

3. Q. Chuanxi and G. Ladas, Linearized oscillations for even-order neutral differential equations. *J. Math. Anal. Appl.*, **159** (1991), 237-250.

4. I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications. Clarendon Press , Oxford, 1991.

5. G. Ladas, Linearized Oscillation for Neutral Equations, Lecture Notes in Pure and Applied Mathematics, Differential Equations. Marcel Dekker, Inc., New York, 1990, pp. 379-387.

6. J.S. Yu and M. P. Chen, Linearized oscillation for first order neutral delay differential equations, *Panamer. Math. J.*, **2** (1993), 90-105.

7. J.H. Shen, J.S. Yu and X.Z. Qian, A linearized oscillation result for odd-order neutral differential equations, *J. Math. Anal. Appl.*, **185** (1994), 365-374.

8. Q. Chuanxi and G. Ladas, Oscillations of neutral differential equations with variable coefficients, *Appl. Anal.*, **32** (1989), 215-228.

9. J.H. Shen, J.S. Yu, Linearized oscillation results for even-order neutral differential equations, *Czechoslavak. Math. J.*, **51** (126) (2001),1-13.

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