Applied Mathematical Sciences, Vol. 1, 2007, no. 10, 471 - 490

Delay-Dependent Robust Stability and Stabilization of Neutral Systems

Magdi S. Mahmoud

Graduate Studies Program, Cairo University
43 Sheikh AlGhazzaly Street
Postal Code 12311-Dokki, Giza, Egypt
magdim@yahoo.com

Abdulla Ismail

Senior Education Manager
Dubai Silicon Oasis Authority
P.O. Box 491, Dubai, UAE
aalzarouni@dso.ae

Abstract

Complete results for the delay-dependent robust stability and feedback stabilization of linear neutral systems with unknown-but-bounded uncertainties are developed. New linear matrix inequalities-based delay-dependent stability criteria are derived in a systematic way using a new expanded state-space representation and a new Lyapunov-Krasovskii functional. The results are established without relying on overbounding. Solution to delay-dependent state-feedback stabilization and $\mathcal{H}_\infty$ synthesis are then obtained. Numerical examples are presented to illustrate the theory.

Keywords: Neutral systems, Delay-dependent stability, Delay-dependent stabilization, $\mathcal{H}_\infty$ synthesis, LMIs

1 Introduction

Stability and stabilization of time-delay systems have been topics of recurring interest over the past decades since delays are often the main causes of instability and poor performance of dynamic systems and encountered in various engineering and physical systems. Recently, the problems of robust stability analysis and robust stabilization of uncertain time-delay systems has been studied, see [11, 10]. It turns out that the choice of an appropriate Lyapunov-Krasovskii functional (LKF) is crucial for developing sufficient stability conditions. General LKF forms might lead to a complicated system of inequalities [9, 6, 7] and therefore approaches to construct new and effective LKF forms are needed. In this regard, stability criteria for linear neutral systems can be broadly classified into two categories: delay-independent, which are applicable to delays of arbitrary size [8] and delay-dependent, which include information on the size of the delay, see [3, 5, 13] and their references. Several model transformation methods and parameterization schemes have been used to derive delay-dependent stability conditions [12].

In this paper, we focus on the robust problems of delay-dependent stability and delay-dependent stabilization using state-feedback and $\mathcal{H}_\infty$ stabilization for linear neutral systems. A new expanded state-space representation is established which transforms the time-delay system into an equivalent system in which all the original system matrices are grouped into the new system matrix and the original delay system becomes easier to handle. The benefit gained is that we do not require overbounding of the quantities involved. The equivalence with the original system is preserved and hence the conservatism of the results will be reduced. Together with the introduction of a new LKF, these advantages simplify the derivation of new delay-dependent stability criteria and feedback stabilization results. All the results are formulated as linear matrix inequalities. A numerical example is worked out to illustrate the theoretical developments.

Notations: In the sequel, the Euclidean norm is used for vectors. We use W^t, W^{-1}, $\lambda(W)$ and $||W||$ to denote, respectively, the transpose, the inverse, the eigenvalues and the induced norm of any square matrix W and $W > 0$ ($W < 0$) stands for a symmetrical and positive- (negative-) definite matrix W. The n-dimensional Euclidean space is denoted by $\mathbb{R}^{n\times n}$ and I stands for unit matrix with appropriate dimension. The symbol $\bullet$ will be used in some matrix expressions to induce a symmetric structure, that is if given matrices $L = L^t$ and $R = R^t$ of appropriate dimensions, then

$$\begin{bmatrix} L & \bullet \\ N & R \end{bmatrix} = \begin{bmatrix} L & N^t \\ N & R \end{bmatrix}$$

Sometimes, the arguments of a function will be omitted when no confusion can arise.

Fact 1: Given a scalar $\epsilon > 0$ and matrices Σ_1, Σ_2 and Φ such that $\Phi^t\,\Phi \le I$, then

$$\Sigma_1\Phi\Sigma_2 + \Sigma_2^t\Phi^t\Sigma_1^t \le \epsilon^{-1}\,\Sigma_1\Sigma_1^t + \epsilon\,\Sigma_2^t\Sigma_2$$

2 Problem Statement and Definitions

We consider the following class of linear neutral systems with parametric uncertainties:

$$\begin{aligned}
\dot{x}(t) &= A_{\Delta o}x(t) + A_{\Delta d}x(t-\tau) + D_{\Delta d}\dot{x}(t-\psi) + B_{\Delta o}u(t) + \Gamma w(t), \quad x_0 \;=\; \theta_0 \\
z(t) &= C_o x(t) \;+\; F_o u(t) \qquad (2.1)
\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector , $u(t) \in \mathbb{R}^p$ is the control input , $w(t) \in \mathbb{R}^q$ is the disturbance input , $z(t) \in \mathbb{R}^q$ is the observed output and $0 \leq \tau \leq \tau^*$, $0 \leq \psi \leq \psi^*$ are an unknown constant time delay factors and $\check{\tau}, \hat{\tau}$ are known bounds. The matrices $A_{\Delta o} \in \mathbb{R}^{n\times n}$, $A_{\Delta d} \in \mathbb{R}^{n\times n}$ and $B_{\Delta o} \in \mathbb{R}^{n\times p}$ are represented by

$$[A_{\Delta o} \quad A_{\Delta d} \quad D_{\Delta d} \quad B_{\Delta o}] \;=\; [A_o \quad A_d \quad D_d \quad B_o] + M\Delta_t[N_a \quad N_d \quad N_x \quad N_b] \qquad (2.2)$$

where $A_o \in \mathbb{R}^{n\times n}$, $B_o \in \mathbb{R}^{n\times p}$, $C_o \in \mathbb{R}^{q\times n}$, $F_o \in \mathbb{R}^{q\times p}$, $D_d \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times n_m}$, $\Gamma \in \mathbb{R}^{n\times r}$, $N_a \in \mathbb{R}^{n_n\times n}$, $N_x \in \mathbb{R}^{n_n\times n}$, $N_d \in \mathbb{R}^{n_n\times n}$ and $N_b \in \mathbb{R}^{n_n\times p}$, are real and known constant matrices with Δ_t is a bounded matrix of uncertainties satisfying $\Delta_t^t\Delta_t \;<\; I$. The uncertainties that satisfy (2.2) are referred to as admissible uncertainties.

Observe that model (2.1) can represent a time-domain formulation of a partial element equivalent circuit (PEEC) [1]. The objective of this paper is to develop delay-dependent methodologies for robust stability and stabilization for the class of uncertain neutral-delay systems of the type (2.1). This will be accomplished in Section 3 (delay-dependent stability) and Section 4 (delay-dependent stabilization) through the establishment of a new expanded state-space representation in which converts the neutral-delay system into an equivalent system in which the system matrix contains all the matrices of the original and the delay state has simple, certain and fixed matrix even if the original delay matrix is uncertain.

3 Delay-Dependent Stability

In the sequel, we write system (2.1):

$$\begin{aligned}
\dot{x}(t) &= y(t) \qquad (3.1) \\
0 &= -y(t) + A_{\Delta o}x(t) + A_{\Delta d}x(t-\tau) + B_o u(t) + \Gamma w(t) \\
0 &= -y(t) + (A_{\Delta o} + A_{\Delta d})x(t) - A_{\Delta d}\left[\int_{t-\tau}^{t} y(s)\, ds\right] + D_{\Delta d}y(t-\psi) \\
&+ \; B_o u(t) + \Gamma w(t) \qquad (3.2)
\end{aligned}$$

Define

$$\sigma(t) = \int_{t-\tau}^{t} y(s)\, ds$$

then it follows that

$$\dot{\sigma}(t) = y(t) - y(t-\tau)$$

and introducing the new composite vector

$$\xi(t) = [x^t(t) \quad \sigma^t(t) \quad y^t(t)]^t$$

we readily obtain the new expanded state-space system

$$\begin{aligned}
(\Sigma_2): \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\xi}(t) &= \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & I \\ A_{\Delta od} & -A_{\Delta d} & -I \end{bmatrix} \xi(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & 0 & 0 \end{bmatrix} \xi(t-\tau) \\
&+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_{\Delta d} \end{bmatrix} \xi(t-\psi) + \begin{bmatrix} 0 \\ 0 \\ B_o \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \Gamma \end{bmatrix} w(t) \\
U\, \dot{\xi}(t) &= \bar{A}_{\Delta\xi}\, \xi(t) + \bar{A}_{\xi d}\, \xi(t-\tau) + \bar{D}_{\Delta d}\, \xi(t-\psi) \\
&+ \bar{B}_o u(t) + \bar{\Gamma} w(t), \\
z(t) &= [C_o \ \ 0 \ \ 0]\xi(t) + F_o u(t) = \bar{C}_o \xi(t) + F_o u(t) \qquad (3.3)
\end{aligned}$$

with $A_{\Delta od} = A_{\Delta o} + A_{\Delta d}$ and where the initial conditions are characterized by

$$\xi_0 = \left[\begin{array}{ccc} x_0^t & \sigma_0^t & y_0^t \end{array} \right]^t \qquad (3.4)$$

Remark 3.1 *In short, if $x(t)$ is a solution of uncertain delay system (2.1) with $\Delta_t \equiv 0$ and $u(t) \equiv 0$, then $\xi(t)$ is a solution of the new expanded state-space system (3.3) subject to (3.4) and the reverse is true. This is the essence of descriptor transformation. It is significant to observe that in system (3.3) all the matrices of the original neutral system are grouped into the new system matrices and henceforth we call it the "Compact Form (CF)".*

We rewrite the CF matrices

$$\bar{A}_{\Delta\xi} = \bar{A}_{\xi o} + \bar{M}\, \Delta_t\, \bar{N}, \quad \bar{D}_{\Delta d} = \bar{D}_o + \bar{M}\, \Delta_t\, \bar{N}_x, \qquad (3.5)$$

where

$$\begin{aligned}
\bar{A}_{\xi o} &= \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & I \\ A_{od} & -A_d & -I \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix}, \quad \bar{D}_o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_d \end{bmatrix}, \\
\bar{N} &= [N_{ad} \ \ -N_d \ \ 0], \ \bar{N}_x = [0 \ \ 0 \ \ N_x], \ N_{ad} = N_a + N_d, \ A_{od} = A_o + A_d. \qquad (3.6)
\end{aligned}$$

Now to derive tractable conditions for stability, we let $u(.) \equiv 0,\ w(.) \equiv 0$ and introduce the following Lyapunov-Krasovskii functional (LKF):

$$V(\xi) \;=\; V_a(\xi) + V_b(\xi) + V_c(\xi) \tag{3.7}$$

with

$$\begin{aligned}
V_a(\xi) &= \xi^t(t)U^t\mathcal{P}\xi(t), \quad \mathcal{P} \in \mathbb{R}^{3n\times 3n},\ U^t\mathcal{P} = \mathcal{P}^tU \geq 0 \\
V_b(\xi) &= \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{\xi}^t(r)U^t\hat{\mathcal{F}}U\dot{\xi}(r)\,dr\,d\beta + \int_{-\psi}^{0}\int_{t+\beta}^{t} \dot{\xi}^t(r)U^t\hat{\mathcal{G}}U\dot{\xi}(r)\,dr\,d\beta, \\
& \quad 0 < \hat{\mathcal{F}}^t = \hat{\mathcal{F}} \in \mathbb{R}^{3n\times 3n},\ 0 < \hat{\mathcal{G}}^t = \hat{\mathcal{F}} \in \mathbb{R}^{3n\times 3n}, \\
V_c(\xi) &= \int_{0}^{t}\int_{\beta-\tau}^{\beta} \eta^t(s)\hat{\mathcal{S}}\eta(s)\,ds\,d\beta + \int_{0}^{t}\int_{\beta-\psi}^{\beta} \zeta^t(s)\hat{\mathcal{Z}}\zeta(s)\,ds\,d\beta, \\
& \quad 0 < \hat{\mathcal{S}}^t = \hat{\mathcal{S}} \in \mathbb{R}^{9n\times 9n},\ 0 < \hat{\mathcal{Z}}^t = \hat{\mathcal{Z}} \in \mathbb{R}^{9n\times 9n} \\
\eta &= [\xi^t(\beta) \quad \xi^t(\beta-\tau) \quad \dot{\xi}^t(s)]^t,\ \zeta = [\xi^t(\beta) \quad \xi^t(\beta-\psi) \quad \dot{\xi}^t(s)]^t
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\mathcal{P} &= \begin{bmatrix} \mathcal{P}_x & \mathcal{P}_f & 0 \\ \bullet & \mathcal{P}_d & 0 \\ \mathcal{P}_g & \mathcal{P}_h & \mathcal{P}_s \end{bmatrix},\ \hat{\mathcal{S}} = \begin{bmatrix} \mathcal{S}_x & \mathcal{S}_f & \mathcal{S}_g \\ \bullet & \mathcal{S}_d & \mathcal{S}_h \\ \bullet & \bullet & U^t\hat{\mathcal{F}}U \end{bmatrix},\ \hat{\mathcal{Z}} = \begin{bmatrix} \mathcal{Z}_x & \mathcal{Z}_f & \mathcal{Z}_g \\ \bullet & \mathcal{Z}_d & \mathcal{Z}_h \\ \bullet & \bullet & U^t\hat{\mathcal{G}}U \end{bmatrix} \\
0 < \mathcal{P}_x &= \mathcal{P}_x^t \in \mathbb{R}^{n\times n},\ 0 < \mathcal{P}_d = \mathcal{P}_d^t \in \mathbb{R}^{n\times n},\ \mathcal{P}_h \in \mathbb{R}^{n\times n},\ 0 < \mathcal{P}_s = \mathcal{P}_s^t \in \mathbb{R}^{n\times n}, \\
& \quad \mathcal{P}_g \in \mathbb{R}^{n\times n},\ \mathcal{P}_f \in \mathbb{R}^{n\times n}, \\
0 < \mathcal{Z}_x &= \mathcal{Z}_x^t \in \mathbb{R}^{3n\times 3n},\ 0 < \mathcal{Z}_d = \mathcal{Z}_d^t \in \mathbb{R}^{3n\times 3n},\ \mathcal{Z}_h \in \mathbb{R}^{3n\times 3n},\ \mathcal{Z}_g \in \mathbb{R}^{3n\times 3n},\ \mathcal{Z}_f \in \mathbb{R}^{3n\times 3n}, \\
0 < \mathcal{S}_x &= \mathcal{S}_x^t \in \mathbb{R}^{3n\times 3n},\ 0 < \mathcal{S}_d = \mathcal{S}_d^t \in \mathbb{R}^{3n\times 3n},\ \mathcal{S}_h \in \mathbb{R}^{3n\times 3n},\ \mathcal{S}_g \in \mathbb{R}^{3n\times 3n},\ \mathcal{S}_f \in \mathbb{R}^{3n\times 3n}, \\
& \quad \hat{\mathcal{G}} = diag[\mathcal{G}_x \ \ \mathcal{G}_d \ \ \mathcal{G}_s],\ \ \hat{\mathcal{G}}^{-1} = \hat{\mathcal{Q}} = diag[\mathcal{Q}_x \ \ \mathcal{Q}_d \ \ \mathcal{Q}_s], \\
& \quad \hat{\mathcal{F}} = diag[\mathcal{F}_x \ \ \mathcal{F}_d \ \ \mathcal{F}_s],\ \ \hat{\mathcal{F}}^{-1} = \hat{\mathcal{W}} = diag[\mathcal{W}_x \ \ \mathcal{W}_d \ \ \mathcal{W}_s]
\end{aligned} \tag{3.9}$$

Define

$$\begin{aligned}
\mathcal{X} = \mathcal{P}^{-1} &= \begin{bmatrix} \mathcal{X}_x & \mathcal{X}_f & 0 \\ \bullet & \mathcal{X}_d & 0 \\ \mathcal{X}_g & \mathcal{X}_h & \mathcal{X}_s \end{bmatrix} \in \mathbb{R}^{3n\times 3n},\quad 0 < \mathcal{X}_x = \mathcal{X}_x^t \in \mathbb{R}^{n\times n},\ 0 < \mathcal{X}_s = \mathcal{X}_s^t \in \mathbb{R}^{n\times n}, \\
0 < \mathcal{X}_d &= \mathcal{X}_d^t \in \mathbb{R}^{n\times n},\ \ \mathcal{X}_g \in \mathbb{R}^{n\times n},\ \ \mathcal{X}_h \in \mathbb{R}^{n\times n},\ \ 0 < \mathcal{X}_f^t = \mathcal{X}_f \in \mathbb{R}^{n\times n}
\end{aligned} \tag{3.10}$$

and introduce the linearizations

$$\begin{aligned}
\mathcal{X}^t\mathcal{S}_x\mathcal{X} &= \mathcal{M} = \begin{bmatrix} \mathcal{M}_x & \mathcal{M}_f & \mathcal{M}_q \\ \bullet & \mathcal{M}_d & \mathcal{M}_e \\ \bullet & \bullet & \mathcal{M}_s \end{bmatrix},\ \mathcal{X}^t\mathcal{Z}_x\mathcal{X} = \mathcal{N} = \begin{bmatrix} \mathcal{N}_x & \mathcal{N}_f & \mathcal{N}_q \\ \bullet & \mathcal{N}_d & \mathcal{N}_e \\ \bullet & \bullet & \mathcal{N}_s \end{bmatrix} \\
\mathcal{X}^t[\mathcal{S}_g + \mathcal{S}_g^t + \mathcal{Z}_g + \mathcal{Z}_g^t]\mathcal{X} &= \mathcal{L} = \begin{bmatrix} \mathcal{L}_x & \mathcal{L}_f & \mathcal{L}_q \\ \bullet & \mathcal{L}_d & \mathcal{L}_e \\ \bullet & \bullet & \mathcal{L}_s \end{bmatrix} \\
\mathcal{X}^t\mathcal{S}_f &= \bar{\mathcal{S}}_f,\ \mathcal{X}^t\mathcal{Z}_f = \bar{\mathcal{Z}}_f,\ \mathcal{X}^t[\mathcal{S}_g + \mathcal{S}_g^t] = \bar{\mathcal{S}}_g,\ \mathcal{X}^t[\mathcal{Z}_g + \mathcal{Z}_g^t] = \bar{\mathcal{Z}}_g
\end{aligned} \tag{3.11}$$

where the respective dimensions are

$$\mathcal{M}_x^t = \mathcal{M}_x \in \mathbb{R}^{n\times n},\ \mathcal{M}_f \in \mathbb{R}^{n\times n},\ \mathcal{M}_q \in \mathbb{R}^{n\times n},\ \mathcal{M}_d^t = \mathcal{M}_d \in \mathbb{R}^{n\times n},\ \mathcal{M}_s^t = \mathcal{M}_s \in \mathbb{R}^{n\times n},$$
$$\mathcal{M}_e \in \mathbb{R}^{n\times n},\ \mathcal{N}_x^t = \mathcal{N}_x \in \mathbb{R}^{n\times n},\ \mathcal{N}_f \in \mathbb{R}^{n\times n},\ \mathcal{N}_q \in \mathbb{R}^{n\times n},\ \mathcal{N}_d^t = \mathcal{N}_d \in \mathbb{R}^{n\times n},\ \mathcal{N}_e \in \mathbb{R}^{n\times n},$$
$$\mathcal{N}_s^t = \mathcal{N}_s \in \mathbb{R}^{n\times n},\ \mathcal{L}_x^t = \mathcal{L}_x \in \mathbb{R}^{n\times n},\ \mathcal{L}_f \in \mathbb{R}^{n\times n},\ \mathcal{L}_q \in \mathbb{R}^{n\times n},\ \mathcal{L}_d^t = \mathcal{L}_d \in \mathbb{R}^{n\times n},$$
$$\mathcal{L}_s^t = \mathcal{L}_s \in \mathbb{R}^{n\times n},\ \mathcal{L}_e \in \mathbb{R}^{n\times n}, \tag{3.12}$$

For convenience, define the following vectors:

$$\begin{aligned}
\kappa_1 &= \begin{bmatrix} \bar{M}^t\mathcal{P} & 0 & 0 & 0 & 0 \end{bmatrix}^t,\ \kappa_2 = \begin{bmatrix} \bar{N} & 0 & 0 & 0 & 0 \end{bmatrix}^t, \\
\kappa_3 &= \begin{bmatrix} 0 & 0 & \bar{N}_x & 0 & 0 \end{bmatrix}^t,\ \kappa_4 = \begin{bmatrix} 0 & 0 & \bar{M}^t & 0 & 0 \end{bmatrix}^t, \\
\kappa_5 &= \begin{bmatrix} \tau^*\bar{N} & 0 & \tau^*\bar{N}t_x & 0 & 0 \end{bmatrix}^t,\ \kappa_6 = \begin{bmatrix} \psi^*\bar{N} & 0 & \psi^*\bar{N}t_x & 0 & 0 \end{bmatrix}^t
\end{aligned} \tag{3.13}$$

The following theorem establishes LMI-based sufficient conditions for delay-dependent robust stability of system (Σ_2).

Theorem 3.1 *System (Σ_2) with $u(.) \equiv 0,\ w(.) \equiv 0$ is delay-dependent robustly stable if there exist matrices $0 < \mathcal{X}_x = \mathcal{X}_x^t \in I\!R^{n\times n},\ 0 < \mathcal{X}_d = \mathcal{X}_d^t \in I\!R^{n\times n},\ 0 < \mathcal{X}_s = \mathcal{X}_s^t \in I\!R^{n\times n},\ 0 < \mathcal{X}_f^t = \mathcal{X}_f \in I\!R^{n\times n},\ \mathcal{X}_h \in I\!R^{n\times n},\ \mathcal{X}_g \in I\!R^{n\times n},\ \mathcal{M}_d = \mathcal{M}_d^t \in I\!R^{n\times n},\ \mathcal{M}_e \in I\!R^{n\times n},\ \mathcal{M}_q \in I\!R^{n\times n},\ \mathcal{M}_f \in I\!R^{n\times n},\ \mathcal{M}_x = \mathcal{M}_x^t \in I\!R^{n\times n},\ \mathcal{M}_s = \mathcal{M}_s^t \in I\!R^{n\times n},\ \mathcal{N}_d^t = \mathcal{N}_d \in I\!R^{n\times n},\ \mathcal{N}_e^t = \mathcal{N}_e \in I\!R^{n\times n},\ \mathcal{N}_q^t = \mathcal{N}_q \in I\!R^{n\times n},\ \mathcal{N}_f^t = \mathcal{N}_f \in I\!R^{n\times n},\ \mathcal{N}_x = \mathcal{N}_x \in I\!R^{n\times n},\ \mathcal{N}_s^t = \mathcal{N}_s \in I\!R^{n\times n},\ \mathcal{L}_x^t = \mathcal{L}_x \in I\!R^{n\times n},\ \mathcal{L}_f \in I\!R^{n\times n},\ \mathcal{L}_q \in I\!R^{n\times n},\ \mathcal{L}_d^t = \mathcal{L}_d \in I\!R^{n\times n},\ \mathcal{L}_s^t = \mathcal{L}_s \in I\!R^{n\times n},\ \mathcal{L}_e \in I\!R^{n\times n},\ \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n},\ \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n},\ \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$ and scalars $\delta > 0,\ \mu > 0,\ \varepsilon > 0,\ \varrho > 0,\ \omega > 0$ such that the following inequalities hold for all admissible uncertainties*

$$\begin{bmatrix}
\Omega_a & \begin{matrix}\Omega_g + \\ \tau^*\bar{\mathcal{S}}_f + \psi^*\bar{\mathcal{Z}}_f + \bar{\mathcal{S}}_g\end{matrix} & \bar{D}_o - \bar{\mathcal{Z}}_g & \tau^*\Omega_d & \psi^*\Omega_d & \Omega_n & 0 & \tau^*\Omega_n & \psi^*\Omega_n \\
\bullet & \Upsilon_d & 0 & \tau^*\bar{A}_{\xi d}^t & \psi^*\bar{A}_{\xi d}^t & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bar{\Upsilon}_f & \tau^*\bar{D}_o^t & \psi^*\bar{D}_o^t & 0 & \bar{N}_x^t & \tau^*\bar{N}_x^t & \psi^*\bar{N}_x^t \\
\bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -\delta I & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mu I & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon I & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varrho I
\end{bmatrix} < 0$$

$$\mathcal{X}^t U^t = U\mathcal{X} \geq 0 \tag{3.14}$$

where

$$\Omega_a = \begin{bmatrix}
\begin{matrix}-\omega I + \mathcal{X}_g + \mathcal{X}_g^t + \\ \tau^*\mathcal{M}_x + \psi^*\mathcal{N}_x + \mathcal{L}_x\end{matrix} & \begin{matrix}\mathcal{X}_h + \mathcal{X}_g^t + \\ \tau^*\mathcal{M}_f + \psi^*\mathcal{N}_f + \mathcal{L}_f\end{matrix} & \begin{matrix}\mathcal{X}_s + \mathcal{X}_s\bar{A}_{od}^t \\ -\mathcal{X}_f\bar{A}_d^t - \mathcal{X}_g^t \\ +\tau^*\mathcal{M}_q + \psi^*\mathcal{N}_q + \mathcal{L}_q\end{matrix} \\
\bullet & \begin{matrix}-\omega I + \mathcal{X}_h + \mathcal{X}_h^t + \\ \tau^*\mathcal{M}_d + \psi^*\mathcal{N}_d + \mathcal{L}_d\end{matrix} & \begin{matrix}\mathcal{X}_s + \mathcal{X}_f^t\bar{A}_{od}^t \\ -\mathcal{X}_d\bar{A}_d^t - \mathcal{X}_h^t \\ \tau^*\mathcal{M}_e + \psi^*\mathcal{N}_e + \mathcal{L}_e\end{matrix} \\
\bullet & \bullet & \begin{matrix}-\mathcal{X}_s - \mathcal{X}_s \\ -\omega I + \delta MM^t + \mu MM^t \\ \tau^*\mathcal{M}_s + \psi^*\mathcal{N}_s + \mathcal{L}_s\end{matrix}
\end{bmatrix},$$

$$\Omega_n = \begin{bmatrix} \mathcal{X}_x N_{ad}^t - \mathcal{X}_f^t N_d^t \\ \mathcal{X}_f N_{ad}^t - \mathcal{X}_d N_d^t \\ 0 \end{bmatrix},\ \Omega_g = \begin{bmatrix} 0 & 0 & 0 \\ -\mathcal{X}_g & -\mathcal{X}_h & -\mathcal{X}_s \\ 0 & 0 & 0 \end{bmatrix},\ \Omega_d = \begin{bmatrix} \mathcal{X}_g^t & \mathcal{X}_g^t & \mathcal{X}_g^t \\ \mathcal{X}_h^t & \mathcal{X}_h^t & \mathcal{X}_h^t \\ \mathcal{X}_s & \mathcal{X}_s & \mathcal{X}_s \end{bmatrix} \tag{3.15}$$

Proof : We consider $V(\xi)$ and evaluate the derivative of the functionals $V_a,\ V_b$ and V_c. For $V_a,$ we have

$$\begin{aligned}
\dot{V}_a(\xi) &= \dot{\xi}^t(t)U^t\mathcal{P}\xi(t)+\xi^t(t)\mathcal{P}^tU\dot{\xi}^t(t)\\
&= \xi^t(t)\Big[\mathcal{P}^t\bar{A}_{\Delta\xi}+\bar{A}^t_{\Delta\xi}\mathcal{P}\Big]\xi(t)+\xi^t\mathcal{P}^t\bar{A}_{\xi d}\xi(t-\tau)+\xi^t(t-\tau)\bar{A}^t_{\xi d}\mathcal{P}\xi(t)\\
&+ \xi^t\mathcal{P}^t\bar{D}_{\Delta d}\xi(t-\psi)+\xi^t(t-\psi)\bar{D}^t_{\Delta d}\mathcal{P}\xi(t) \qquad (3.16)
\end{aligned}$$

For V_b, we get

$$\begin{aligned}
\dot{V}_b(\xi) &= \tau\,\dot{\xi}^t(t)U^t\hat{\mathcal{F}}U\dot{\xi}(t)\ +\ \psi\,\dot{\xi}^t(t)U^t\hat{\mathcal{G}}U\dot{\xi}(t)\ -\ \int_{t-\tau}^t\dot{\xi}^t(r)U^t\hat{\mathcal{F}}U\dot{\xi}(r)\,dr\\
&- \int_{t-\psi}^t\dot{\xi}^t(r)U^t\hat{\mathcal{G}}U\dot{\xi}(r)\,dr\\
&\le \tau^*\,\dot{\xi}^t(t)U^t\hat{\mathcal{F}}U\dot{\xi}(t)\ +\ \psi^*\,\dot{\xi}^t(t)U^t\hat{\mathcal{G}}U\dot{\xi}(t)\ -\ \int_{t-\tau}^t\dot{\xi}^t(r)U^t\hat{\mathcal{F}}U\dot{\xi}(r)\,dr\\
&- \int_{t-\psi}^t\dot{\xi}^t(r)U^t\hat{\mathcal{G}}U\dot{\xi}(r)\,dr \qquad (3.17)
\end{aligned}$$

For $V_c,$ we obtain

$$\begin{aligned}
\dot{V}_c(\xi) &= \tau\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}^t\begin{bmatrix}\mathcal{S}_x & \mathcal{S}_f\\ \bullet & \mathcal{S}_d\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}+\psi\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}^t\begin{bmatrix}\mathcal{Z}_x & \mathcal{Z}_f\\ \bullet & \mathcal{Z}_d\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}\\
&+ 2\int_{t-\tau}^t\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}^t\begin{bmatrix}\mathcal{S}_g\\ \mathcal{S}_h\end{bmatrix}\dot{\xi}(s)\,ds\ +\ 2\int_{t-\psi}^t\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}^t\begin{bmatrix}\mathcal{Z}_g\\ \mathcal{Z}_h\end{bmatrix}\dot{\xi}(s)\,ds\\
&+ \int_{t-\tau}^t\dot{\xi}^t(r)U^t\hat{\mathcal{W}}U\dot{\xi}(r)\,dr\ +\ \int_{t-\psi}^t\dot{\xi}^t(r)U^t\hat{\mathcal{Q}}U\dot{\xi}(r)\,dr\\
&\le \tau^*\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}^t\begin{bmatrix}\mathcal{S}_x & \mathcal{S}_f\\ \bullet & \mathcal{S}_d\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}+\int_{t-\tau}^t\dot{\xi}^t(r)U^t\hat{\mathcal{W}}U\dot{\xi}(r)\,dr\\
&+ \psi^*\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}^t\begin{bmatrix}\mathcal{Z}_x & \mathcal{Z}_f\\ \bullet & \mathcal{Z}_d\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}+\int_{t-\psi}^t\dot{\xi}^t(r)U^t\hat{\mathcal{Q}}U\dot{\xi}(r)\,dr,\\
&+ 2\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}^t\begin{bmatrix}\mathcal{S}_g\\ \mathcal{S}_h\end{bmatrix}\bar{I}\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}\\
&+ 2\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}^t\begin{bmatrix}\mathcal{Z}_g\\ \mathcal{Z}_h\end{bmatrix}\bar{I}\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix},\ \bar{I}=[I\quad -I] \qquad (3.18)
\end{aligned}$$

It follows from (3.3) with $(u(.)\equiv 0, w(.)\equiv 0)$ and (3.16)-(3.18) that

$$\begin{aligned}
&\dot{V}(\xi)\le\\
&\xi^t(t)\Big[\mathcal{P}^t\bar{A}_{\Delta\xi}+\bar{A}^t_{\Delta\xi}\mathcal{P}\Big]\xi(t)+\xi^t\mathcal{P}^t\bar{A}_{\xi d}\xi(t-\tau)+\xi^t(t-\tau)\bar{A}^t_{\xi d}\mathcal{P}\xi(t)+\xi^t\mathcal{P}^t\bar{D}_{\Delta d}\xi(t-\psi)\\
+\ &\xi^t(t-\psi)\bar{D}^t_{\Delta d}\mathcal{P}\xi(t)+\tau^*\dot{\xi}^t(t)U^t\hat{\mathcal{F}}U\dot{\xi}(t)+\psi^*\dot{\xi}^t(t)U^t\hat{\mathcal{G}}U\dot{\xi}(t)\\
+\ &\tau^*\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}^t\begin{bmatrix}\mathcal{S}_x & \mathcal{S}_f\\ \bullet & \mathcal{S}_d\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}+2\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}^t\begin{bmatrix}\mathcal{S}_g\\ \mathcal{S}_h\end{bmatrix}\bar{I}\begin{bmatrix}\xi(t)\\ \xi(t-\tau)\end{bmatrix}\\
+\ &\psi^*\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}^t\begin{bmatrix}\mathcal{Z}_x & \mathcal{Z}_f\\ \bullet & \mathcal{Z}_d\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}+2\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}^t\begin{bmatrix}\mathcal{Z}_g\\ \mathcal{Z}_h\end{bmatrix}\bar{I}\begin{bmatrix}\xi(t)\\ \xi(t-\psi)\end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \\ \xi(t-\psi) \end{bmatrix}^t \Upsilon \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \\ \xi(t-\psi) \end{bmatrix} \tag{3.19}$$

where

$$\Upsilon = \begin{bmatrix} \begin{matrix}\Upsilon_a + \\ \tau^* \bar{\mathcal{A}}^t_{\Delta\xi}\hat{\mathcal{F}}\bar{\mathcal{A}}_{\Delta\xi} + \\ \psi^* \bar{\mathcal{A}}^t_{\Delta\xi}\hat{\mathcal{G}}\bar{\mathcal{A}}_{\Delta\xi}\end{matrix} & \begin{matrix}\Upsilon_b + \\ \tau^* \bar{\mathcal{A}}^t_{\Delta\xi}\hat{\mathcal{F}}\bar{A}_{\xi d} + \\ \psi^* \bar{\mathcal{A}}^t_{\Delta\xi}\hat{\mathcal{G}}\bar{A}_{\xi d}\end{matrix} & \begin{matrix}\Upsilon_c + \\ \tau^* \bar{\mathcal{A}}^t_{\Delta\xi}\hat{\mathcal{F}}\bar{D}_{\Delta d} + \\ \psi^* \bar{\mathcal{A}}^t_{\Delta\xi}\hat{\mathcal{G}}\bar{D}_{\Delta d}\end{matrix} \\ \bullet & \begin{matrix}\Upsilon_d + \\ \tau^* \bar{A}^t_{\xi d}\hat{\mathcal{F}}\bar{A}_{\xi d} + \\ \psi^* \bar{A}^t_{\xi d}\hat{\mathcal{G}}\bar{A}_{\xi d}\end{matrix} & \begin{matrix}\tau^* \bar{A}^t_{\xi d}\hat{\mathcal{F}}\bar{D}_{\Delta d} + \\ \psi^* \bar{A}^t_{\xi d}\hat{\mathcal{G}}\bar{D}_{\Delta d}\end{matrix} \\ \bullet & \bullet & \begin{matrix}\Upsilon_f + \\ \tau^* \bar{D}^t_{\Delta d}\hat{\mathcal{F}}\bar{D}_{\Delta d} + \\ \psi^* \bar{D}^t_{\Delta d}\hat{\mathcal{G}}\bar{D}_{\Delta d}\end{matrix} \end{bmatrix},$$

$$\begin{aligned} \Upsilon_a &= \mathcal{P}^t \bar{\mathcal{A}}_{\Delta\xi} + \bar{\mathcal{A}}^t_{\Delta\xi}\mathcal{P} + \tau^*\mathcal{S}_x + \psi^*\mathcal{Z}_x + \mathcal{S}_g + \mathcal{S}^t_g + \mathcal{Z}_g + \mathcal{Z}^t_g \\ \Upsilon_b &= \mathcal{P}^t \bar{A}_{\xi d} + \tau^*\mathcal{S}_f + \psi^*\mathcal{Z}_f - \mathcal{S}_g - \mathcal{S}^t_g, \;\; \Upsilon_c = \mathcal{P}^t\bar{D}_{\Delta d} - \mathcal{Z}_g - \mathcal{Z}^t_g \\ \Upsilon_d &= -\mathcal{S}_h - \mathcal{S}^t_h + \tau^*\mathcal{S}_d + \psi^*\mathcal{Z}_d, \;\; \Upsilon_f = -\mathcal{Z}_h - \mathcal{Z}^t_h \end{aligned} \tag{3.20}$$

By Laypunov-Krasovskii theory, the existence of $V(\xi) > 0$ such that $(\dot{V}(\xi) < 0, \forall\xi(t) \neq 0)$ guarantees asymptotic stability of system (Σ_2). This implies that $\Upsilon < 0$. On observing that

$$\begin{aligned} \Upsilon_a &= \mathcal{P}^t\bar{A}_{\xi o} + \bar{A}^t_{\xi o}\mathcal{P} + \tau^*\mathcal{S}_x + \psi^*\mathcal{Z}_x + \mathcal{S}_g + \mathcal{S}^t_g + \mathcal{Z}_g + \mathcal{Z}^t_g + \mathcal{P}^t\bar{M}\Delta_t\bar{N} + \bar{N}^t\Delta^t_t\bar{M}\mathcal{P} \\ &= \Upsilon_{ao} + \mathcal{P}^t\bar{M}\Delta_t\bar{N} + \bar{N}^t\Delta^t_t\bar{M}\mathcal{P} \\ \Upsilon_c &= \mathcal{P}^t\bar{D}_o + \mathcal{P}^t\bar{M}\Delta_t\bar{N}_x - \mathcal{Z}_g - \mathcal{Z}^t_g \\ &= \Upsilon_{co} + \mathcal{P}^t\bar{M}\Delta_t\bar{N}_x \end{aligned} \tag{3.21}$$

and by Schur complement and using **Fact 1** along with some matrix manipulations, this stability condition becomes

$$\begin{bmatrix} \Upsilon_a & \Upsilon_b & \Upsilon_c & \tau^*\bar{\mathcal{A}}^t_{\Delta\xi} & \psi^*\bar{\mathcal{A}}^t_{\Delta\xi} \\ \bullet & \Upsilon_d & 0 & \tau^*\bar{A}^t_{\xi d} & \psi^*\bar{A}^t_{\xi d} \\ \bullet & \bullet & \Upsilon_f & \tau^*\bar{D}^t_{\Delta d} & \psi^*\bar{D}^t_{\Delta d} \\ \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 \\ \bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} \end{bmatrix} = \begin{bmatrix} \Upsilon_{ao} & \Upsilon_b & \Upsilon_{co} & \tau^*\bar{A}^t_{\xi o} & \psi^*\bar{A}^t_{\xi o} \\ \bullet & \Upsilon_d & 0 & \tau^*\bar{A}^t_{\xi d} & \psi^*\bar{A}^t_{\xi d} \\ \bullet & \bullet & \Upsilon_f & \tau^*\bar{D}^t_o & \psi^*\bar{D}^t_o \\ \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 \\ \bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} \end{bmatrix}$$

$$+ \begin{bmatrix} \mathcal{P}^t\bar{M}\Delta_t\bar{N} + \bar{N}^t\Delta^t_t\bar{M}\mathcal{P} & 0 & \mathcal{P}^t\bar{M}\Delta_t\bar{N}_x & \tau^*\bar{N}^t\Delta^t_t\bar{M} & \psi^*\bar{N}^t\Delta^t_t\bar{M} \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & \tau^*\bar{N}^t_x\Delta^t_t\bar{M} & \psi^*\bar{N}^t_x\Delta^t_t\bar{M} \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 \end{bmatrix}$$

$$\leq \begin{bmatrix} \Upsilon_{ao} & \Upsilon_b & \Upsilon_{co} & \tau^* \bar{A}^t_{\xi o} & \psi^* \bar{A}^t_{\xi o} \\ \bullet & \Upsilon_d & 0 & \tau^* \bar{A}^t_{\xi d} & \psi^* \bar{A}^t_{\xi d} \\ \bullet & \bullet & \Upsilon_f & \tau^* \bar{D}^t_o & \psi^* \bar{D}^t_o \\ \bullet & \bullet & \bullet & -\tau^* \hat{\mathcal{W}} & 0 \\ \bullet & \bullet & \bullet & \bullet & -\psi^* \hat{\mathcal{Q}} \end{bmatrix} + \delta\kappa_1\kappa_1^t + \delta^{-1}\kappa_2\kappa_2^t + \mu\kappa_1\kappa_1^t$$

$$+ \quad \mu^{-1}\kappa_3\kappa_3^t + \varepsilon\kappa_4\kappa_4^t + \varepsilon^{-1}\kappa_5\kappa_5^t + \varrho\kappa_4\kappa_4^t + \varrho^{-1}\kappa_6\kappa_6^t$$

$$= \begin{bmatrix} \bar{\Upsilon}_{ao} & \Upsilon_b & \Upsilon_{co} & \tau^* \bar{A}^t_{\xi o} & \psi^* \bar{A}^t_{\xi o} & \bar{N}^t & 0 & \tau^* \bar{N}^t & \psi^* \bar{N}^t \\ \bullet & \Upsilon_d & 0 & \tau^* \bar{A}^t_{\xi d} & \psi^* \bar{A}^t_{\xi d} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bar{\Upsilon}_f & \tau^* \bar{D}^t_o & \psi^* \bar{D}^t_o & 0 & \bar{N}^t_x & \tau^* \bar{N}^t_x & \psi^* \bar{N}^t_x \\ \bullet & \bullet & \bullet & -\tau^* \hat{\mathcal{W}} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\psi^* \hat{\mathcal{Q}} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\delta I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mu I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varrho I \end{bmatrix} < 0 \quad (3.22)$$

where

$$\bar{\Upsilon}_{ao} \;=\; \Upsilon_{ao} + \delta\mathcal{P}^t \bar{M}\bar{M}^t\mathcal{P} + \mu\mathcal{P}^t\bar{M}\bar{M}^t\mathcal{P}, \;\; \bar{\Upsilon}_f = \Upsilon_f + \varepsilon\bar{M}\bar{M}^t + \varrho\bar{M}\bar{M}^t \quad (3.23)$$

Let

$$\mathcal{B} \;=\; \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \;\; \mathbf{B} = \begin{bmatrix} \mathcal{B} & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad (3.24)$$

Premultiplying (3.22) by $diag[\mathcal{X}^t \; I \; I \; I \; I \; I \; I \; I \; I]$, postmultiplying the result by $diag[\mathcal{X} \; I \; I \; I \; I \; I \; I \; I \; I]$, and applying the **S**-procedure [2] we arrive at the LMI (3.14) as desired. $\nabla\nabla\nabla$

In the absence of uncertainties we get the following corollary

Corollary 3.1 *System* (Σ_2) *with* $u(.) \equiv 0$, $w(.) \equiv 0$ *is delay-dependent stable if there exist matrices* $0 < \mathcal{X}_x = \mathcal{X}_x^t \in I\!R^{n\times n}$, $0 < \mathcal{X}_d = \mathcal{X}_d^t \in I\!R^{n\times n}$, $0 < \mathcal{X}_s = \mathcal{X}_s^t \in I\!R^{n\times n}$, $0 < \mathcal{X}_f^t = \mathcal{X}_f \in I\!R^{n\times n}$, $\mathcal{X}_h \in I\!R^{n\times n}$, $\mathcal{X}_g \in I\!R^{n\times n}$, $\mathcal{M}_d = \mathcal{M}_d^t \in I\!R^{n\times n}$, $\mathcal{M}_e \in I\!R^{n\times n}$, $\mathcal{M}_q \in I\!R^{n\times n}$, $\mathcal{M}_f \in I\!R^{n\times n}$, $\mathcal{M}_x = \mathcal{M}_x^t \in I\!R^{n\times n}$, $\mathcal{M}_s = \mathcal{M}_s^t \in I\!R^{n\times n}$, $\mathcal{N}_d^t = \mathcal{N}_d \in I\!R^{n\times n}$, $\mathcal{N}_e^t = \mathcal{N}_e \in I\!R^{n\times n}$, $\mathcal{N}_q^t = \mathcal{N}_q \in I\!R^{n\times n}$, $\mathcal{N}_f^t = \mathcal{N}_f \in I\!R^{n\times n}$, $\mathcal{N}_x = \mathcal{N}_x \in I\!R^{n\times n}$, $\mathcal{N}_s^t = \mathcal{N}_s \in I\!R^{n\times n}$, $\mathcal{L}_x^t = \mathcal{L}_x \in I\!R^{n\times n}$, $\mathcal{L}_f \in I\!R^{n\times n}$, $\mathcal{L}_q \in I\!R^{n\times n}$, $\mathcal{L}_d^t = \mathcal{L}_d \in I\!R^{n\times n}$, $\mathcal{L}_s^t = \mathcal{L}_s \in I\!R^{n\times n}$, $\mathcal{L}_e \in I\!R^{n\times n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in$

$I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$ *and a scalar* $\omega > 0$ *such that the following inequalities hold*

$$\begin{bmatrix} \Omega_a & \begin{matrix}\Omega_g+\\ \tau^*\bar{\mathcal{S}}_f+\psi^*\bar{\mathcal{Z}}_f+\bar{\mathcal{S}}_g\end{matrix} & \bar{D}_o-\bar{\mathcal{Z}}_g & \tau^*\Omega_d & \psi^*\Omega_d \\ \bullet & \Upsilon_d & 0 & \tau^*\bar{A}^t_{\xi d} & \psi^*\bar{A}^t_{\xi d} \\ \bullet & \bullet & \bar{\Upsilon}_f & \tau^*\bar{D}^t_o & \psi^*\bar{D}^t_o \\ \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 \\ \bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} \end{bmatrix} < 0$$

$$\mathcal{X}^t U^t \;=\; U\mathcal{X} \;\geq\; 0 \tag{3.25}$$

Remark 3.2 *The results of* **Theorem** *3.1 and* **Corollary** *3.1 contribute to the stability theory of time-delay systems in many respects: First, system* (Σ_2) *is a new state-space representation in which the system elements are grouped into a compact matrix which greatly simplifies the analysis. Second, the matrix multiplying the state derivative is constant and singular. Third, the form of the LFK (3.7)-(3.8) is new and facilitates the derivation of the stability condition without overbounding expressions, a fact lends our methodology superior to the existing techniques in the literature*

Remark 3.3 *By setting* $D_{\Delta d} \equiv 0,\ \psi \equiv 0$*, we obtain the class of retarded systems*

$$\begin{aligned} E_o\,\dot{x}(t) &= A_{\Delta o}x(t) + A_{\Delta d}x(t-\tau) + B_{\Delta o}u(t) + \Gamma w(t), \quad x_0 \;=\; \theta_0 \\ z(t) &= C_o x(t) + F_o u(t) \end{aligned} \tag{3.26}$$

Delay-dependent stability results for system (3.26)are expressed by the following theorems

Theorem 3.2 *System (3.26) with* $u(.) \equiv 0,\ w(.) \equiv 0$ *is delay-dependent robustly stable if there exist matrices* $0 < \mathcal{X}_x = \mathcal{X}_x^t \in I\!R^{n\times n}$, $0 < \mathcal{X}_d = \mathcal{X}_d^t \in I\!R^{n\times n}$, $0 < \mathcal{X}_s = \mathcal{X}_s^t \in I\!R^{n\times n}$, $0 < \mathcal{X}_f^t = \mathcal{X}_f \in I\!R^{n\times n}$, $\mathcal{X}_h \in I\!R^{n\times n}$, $\mathcal{X}_g \in I\!R^{n\times n}$, $\mathcal{M}_d = \mathcal{M}_d^t \in I\!R^{n\times n}$, $\mathcal{M}_e \in I\!R^{n\times n}$, $\mathcal{M}_q \in I\!R^{n\times n}$, $\mathcal{M}_f \in I\!R^{n\times n}$, $\mathcal{M}_x = \mathcal{M}_x^t \in I\!R^{n\times n}$, $\mathcal{M}_s = \mathcal{M}_s^t \in I\!R^{n\times n}$, $\mathcal{N}_d^t = \mathcal{N}_d \in I\!R^{n\times n}$, $\mathcal{N}_e^t = \mathcal{N}_e \in I\!R^{n\times n}$, $\mathcal{N}_q^t = \mathcal{N}_q \in I\!R^{n\times n}$, $\mathcal{N}_f^t = \mathcal{N}_f \in I\!R^{n\times n}$, $\mathcal{N}_x = \mathcal{N}_x \in I\!R^{n\times n}$, $\mathcal{N}_s^t = \mathcal{N}_s \in I\!R^{n\times n}$, $\mathcal{L}_x^t = \mathcal{L}_x \in I\!R^{n\times n}$, $\mathcal{L}_f \in I\!R^{n\times n}$, $\mathcal{L}_q \in I\!R^{n\times n}$, $\mathcal{L}_d^t = \mathcal{L}_d \in I\!R^{n\times n}$, $\mathcal{L}_s^t = \mathcal{L}_s \in I\!R^{n\times n}$, $\mathcal{L}_e \in I\!R^{n\times n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$ *and scalars* $\delta > 0,\ \mu > 0,\ \varepsilon > 0,\ \varrho > 0,\ \omega > 0$ *such that the following inequalities hold for all admissible uncertainties*

$$\begin{bmatrix} \hat{\Omega}_a & \begin{matrix}\Omega_g+\\ \tau^*\bar{\mathcal{S}}_f+\psi^*\bar{\mathcal{Z}}_f+\bar{\mathcal{S}}_g\end{matrix} & \bar{D}_o-\bar{\mathcal{Z}}_g & \tau^*\Omega_d & \Omega_n & 0 & \tau^*\Omega_n \\ \bullet & \Upsilon_d & 0 & \tau^*\bar{A}^t_{\xi d} & 0 & 0 & 0 \\ \bullet & \bullet & \bar{\Upsilon}_f & \tau^*\bar{D}^t_o & 0 & \bar{N}^t_x & \tau^*\bar{N}^t_x \\ \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\delta I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mu I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon I \end{bmatrix} < 0$$

$$\mathcal{X}^t U^t \;=\; U\mathcal{X} \;\geq\; 0 \tag{3.27}$$

where

$$\hat{\Omega}_a \;=\; \begin{bmatrix} \begin{array}{c} -\omega I + \mathcal{X}_g + \mathcal{X}_g^t + \\ \tau^*\mathcal{M}_x + \mathcal{L}_x \end{array} & \begin{array}{c} \mathcal{X}_h + \mathcal{X}_g^t + \\ \tau^*\mathcal{M}_f + \mathcal{L}_f \end{array} & \begin{array}{c} \mathcal{X}_s + \mathcal{X}_s\bar{A}_{od}^t \\ -\mathcal{X}_f\bar{A}_d^t - \mathcal{X}_g^t \\ +\tau^*\mathcal{M}_q + \mathcal{L}_q \end{array} \\ \bullet & \begin{array}{c} -\omega I + \mathcal{X}_h + \mathcal{X}_h^t + \\ \tau^*\mathcal{M}_d + \mathcal{L}_d \end{array} & \begin{array}{c} \mathcal{X}_s + \mathcal{X}_f^t\bar{A}_{od}^t \\ -\mathcal{X}_d\bar{A}_d^t - \mathcal{X}_h^t \\ \tau^*\mathcal{M}_e + \mathcal{L}_e \end{array} \\ \bullet & \bullet & \begin{array}{c} -\mathcal{X}_s - \mathcal{X}_s \\ -\omega I + \delta MM^t + \mu MM^t \\ \tau^*\mathcal{M}_s + \mathcal{L}_s \end{array} \end{bmatrix} \tag{3.28}$$

Corollary 3.2 *System (3.26) with $u(.) \equiv 0,\ w(.) \equiv 0$ is delay-dependent stable if there exist matrices $0 < \mathcal{X}_x = \mathcal{X}_x^t \in I\!R^{n\times n},\ 0 < \mathcal{X}_d = \mathcal{X}_d^t \in I\!R^{n\times n},\ 0 < \mathcal{X}_s = \mathcal{X}_s^t \in I\!R^{n\times n},\ 0 < \mathcal{X}_f^t = \mathcal{X}_f \in I\!R^{n\times n},\ \mathcal{X}_h \in I\!R^{n\times n},\ \mathcal{X}_g \in I\!R^{n\times n},\ \mathcal{M}_d = \mathcal{M}_d^t \in I\!R^{n\times n},\ \mathcal{M}_e \in I\!R^{n\times n},\ \mathcal{M}_q \in I\!R^{n\times n},\ \mathcal{M}_f \in I\!R^{n\times n},\ \mathcal{M}_x = \mathcal{M}_x^t \in I\!R^{n\times n},\ \mathcal{M}_s = \mathcal{M}_s^t \in I\!R^{n\times n}$, $\mathcal{N}_d^t = \mathcal{N}_d \in I\!R^{n\times n}$, $\mathcal{N}_e^t = \mathcal{N}_e \in I\!R^{n\times n}$, $\mathcal{N}_q^t = \mathcal{N}_q \in I\!R^{n\times n}$, $\mathcal{N}_f^t = \mathcal{N}_f \in I\!R^{n\times n}$, $\mathcal{N}_x = \mathcal{N}_x \in I\!R^{n\times n}$, $\mathcal{N}_s^t = \mathcal{N}_s \in I\!R^{n\times n},\ \mathcal{L}_x^t = \mathcal{L}_x \in I\!R^{n\times n},\ \mathcal{L}_f \in I\!R^{n\times n},\ \mathcal{L}_q \in I\!R^{n\times n},\ \mathcal{L}_d^t = \mathcal{L}_d \in I\!R^{n\times n},\ \mathcal{L}_s^t = \mathcal{L}_s \in I\!R^{n\times n},\ \mathcal{L}_e \in I\!R^{n\times n},\ \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n},\ \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n},\ \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$ and a scalar $\omega > 0$ such that the following inequalities hold*

$$\begin{bmatrix} \hat{\Omega}_a & \Omega_g + \tau^*\bar{\mathcal{S}}_f + \bar{\mathcal{S}}_g & -\bar{\mathcal{Z}}_g & \tau^*\Omega_d \\ \bullet & \Upsilon_d & 0 & -\tau^*\bar{A}_{\xi d}^t \\ \bullet & \bullet & \bar{\Upsilon}_f & \\ \bullet & \bullet & \bullet & \tau^*\hat{\mathcal{W}} \end{bmatrix} < 0$$

$$\mathcal{X}^t U^t \;=\; U\mathcal{X} \;\geq\; 0 \tag{3.29}$$

4 Delay-Dependent Stabilization

In the sequel, we consider two stabilization schemes: one uses state feedback and the other is based on $\mathcal{H}_\infty$ control approach.

4.1 State-Feedback Stabilization

Application of the state-feedback control law

$$u = K_o\, x(t) = K_o\, \tilde{I}\, \xi(t), \quad \tilde{I} = [I \;\; 0 \;\; 0] \tag{4.1}$$

to system (3.3) with $w_k \equiv 0$ yields the following closed-loop system

$$(\Sigma_3):\;\; U\,\dot{\xi}(t) \;\;=\;\; \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & I \\ A_{\Delta odk} & -A_{\Delta d} & -I \end{bmatrix} \;+\; \bar{A}_{\xi d}\,\xi(t-\tau) \;+\; \bar{D}_{\Delta d}\,\xi(t-\psi)$$

$$\begin{aligned} &= \bar{\mathcal{A}}_{\Delta\xi k}\xi(t) \ + \ \bar{A}_{\xi d}\,\xi(t-\tau) \ + \ \bar{D}_{\Delta d}\,\xi(t-\psi) \\ &= (\bar{A}_{\xi k} + \bar{M}\Delta_t\bar{N})\xi(t) \ + \ \bar{A}_{\xi d}\,\xi_(t-\tau) \ + \ \bar{D}_{\Delta d}\,\xi(t-\psi), \\ z(t) &= \bar{C}_k\xi(t), \quad \bar{C}_k \ = \ (C_o + F_oK_o)\tilde{I} \end{aligned} \tag{4.2}$$

with

$$\bar{A}_{\xi k} = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & I \\ A_{od} + B_oK_o & -A_d & -I \end{bmatrix}$$

It follows from inequality that system (4.2) is delay-dependent robustly stablizable by the feedback control law (4.1) if the following LMI is satisfied for all admissible uncertainties:

$$\begin{bmatrix} \bar{\Upsilon}_{ak} & \Upsilon_b & \Upsilon_{co} & \tau^*\bar{\mathcal{A}}^t_{\xi k} & \psi^*\bar{\mathcal{A}}^t_{\xi k} & \bar{N}^t & 0 & \tau^*\bar{N}^t & \psi^*\bar{N}^t \\ \bullet & \Upsilon_d & 0 & \tau^*\bar{A}^t_{\xi d} & \psi^*\bar{A}^t_{\xi d} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bar{\Upsilon}_f & \tau^*\bar{D}^t_o & \psi^*\bar{D}^t_o & 0 & \bar{N}^t_x & \tau^*\bar{N}^t_x & \psi^*\bar{N}^t_x \\ \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\delta I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mu I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varrho I \end{bmatrix} < 0 \tag{4.3}$$

where

$$\begin{aligned} \bar{\Upsilon}_{ak} &= \Upsilon_{ak} + \delta\mathcal{P}^t\bar{M}\bar{M}^t\mathcal{P} + \mu\mathcal{P}^t\bar{M}\bar{M}^t\mathcal{P} \\ \Upsilon_{ak} &= \mathcal{P}^t\bar{A}_{\xi k} + \bar{A}^t_{\xi k}\mathcal{P} + \tau^*\mathcal{S}_x + \psi^*\mathcal{Z}_x + \mathcal{S}_g + \mathcal{S}^t_g + \mathcal{Z}_g + \mathcal{Z}^t_g \end{aligned} \tag{4.4}$$

Following the steps of section 3, we get the stabilization result which is summarized by the following theorem:

Theorem 4.1 *System* (Σ_3) *is delay-dependent robustly stablizable by the feedback controller (4.1) if there exist matrices* $0 < \mathcal{X}_x = \mathcal{X}^t_x \in I\!R^{n\times n}$, $0 < \mathcal{X}_d = \mathcal{X}^t_d \in I\!R^{n\times n}$, $0 < \mathcal{X}_s = \mathcal{X}^t_s \in I\!R^{n\times n}$, $0 < \mathcal{X}^t_f = \mathcal{X}_f \in I\!R^{n\times n}$, $\mathcal{X}_h \in I\!R^{n\times n}$, $\mathcal{X}_g \in I\!R^{n\times n}$, $\mathcal{M}_d = \mathcal{M}^t_d \in I\!R^{n\times n}$, $\mathcal{M}_e \in I\!R^{n\times n}$, $\mathcal{M}_q \in I\!R^{n\times n}$, $\mathcal{M}_f \in I\!R^{n\times n}$, $\mathcal{M}_x = \mathcal{M}^t_x \in I\!R^{n\times n}$, $\mathcal{M}_s = \mathcal{M}^t_s \in I\!R^{n\times n}$, $\mathcal{N}^t_d = \mathcal{N}_d \in I\!R^{n\times n}$, $\mathcal{N}^t_e = \mathcal{N}_e \in I\!R^{n\times n}$, $\mathcal{N}^t_q = \mathcal{N}_q \in I\!R^{n\times n}$, $\mathcal{N}^t_f = \mathcal{N}_f \in I\!R^{n\times n}$, $\mathcal{N}_x = \mathcal{N}_x \in I\!R^{n\times n}$, $\mathcal{N}^t_s = \mathcal{N}_s \in I\!R^{n\times n}$, $\mathcal{L}^t_x = \mathcal{L}_x \in I\!R^{n\times n}$, $\mathcal{L}_f \in I\!R^{n\times n}$, $\mathcal{L}_q \in I\!R^{n\times n}$, $\mathcal{L}^t_d = \mathcal{L}_d \in I\!R^{n\times n}$, $\mathcal{L}^t_s = \mathcal{L}_s \in I\!R^{n\times n}$, $\mathcal{L}_e \in I\!R^{n\times n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$ *and scalars* $\delta > 0$, $\mu > 0$, $\varepsilon >$

0, $\varrho > 0$, $\omega > 0$ *such that the following LMIs hold for all admissible uncertainties*

$$\begin{bmatrix} \Omega_k & \begin{matrix}\Omega_g + \bar{\mathcal{S}}_g + \\ \tau^*\bar{\mathcal{S}}_f + \psi^*\bar{\mathcal{Z}}_f\end{matrix} & \bar{D}_o - \bar{\mathcal{Z}}_g & \tau^*\Omega_d & \psi^*\Omega_d & \Omega_n & 0 & \tau^*\Omega_n & \psi^*\Omega_n \\ \bullet & \Upsilon_d & 0 & \tau^*\bar{A}^t_{\xi d} & \psi^*\bar{A}^t_{\xi d} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bar{\Upsilon}_f & \tau^*\bar{D}^t_o & \psi^*\bar{D}^t_o & 0 & \bar{N}^t_x & \tau^*\bar{N}^t_x & \psi^*\bar{N}^t_x \\ \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\delta I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mu I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varrho I \end{bmatrix} < 0$$

$$\mathcal{X}^t U^t \;=\; U\mathcal{X} \;\geq\; 0 \tag{4.5}$$

where

$$\Omega_k \;=\; \begin{bmatrix} \begin{matrix}-\omega I + \mathcal{X}_g + \mathcal{X}^t_g + \\ \tau^*\mathcal{M}_x + \psi^*\mathcal{N}_x + \mathcal{L}_x\end{matrix} & \begin{matrix}\mathcal{X}_h + \mathcal{X}^t_g + \\ \tau^*\mathcal{M}_f + \psi^*\mathcal{N}_f + \mathcal{L}_f\end{matrix} & \begin{matrix}\mathcal{X}_s + \mathcal{X}_s\bar{A}^t_{od} + \mathcal{Y}^t_s B^t_o \\ -\mathcal{X}_f\bar{A}^t_d - \mathcal{X}^t_g \\ +\tau^*\mathcal{M}_q + \psi^*\mathcal{N}_q + \mathcal{L}_q\end{matrix} \\ \bullet & \begin{matrix}-\omega I + \mathcal{X}_h + \mathcal{X}^t_h + \\ \tau^*\mathcal{M}_d + \psi^*\mathcal{N}_d + \mathcal{L}_d\end{matrix} & \begin{matrix}\mathcal{X}_s + \mathcal{X}^t_f\bar{A}^t_{od} + \mathcal{Y}^t_f B^t_o \\ -\mathcal{X}_d\bar{A}^t_d - \mathcal{X}^t_h \\ \tau^*\mathcal{M}_e + \psi^*\mathcal{N}_e + \mathcal{L}_e\end{matrix} \\ \bullet & \bullet & \begin{matrix}-\mathcal{X}_s - \mathcal{X}_s \\ -\omega I + \delta MM^t + \mu MM^t \\ \tau^*\mathcal{M}_s + \psi^*\mathcal{N}_s + \mathcal{L}_s\end{matrix} \end{bmatrix}, \tag{4.6}$$

The feedback gain is given by $K_o = \mathcal{Y}_s\mathcal{X}_x^{-1}$.

Proof: Follows from **Theorem** 3.1 after taking $\mathcal{Y}_x = K_o\mathcal{X}_x$ and $\mathcal{Y}_f = K_o\mathcal{X}_f$. $\nabla\nabla\nabla$

In the absence of uncertainties we have the following corollary

Corollary 4.1 *System* (Σ_3) *with* $M \equiv 0$, $N_a \equiv 0$ *and* $N_d \equiv 0$ *is delay-dependent robustly stablizable by the feedback controller (4.1) if there exist matrices* $0 < \mathcal{X}_x = \mathcal{X}^t_x \in I\!R^{n\times n}$, $0 < \mathcal{X}_d = \mathcal{X}^t_d \in I\!R^{n\times n}$, $0 < \mathcal{X}_s = \mathcal{X}^t_s \in I\!R^{n\times n}$, $0 < \mathcal{X}^t_f = \mathcal{X}_f \in I\!R^{n\times n}$, $\mathcal{X}_h \in I\!R^{n\times n}$, $\mathcal{X}_g \in I\!R^{n\times n}$, $\mathcal{M}_d = \mathcal{M}^t_d \in I\!R^{n\times n}$, $\mathcal{M}_e \in I\!R^{n\times n}$, $\mathcal{M}_q \in I\!R^{n\times n}$, $\mathcal{M}_f \in I\!R^{n\times n}$, $\mathcal{M}_x = \mathcal{M}^t_x \in I\!R^{n\times n}$, $\mathcal{M}_s = \mathcal{M}^t_s \in I\!R^{n\times n}$, $\mathcal{N}^t_d = \mathcal{N}_d \in I\!R^{n\times n}$, $\mathcal{N}^t_e = \mathcal{N}_e \in I\!R^{n\times n}$, $\mathcal{N}^t_q = \mathcal{N}_q \in I\!R^{n\times n}$, $\mathcal{N}^t_f = \mathcal{N}_f \in I\!R^{n\times n}$, $\mathcal{N}_x = \mathcal{N}_x \in I\!R^{n\times n}$, $\mathcal{N}^t_s = \mathcal{N}_s \in I\!R^{n\times n}$, $\mathcal{L}^t_x = \mathcal{L}_x \in I\!R^{n\times n}$, $\mathcal{L}_f \in I\!R^{n\times n}$, $\mathcal{L}_q \in I\!R^{n\times n}$, $\mathcal{L}^t_d = \mathcal{L}_d \in I\!R^{n\times n}$, $\mathcal{L}^t_s = \mathcal{L}_s \in I\!R^{n\times n}$, $\mathcal{L}_e \in I\!R^{n\times n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$ *and a scalar* $\omega > 0$ *such that the following LMIs hold*

$$\begin{bmatrix} \Omega_k & \begin{matrix}\Omega_g + \\ \tau^*\bar{\mathcal{S}}_f + \psi^*\bar{\mathcal{Z}}_f + \bar{\mathcal{S}}_g\end{matrix} & \bar{D}_o - \bar{\mathcal{Z}}_g & \tau^*\Omega_d & \psi^*\Omega_d \\ \bullet & \Upsilon_d & 0 & \tau^*\bar{A}^t_{\xi d} & \psi^*\bar{A}^t_{\xi d} \\ \bullet & \bullet & \bar{\Upsilon}_f & \tau^*\bar{D}^t_o & \psi^*\bar{D}^t_o \\ \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 \\ \bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} \end{bmatrix} < 0$$

$$\mathcal{X}^t U^t \;=\; U\mathcal{X} \;\geq\; 0 \tag{4.7}$$

The feedback gain is given by $K_o = \mathcal{Y}_s \mathcal{X}_x^{-1}$.

4.2 $\mathcal{H}_\infty$ Synthesis

In the sequel, we extend the results attained in the forgoing section to the case of $\mathcal{H}_\infty$ control. Application of (4.1) to system (Σ_2) yields the closed-loop system

$$\begin{aligned}
(\Sigma_3): \quad U\,\dot{\xi}(t) &= \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & I \\ A_{\Delta odk} & -A_{\Delta d} & -I \end{bmatrix} + \bar{A}_{\xi d}\,\xi_{k-d} + \bar{\Gamma}w(t) \\
&= (\bar{A}_{\xi k} + \bar{M}\Delta_t \bar{N})\xi(t) + \bar{A}_{\xi d}\,\xi(t-\tau) + \bar{D}_{\Delta d}\,\xi(t-\psi) + \bar{\Gamma}w(t), \\
z(t) &= \bar{C}_k \xi(t), \quad \bar{C}_k = (C_o + F_o K_o)\tilde{I}
\end{aligned} \tag{4.8}$$

For system (Σ_3), let $\{z(t)\}$, $\{w(t)\}$ be the trajectories of the observed output and external disturbances with respective norms $||z(t)||$, $||w(t)||$. Given a disturbance attenuation level $\gamma > 0$, we define the performance index

$$\mathsf{J}(w) = \int_0^\infty (z^t z - \gamma^2\, w^t w)\, ds$$

It is required to achieve $\mathsf{J}(w) < 0\ \forall\, 0 \neq \{w(t)\} \in \mathcal{L}_2$, $\theta_0 = 0$. Equivalently stated, we seek to develop conditions for the state-feedback controller (4.1) that render the closed-looped system (4.8) robustly stable for all admissible uncertainties while satisfying $||z(t)||, < \gamma^2\, ||w(t)||$.

Proceeding to reach $\mathsf{J}(w) < 0$, we consider $V(\xi)$ of (3.7) and evaluate the derivative $\dot{V}(\xi)$ along the trajectories of system (Σ_3). Thus we have

$$\begin{aligned}
& \dot{V}(\xi) + z^t(t)z(t) - \gamma^2\, w^t(t)w(t) \le \\
& \xi^t(t)\left[\mathcal{P}^t \bar{A}_{\Delta\xi k} + \bar{A}^t_{\Delta\xi k}\mathcal{P}\right]\xi(t) + \xi^t \mathcal{P}^t \bar{A}_{\xi d}\xi(t-\tau) + \xi^t(t-\tau)\bar{A}^t_{\xi d}\mathcal{P}\xi(t) + \xi^t \mathcal{P}^t \bar{D}_{\Delta d}\xi(t-\psi) \\
+ \;& \xi^t(t-\psi)\bar{D}^t_{\Delta d}\mathcal{P}\xi(t) + \xi^t \mathcal{P}^t \bar{\Gamma}w(t) + w^t(t)\bar{\Gamma}^t \mathcal{P}\xi(t) + \tau^* \dot{\xi}^t(t) U^t \hat{\mathcal{F}} U \dot{\xi}(t) \\
+ \;& \psi^* \dot{\xi}^t(t) U^t \hat{\mathcal{G}} U \dot{\xi}(t) + \xi^t(t)\bar{C}_k^t \bar{C}_k \xi(t) - \gamma^2\, w^t(t)w(t) \\
+ \;& \tau^* \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \end{bmatrix}^t \begin{bmatrix} \mathcal{S}_x & \mathcal{S}_f \\ \bullet & \mathcal{S}_d \end{bmatrix} \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \end{bmatrix} + 2 \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \end{bmatrix}^t \begin{bmatrix} \mathcal{S}_g \\ \mathcal{S}_h \end{bmatrix} \bar{I} \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \end{bmatrix} \\
+ \;& \psi^* \begin{bmatrix} \xi(t) \\ \xi(t-\psi) \end{bmatrix}^t \begin{bmatrix} \mathcal{Z}_x & \mathcal{Z}_f \\ \bullet & \mathcal{Z}_d \end{bmatrix} \begin{bmatrix} \xi(t) \\ \xi(t-\psi) \end{bmatrix} + 2 \begin{bmatrix} \xi(t) \\ \xi(t-\psi) \end{bmatrix}^t \begin{bmatrix} \mathcal{Z}_g \\ \mathcal{Z}_h \end{bmatrix} \bar{I} \begin{bmatrix} \xi(t) \\ \xi(t-\psi) \end{bmatrix} \\
= \;& \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \\ \xi(t-\psi) \\ w(t) \end{bmatrix}^t \Xi \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \\ \xi(t-\psi) \\ w(t) \end{bmatrix}
\end{aligned} \tag{4.9}$$

where

$$\Xi = \left[\begin{array}{cccc} \begin{array}{c}\Upsilon_h + \\ \tau^* \bar{\mathcal{A}}^t_{\Delta\xi k} \hat{\mathcal{F}} \bar{\mathcal{A}}_{\Delta\xi k} + \\ \psi^* \bar{\mathcal{A}}^t_{\Delta\xi k} \hat{\mathcal{G}} \bar{\mathcal{A}}_{\Delta\xi} \\ + \bar{C}^t_k \bar{C}_k\end{array} & \begin{array}{c}\Upsilon_b + \\ \tau^* \bar{\mathcal{A}}^t_{\Delta\xi k} \hat{\mathcal{F}} \bar{A}_{\xi d} + \\ \psi^* \bar{\mathcal{A}}^t_{\Delta\xi k} \hat{\mathcal{G}} \bar{A}_{\xi d}\end{array} & \begin{array}{c}\Upsilon_c + \\ \tau^* \bar{A}^t_{\Delta\xi} \hat{\mathcal{F}} \bar{D}_{\Delta d} + \\ \psi^* \bar{A}^t_{\Delta\xi} \hat{\mathcal{G}} \bar{D}_{\Delta d}\end{array} & \begin{array}{c}\mathcal{P}^t \bar{\Gamma} + \\ \tau^* \bar{\mathcal{A}}^t_{\Delta\xi k} \hat{\mathcal{F}} \bar{\Gamma} + \\ \psi^* \bar{\mathcal{A}}^t_{\Delta\xi k} \hat{\mathcal{F}} \bar{\Gamma}\end{array} \\ \bullet & \begin{array}{c}\Upsilon_d + \\ \tau^* \bar{A}^t_{\xi d} \hat{\mathcal{F}} \bar{A}_{\xi d} + \\ \psi^* \bar{A}^t_{\xi d} \hat{\mathcal{G}} \bar{A}_{\xi d}\end{array} & \begin{array}{c}\tau^* \bar{A}^t_{\xi d} \hat{\mathcal{F}} \bar{D}_{\Delta d} + \\ \psi^* \bar{A}^t_{\xi d} \hat{\mathcal{G}} \bar{D}_{\Delta d}\end{array} & \begin{array}{c}\tau^* \bar{A}^t_{\xi d} \hat{\mathcal{F}} \bar{\Gamma} + \\ \psi^* \bar{A}^t_{\xi d} \hat{\mathcal{G}} \bar{\Gamma}\end{array} \\ \bullet & \bullet & \begin{array}{c}\Upsilon_f + \\ \tau^* \bar{D}^t_{\Delta d} \hat{\mathcal{F}} \bar{D}_{\Delta d} + \\ \psi^* \bar{D}^t_{\Delta d} \hat{\mathcal{G}} \bar{D}_{\Delta d}\end{array} & \begin{array}{c}\tau^* \bar{D}^t_{\Delta d} \hat{\mathcal{F}} \bar{\Gamma} + \\ \psi^* \bar{D}^t_{\Delta d} \hat{\mathcal{G}} \bar{\Gamma}\end{array} \\ \bullet & \bullet & \bullet & \begin{array}{c}-\gamma^2 I + \\ \tau^* \bar{\Gamma}^t \hat{\mathcal{F}} \bar{\Gamma} + \\ \psi^* \bar{\Gamma}^t \hat{\mathcal{G}} \bar{\Gamma}\end{array} \end{array}\right],$$

$$\Upsilon_h = \mathcal{P}^t \bar{\mathcal{A}}_{\Delta\xi k} + \bar{\mathcal{A}}^t_{\Delta\xi k} \mathcal{P} + \tau^* \mathcal{S}_x + \psi^* \mathcal{Z}_x + \mathcal{S}_g + \mathcal{S}^t_g + \mathcal{Z}_g + \mathcal{Z}^t_g \tag{4.10}$$

The following theorem summarizes the corresponding result.

Theorem 4.2 *System (Σ_3) is delay-dependent robustly stablizable by the feedback controller (4.1) with disturbance attenuation $\gamma > 0$ if there exist matrices $0 < \mathcal{X}_x = \mathcal{X}^t_x \in I\!R^{n\times n}$, $0 < \mathcal{X}_d = \mathcal{X}^t_d \in I\!R^{n\times n}$, $0 < \mathcal{X}_s = \mathcal{X}^t_s \in I\!R^{n\times n}$, $0 < \mathcal{X}^t_f = \mathcal{X}_f \in I\!R^{n\times n}$, $\mathcal{X}_h \in I\!R^{n\times n}$, $\mathcal{X}_g \in I\!R^{n\times n}$, $\mathcal{M}_d = \mathcal{M}^t_d \in I\!R^{n\times n}$, $\mathcal{M}_e \in I\!R^{n\times n}$, $\mathcal{M}_q \in I\!R^{n\times n}$, $\mathcal{M}_f \in I\!R^{n\times n}$, $\mathcal{M}_x = \mathcal{M}^t_x \in I\!R^{n\times n}$, $\mathcal{M}_s = \mathcal{M}^t_s \in I\!R^{n\times n}$, $\mathcal{N}^t_d = \mathcal{N}_d \in I\!R^{n\times n}$, $\mathcal{N}^t_e = \mathcal{N}_e \in I\!R^{n\times n}$, $\mathcal{N}^t_q = \mathcal{N}_q \in I\!R^{n\times n}$, $\mathcal{N}^t_f = \mathcal{N}_f \in I\!R^{n\times n}$, $\mathcal{N}_x = \mathcal{N}_x \in I\!R^{n\times n}$, $\mathcal{N}^t_s = \mathcal{N}_s \in I\!R^{n\times n}$, $\mathcal{L}^t_x = \mathcal{L}_x \in I\!R^{n\times n}$, $\mathcal{L}_f \in I\!R^{n\times n}$, $\mathcal{L}_q \in I\!R^{n\times n}$, $\mathcal{L}^t_d = \mathcal{L}_d \in I\!R^{n\times n}$, $\mathcal{L}^t_s = \mathcal{L}_s \in I\!R^{n\times n}$, $\mathcal{L}_e \in I\!R^{n\times n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$, $\bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$ and scalars $\gamma > 0$, $\delta > 0$, $\mu > 0$, $\varepsilon > 0$, $\varrho > 0$, $\omega > 0$ such that the following LMI holds for all admissible uncertainties*

$$\left[\begin{array}{ccccccccccc} \Omega_k & \begin{array}{c}\Omega_g + \bar{\mathcal{S}}_g \\ \tau^* \bar{\mathcal{S}}_f + \psi^* \bar{\mathcal{Z}}_f\end{array} & \bar{D}_o - \bar{\mathcal{Z}}_g & \bar{\Gamma} & \tau^* \Omega_d & \psi^* \Omega_d & \begin{array}{c}\mathcal{X}^t_s C^t_o + \\ \mathcal{Y}^t_s F^t_o\end{array} & \Omega_n & 0 & \tau^* \Omega_n & \psi^* \Omega_n \\ \bullet & \Upsilon_d & 0 & 0 & \tau^* \bar{A}^t_{\xi d} & \psi^* \bar{A}^t_{\xi d} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bar{\Upsilon}_f & 0 & \tau^* \bar{D}^t_o & \psi^* \bar{D}^t_o & 0 & 0 & \bar{N}^t_x & \tau^* \bar{N}^t_x & \psi^* \bar{N}^t_x \\ \bullet & \bullet & \bullet & -\gamma^2 I & \tau^* \bar{\Gamma}^t & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\tau^* \hat{\mathcal{W}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\psi^* \hat{\mathcal{Q}} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\delta I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mu I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varrho I \end{array}\right]$$

$$< 0$$

$$\mathcal{X}^t U^t = U \mathcal{X} \geq 0 \tag{4.11}$$

The feedback gain is given by $K_o = \mathcal{Y}_s \mathcal{X}^{-1}_x$.

Proof: It is evedient that $\mathsf{J}(w) < 0$ implies that $\Xi < 0$ and by following parallel development to **Theorem** (3.1) the latter condition corresponds to

$$\begin{bmatrix} \bar{\Upsilon}_{ao} & \Upsilon_b & \Upsilon_{co} & \mathcal{P}^t\bar{\Gamma} & \tau^*\bar{A}^t_{\xi o} & \psi^*\bar{A}^t_{\xi o} & \bar{C}^t_k & \bar{N}^t & 0 & \tau^*\bar{N}^t & \psi^*\bar{N}^t \\ \bullet & \Upsilon_d & 0 & 0 & \tau^*\bar{A}^t_{\xi d} & \psi^*\bar{A}^t_{\xi d} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bar{\Upsilon}_f & 0 & \tau^*\bar{D}^t_o & \psi^*\bar{D}^t_o & 0 & 0 & \bar{N}^t_x & \tau^*\bar{N}^t_x & \psi^*\bar{N}^t_x \\ \bullet & \bullet & \bullet & -\gamma^2 I & \tau^*\bar{\Gamma}^t & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\delta I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mu I & 0 & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varrho I \end{bmatrix} < 0 \qquad (4.12)$$

where

$$\bar{\Upsilon}_{ao} \;=\; \Upsilon_{ao} + \delta\mathcal{P}^t\bar{M}\bar{M}^t\mathcal{P} + \mu\mathcal{P}^t\bar{M}\bar{M}^t\mathcal{P}, \;\; \bar{\Upsilon}_f = \Upsilon_f + \varepsilon\bar{M}\bar{M}^t + \varrho\bar{M}\bar{M}^t \quad (4.13)$$

Premultiplying (3.22) by $diag[\mathcal{X}^t\; I\; I\; I\; I\; I\; I\; I\; I\; I\; I]$, postmultiplying the result by $diag[\mathcal{X}\; I\; I\; I\; I\; I\; I\; I\; I\; I\; I]$, and applying the **S**-procedure [2] we arrive at the LMI (4.11) as desired. $\nabla\nabla\nabla$

Finally, in the absence of uncertainties we have the following corollary

Corollary 4.2 *System* (Σ_3) *with* $M \equiv 0,\; N_a \equiv 0$ *and* $N_d \equiv 0$ *is delay-dependent stablizable by the feedback controller (4.1) law with disturbance attenuation* $\gamma > 0$ *if there exist matrices* $0 < \mathcal{X}_x = \mathcal{X}_x^t \in I\!R^{n\times n},\; 0 < \mathcal{X}_d = \mathcal{X}_d^t \in I\!R^{n\times n},\; 0 < \mathcal{X}_s = \mathcal{X}_s^t \in I\!R^{n\times n},\; 0 < \mathcal{X}_f^t = \mathcal{X}_f \in I\!R^{n\times n},\; \mathcal{X}_h \in I\!R^{n\times n},\; \mathcal{X}_g \in I\!R^{n\times n},\; \mathcal{M}_d = \mathcal{M}_d^t \in I\!R^{n\times n},\; \mathcal{M}_e \in I\!R^{n\times n},\; \mathcal{M}_q \in I\!R^{n\times n},\; \mathcal{M}_f \in I\!R^{n\times n},\; \mathcal{M}_x = \mathcal{M}_x^t \in I\!R^{n\times n},\; \mathcal{M}_s = \mathcal{M}_s^t \in I\!R^{n\times n},\; \mathcal{N}_d^t = \mathcal{N}_d \in I\!R^{n\times n},\; \mathcal{N}_e^t = \mathcal{N}_e \in I\!R^{n\times n},\; \mathcal{N}_q^t = \mathcal{N}_q \in I\!R^{n\times n},\; \mathcal{N}_f^t = \mathcal{N}_f \in I\!R^{n\times n},\; \mathcal{N}_x = \mathcal{N}_x \in I\!R^{n\times n},\; \mathcal{N}_s^t = \mathcal{N}_s \in I\!R^{n\times n},\; \mathcal{L}_x^t = \mathcal{L}_x \in I\!R^{n\times n},\; \mathcal{L}_f \in I\!R^{n\times n},\; \mathcal{L}_q \in I\!R^{n\times n},\; \mathcal{L}_d^t = \mathcal{L}_d \in I\!R^{n\times n},\; \mathcal{L}_s^t = \mathcal{L}_s \in I\!R^{n\times n},\; \mathcal{L}_e \in I\!R^{n\times n},\; \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n},\; \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n},\; \bar{\mathcal{S}}_f \in I\!R^{3n\times 3n}$ *and scalars* $\gamma > 0,\; \omega > 0$ *such that the following LMIs hold*

$$\begin{bmatrix} \Omega_k & \begin{array}{c}\Omega_g + \bar{\mathcal{S}}_g \\ \tau^*\bar{\mathcal{S}}_f + \psi^*\bar{\mathcal{Z}}_f\end{array} & \bar{D}_o - \bar{\mathcal{Z}}_g & \bar{\Gamma} & \tau^*\Omega_d & \psi^*\Omega_d & \begin{array}{c}\mathcal{X}_s^t C_o^t + \\ \mathcal{Y}_s^t F_o^t\end{array} \\ \bullet & \Upsilon_d & 0 & 0 & \tau^*\bar{A}^t_{\xi d} & \psi^*\bar{A}^t_{\xi d} & 0 \\ \bullet & \bullet & \bar{\Upsilon}_f & 0 & \tau^*\bar{D}^t_o & \psi^*\bar{D}^t_o & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & \tau^*\bar{\Gamma}^t & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\tau^*\hat{\mathcal{W}} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\psi^*\hat{\mathcal{Q}} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0$$

$$\mathcal{X}^t U^t \;=\; U\mathcal{X} \;\geq\; 0 \qquad (4.14)$$

The feedback gain is given by $K_o = \mathcal{Y}_s\mathcal{X}_x^{-1}$.

5 Examples

In the sequel, some examples are worked out to illustrate the theoretical developments.

5.1 Example 1

The first example is motivated by a small PEEC model treated in [1], where the nominal numerical data are

$$A_o = 100 \times \begin{bmatrix} \alpha & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \quad A_d = 100 \times \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix}, \quad D_d = \frac{1}{72}\begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}$$

For $\alpha = -7$ and applying **Corollary** 3.1, using the LMI-MATLAB Control Toolbox, it has been found that a feasible solution is attained for $\tau^* = 1.3527,\ \psi^* = 1.4029$ which means that our method guarantees delay-dependent stability for all system delays less than $\tau = 1.3527,\ \psi = 1.4029$. This result is less conservative that the results of [1, 5] where τ^* were 1 and 0.43 respectively. Note that the result of [1] is delay-independent and those of [5] is delay-dependent. For $\alpha = -2.105$ a feasible solution is attained for $\tau^* = 1.5022,\ \psi^* = 1.5104$ which is less conservative that the result of [13] where τ^* was 1.1413.

On taking the parameter uncertainties commonly existing in system application, we apply **Theorem** 3.1 with the

$$M = \begin{bmatrix} 0.6 \\ 0.7 \\ 0.2 \end{bmatrix}, \quad N_a^t = \begin{bmatrix} 0.4 \\ 0 \\ 0.3 \end{bmatrix}, \quad N_d^t = \begin{bmatrix} 0 \\ 0.8 \\ 0.2 \end{bmatrix}, \quad N_x^t = \begin{bmatrix} 0.5 \\ 0.6 \\ 0 \end{bmatrix}, \quad \alpha = -2.105$$

A feasible solution is attained for $\tau^* = 0.8516,\ \psi^* = 0.9024$ which means that our method guarantees delay-dependent robustly stability for all admissible uncertainties and for all system delays less than $\tau = 0.8516,\ \psi = 0.9024$. In comparison with [13], our results are less conservative since their $\tau^* = 0.4064$.

5.2 Example 2

Consider a linear neutral system of the type (2.1) with

$$A_o = \begin{bmatrix} 0.1 & -0.9 & 0.2 \\ 0 & 0.5 & 0.3 \\ -2 & 0 & 0.5 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.2 & 0 & 0.2 \\ 0.1 & -0.2 & 0.1 \\ 0.2 & 0.1 & 0 \end{bmatrix}, \quad D_d = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & 0.2 \\ 0.1 & 0 & 0.1 \end{bmatrix},$$

$$C_o = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad N_x^t = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \quad F_o = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, \quad N_a^t = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad N_d^t = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

A feasible solution of **Theorem** 4.2 is given by

$$\begin{aligned} K &= \begin{bmatrix} -0.3251 & -0.0011 & -1.0325 \\ -0.0831 & -10.7885 & -1.9215 \end{bmatrix}, \quad \gamma = 1.435, \quad \omega = 1.1276, \\ \tau^* &= 4.9835, \quad \psi^* = 4.7714 \end{aligned}$$

5.3 Example 3

Consider the linear uncertain system treated in [4, 12] with data

$$\begin{aligned} A_o &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad D_d = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \quad N_x^t = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}, \\ M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_a^t = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}, \quad N_d^t = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}, \quad 0 \le c \le 1 \end{aligned}$$

Values of the upper bound on delays τ^* and ψ^* with $\beta = 0.2$ and c varying are listed in Table 1. Observe our system has two delay factors whereas in references [4, 12] there is only one delay factor. It is clearly evident that our method yields delay-dependent results that are significantly improved over those of [4, 12].

Upper Bounds τ^*	*Parameter* c	0	0.1	0.2	0.3	0.4
	Ref [12]	2.43	2.24	2.03	1.78	1.50
	Ref [4]	1.77	1.48	1.16	0.79	0.37
	Theorem 3.1	2.75	2.51	2.36	2.02	1.84
Upper Bounds ψ^*	*Parameter* c	0	0.1	0.2	0.3	0.4
	Ref [12]	2.43	2.24	2.03	1.78	1.50
	Ref [4]	1.77	1.48	1.16	0.79	0.37
	Theorem 3.1	3.04	2.81	2.66	2.42	2.11

Table 1: Upper Bounds on Delays τ^* and ψ^* $\beta = 0.2$

5.4 Example 4

Our last example is characterized by

$$\begin{aligned} A_o &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -2 & -0.5 \\ 0 & -1 \end{bmatrix}, \quad D_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad N_x^t = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_a^t = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad N_d^t = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} \end{aligned}$$

This example was examined in [3, 12]. The upper bound on delays τ^* and ψ^* for which the system is stabilized by state feedback was obtained from the feasible solution of **Theorem** 4.1 to $\tau^* = 0.8875$ and $\psi^* = 0.7966$ and the corresponding state feedback gain is

$$K_o = \begin{bmatrix} -24.6 & -17.7 \end{bmatrix}$$

In [12], $\tau^* = 0.6548$.

Application of **Corollary** 4.1 yields the feasible solution

$$K_o = \begin{bmatrix} -24.9 & -10.9 \end{bmatrix}, \quad \tau^* = 1.2158, \quad \psi^* = 1.3215$$

Note in this nominal case that τ^* in [3] and [12] was = 0.5865 and 0.9518, respectively.

To sum up from the foregoing examples, our delay-dependent design methodology stands superior to all previously published results since it significantly reduces the conservatism and employs new expanded model transformation.

6 Conclusions

For linear neutral systems with norm-bounded uncertainties, this paper has established

1) An expanded state-space representation to exhibit the delay-dependent dynamics while preserving the equivalence with the original system

2) A new delay-dependent stability criteria in a systematic way and without relying on overbounding by using an appropriate Lyapunov-Krasovskii functional, and

4) A new delay-dependent stabilization using state-feedback and $\mathcal{H}_\infty$ approach.

Numerical examples have been presented to illustrate the theoretical developments. Superiority over existing techniques have been illuminated in all the examples.

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Received: September 27, 2006