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Similarity Approach to the Problem of Second Grade Fluid Flows over a Stretching Sheet

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Abstract

This paper presents a study of the two-dimensional boundary layer flows of viscoelastic second grade fluid over a stretching sheet by the similarity method. Exploiting that some features of free-parameter method and the "separation of variables" method are alike, an ordinary differential equation governing the flow, in terms of a similarity parameter is derived. An exact solution to this equation is obtained in dimensional form. The results are discussed.

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Keywords: Two-dimensional boundary layer flows, Viscoelastic second grade fluid, Free-parameter method, Separation of variables, Similarity method

1 Introduction

Morgan [10] proposed first a method for obtaining similarity solutions of partial differential equations. According to him seeking of similarity solutions of a system of partial differential equations is equivalent to the determination of the invariant solutions of these equations under the appropriate one-parameter group of transformations.

Birkhoff [3] applied an one-parameter group of transformation to obtain similarity solutions in some problems of fluid mechanics.

Hansen [8] described the free-parameter method that can employed in finding similarity solutions of many types of problems including that from fluid mechanics. In this method the dependent variable occurring in a particular partial differential equation is expressed as the product of two functions. One function of this product is a function of all of the independent variables except one. The other function is supposed to depend on a parameter, say η, where η is a variable obtained from a transformation of variables including the independent variable not occurring in the first function.

Let ϕ be the dependent variable of a particular partial differential equation and the independent variables are $x_1, x_2, ..., x_n, y$. We can express ϕ as:

$$\phi\left(x_1, x_2, ..., x_n, y\right) = \Phi\left(x_1, x_2, ..., x_n\right) F(\eta)$$

where

$$\eta = \eta\left(x_1, x_2, ..., x_n, y\right).$$

The variables to be involved in the expressions for ϕ and η are choiced depending on the nature of the problem under investigation. The variable y is included in η as the boundary condition on ϕ depend largely on y. The main advantage of this method is that a partial differential equation may reduce to an ordinary differential equation.

Another method of performing similarity analysis is the well-known separation of variables method Abbott and Kline [1]. As far as the role of boundary conditions and formulation of similarity transformations are concerned this method is quite similar to the free-parameter method. The method is applicable for determining the possible similarity solutions of a partial differential equation when some, but not all of the boundary and initial conditions are given.

We attempt to demonstrate the alikeness of the free-parameter method and the separation of variables method in determining the exact solution for the problem of boundary layer flow of an incompressible visco-elastic second grade fluid along a stretching sheet.

2 Formulation of the problem

The boundary layer flow of an incompressible viscous Newtonian fluid on a moving solid surface has been investigated by many authors, e.g. [5],[7],[14]. The results of these investigations are useful to gain insight into polymer processing application such as the continuous extrusion of a polymer sheet from a die. But flows of non-Newtonian fluids have also become more and more important ([11], [13]). Fluids obeying Newton's law where the value of μ is constant are known as Newtonian fluids. Viscoelastic fluids are similar to Newtonian fluids but if there is a sudden large change in shear they behave like plastic, where shear stress must reach a certain minimum before flow commences.

Coleman and Noll [4] originally suggested a constitutive equation for the incompressible viscoelastic second grade fluid, based on the postulate of fading memory, as

$$T = -pI + \mu A_1 + a_1 A_2 + a_2 A_1^2 \tag{1}$$

where

T	:	is the stress tensor,
p	:	is the pressure,
μ	:	is the dynamic viscosity,
a_1, a_2	:	are the first and second normal stress coefficients and
A_1, A_2	:	are the kinematic tensors, expressed as:

$$A_1 = gradV + (gradV)^T \tag{2}$$

$$A_2 = \frac{d}{dt}A_1 + A_1(gradV) + (gradV)^T A_1 \tag{3}$$

where V is the velocity and $\frac{d}{dt}$ is the material time derivative.

Let us consider the flow of second grade fluid, governed by (1), past a plane wall $y = 0$, the flow being confined to the region $y > 0$. The wall is stretched on both sides from a fixed origin along the x-axis, the origin being kept fixed by applying two equal and opposite forces [6].

Beard and Walters [2] derived the steady two-dimensional boundary layer equations for this fluid as:

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0 \tag{4}$$

$$u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \kappa\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\cdot\frac{\partial^2 \upsilon}{\partial y^2} + \upsilon\frac{\partial^3 \upsilon}{\partial y^3}\right] \tag{5}$$

where

$$\begin{array}{lcl} \nu & = & \dfrac{\mu}{\rho}:\ \text{is the kinematic viscosity} \\ \mu & : & \text{is the viscosity of the fluid} \\ \rho & : & \text{is the density and} \\ \kappa & : & \text{is a positive parameter associated with the viscoelastic fluid.} \end{array}$$

The relevant boundary condition for $x \geq 0$ are:

$$\begin{array}{lcl} u & = & ax \;\; and \;\; \upsilon = 0 \;\; at \;\; y = 0 \\ u & \longrightarrow & 0 \;\;\; as \;\;\; y \longrightarrow \infty \end{array} \tag{6}$$

where a is a constant.

We solve equations (4) and (5) by similarity method in the next section.

3 Similarity approach: solution to the problem

We introduce the stream function ψ as:

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \upsilon = -\frac{\partial \psi}{\partial x} \tag{7}$$

Equation (4) is satisfied on substitution of (7) in it. Now, substituting (7) in (5), we obtain easily

$$\frac{\partial \psi}{\partial y}\cdot\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\cdot\frac{\partial^2 \psi}{\partial y^2}$$

$$= \nu \frac{\partial^3\psi}{\partial y^3} - \kappa \left[\frac{\partial^2\psi}{\partial x \partial y} \cdot \frac{\partial^3\psi}{\partial y^3} + \frac{\partial\psi}{\partial y} \cdot \frac{\partial^4\psi}{\partial x \partial y^3} - \frac{\partial^2\psi}{\partial y^2} \cdot \frac{\partial^3\psi}{\partial x \partial y^2} - \frac{\partial\psi}{\partial x} \cdot \frac{\partial^4\psi}{\partial y^4} \right] \tag{8}$$

At this stage we introduce transformations of the independent variables as

$$x = \zeta, \quad y = \frac{\eta}{g(\zeta)} \tag{9}$$

various derivatives appearing in (8) are transformed under (9) as:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\eta} \cdot \eta \cdot \frac{d\ln g}{d\zeta} + \frac{\partial\psi}{\partial\zeta}$$

$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\eta} \cdot g(\zeta)$$

$$\frac{\partial^2\psi}{\partial y^2} = g^2(\zeta) \cdot \frac{\partial^2\psi}{\partial\eta^2}$$

$$\frac{\partial^3\psi}{\partial y^3} = g^3(\zeta) \cdot \frac{\partial^3\psi}{\partial\eta^3}$$

$$\frac{\partial^2\psi}{\partial y \partial x} = \left[g(\zeta) \cdot \frac{\partial^2\psi}{\partial\eta\partial\zeta} + \frac{\partial\psi}{\partial\eta} \cdot g'(\zeta) \right] + \frac{\partial^2\psi}{\partial\eta^2} \cdot g(\zeta) \cdot \eta \cdot \frac{d\ln g}{d\zeta}$$

$$\frac{\partial^2\psi}{\partial x \partial y} \cdot \frac{\partial^3\psi}{\partial y^3} = \left\{ \left[g(\zeta) \cdot \frac{\partial^2\psi}{\partial\eta\partial\zeta} + \frac{\partial\psi}{\partial\eta} \cdot g'(\zeta) \right] + \frac{\partial^2\psi}{\partial\eta^2} \cdot g(\zeta) \cdot \eta \cdot \frac{d\ln g}{d\zeta} \right\} \cdot g^3(\zeta) \cdot \frac{\partial^3\psi}{\partial\eta^3}$$

$$\frac{\partial\psi}{\partial y} \cdot \frac{\partial^4\psi}{\partial x \partial y^3}$$

$$= g(\zeta) \cdot \frac{\partial\psi}{\partial\eta} \left[3g^2(\zeta) \cdot g'(\zeta) \cdot \frac{\partial^3\psi}{\partial\eta^3} + g^3(\zeta) \cdot \frac{\partial^4\psi}{\partial\zeta\partial\eta^3} + g^2(\zeta) \cdot g'(\zeta) \cdot \eta \cdot \frac{\partial^4\psi}{\partial\eta^4} \right]$$

$$\frac{\partial^2\psi}{\partial y^2}\cdot\frac{\partial^3\psi}{\partial y^2\partial x}$$

$$= g^2(\zeta)\cdot\frac{\partial^2\psi}{\partial\eta^2}\left[2g(\zeta)\cdot g\acute{}(\zeta)\cdot\frac{\partial^2\psi}{\partial\eta^2}+g^2(\zeta)\cdot\frac{\partial^3\psi}{\partial\zeta\partial\eta^2}+g^2(\zeta)\cdot\frac{\partial^3\psi}{\partial\eta^3}\cdot\eta\cdot\frac{d\ln g}{d\zeta}\right]$$

$$\frac{\partial\psi}{\partial x}\cdot\frac{\partial^4\psi}{\partial y^4}=\left[\frac{\partial\psi}{\partial\zeta}+\frac{\partial\psi}{\partial\eta}\cdot\eta\cdot\frac{d\ln g}{d\zeta}\right]\cdot\left[g^4(\zeta)\cdot\frac{\partial^4\psi}{\partial\eta^4}\right] \tag{10}$$

Substituting the transformed derivative (10) into equation (8), we obtain

$$\frac{\partial\psi}{\partial\eta}\cdot g(\zeta)\left[g(\zeta)\frac{\partial^2\psi}{\partial\eta\partial\zeta}+\frac{\partial\psi}{\partial\eta}g\acute{}(\zeta)+\frac{\partial^2\psi}{\partial\eta^2}g(\zeta)\eta\frac{d\ln g}{d\zeta}\right]$$

$$-\left(\frac{\partial\psi}{\partial\zeta}+\frac{\partial\psi}{\partial\eta}\cdot\eta\cdot\frac{d\ln g}{d\zeta}\right)g^2(\zeta)\frac{\partial^2\psi}{\partial\eta^2}=$$

$$\nu g^3(\zeta)\frac{\partial^3\psi}{\partial\eta^3}-\kappa\left[\left\{\left[g(\zeta)\frac{\partial^2\psi}{\partial\eta\partial\zeta}+\frac{\partial\psi}{\partial\eta}g\acute{}(\zeta)\right]+\frac{\partial^2\psi}{\partial\eta^2}g(\zeta)\eta\frac{d\ln g}{d\zeta}\right\}g^3(\zeta)\frac{\partial^3\psi}{\partial\eta^3}+\right.$$

$$+\frac{\partial\psi}{\partial\eta}\cdot g(\zeta)\left[3g^2(\zeta)\cdot g\acute{}(\zeta)\frac{\partial^3\psi}{\partial\eta^3}+g^3(\zeta)\frac{\partial^4\psi}{\partial\zeta\partial\eta^3}+g^2(\zeta)\cdot g\acute{}(\zeta)\cdot\eta\cdot\frac{\partial^4\psi}{\partial\eta^4}\right]-$$

$$-g^2(\zeta)\cdot\frac{\partial^2\psi}{\partial\eta^2}\left[2g(\zeta)\cdot g\acute{}(\zeta)\frac{\partial^2\psi}{\partial\eta^2}+g^2(\zeta)\cdot\frac{\partial^3\psi}{\partial\zeta\partial\eta^2}+g^2(\zeta)\cdot\frac{\partial^3\psi}{\partial\eta^3}\cdot\eta\cdot\frac{d\ln g}{d\zeta}\right]-$$

$$\left.-\left[\frac{\partial\psi}{\partial\zeta}+\frac{\partial\psi}{\partial\eta}\cdot\eta\cdot\frac{d\ln g}{d\zeta}\right]\cdot\left[g^4(\zeta)\cdot\frac{\partial^4\psi}{\partial\eta^4}\right]\right] \tag{11}$$

The equation (11) is a complicated one. To proceed further, we adopt the separation of variables technique and accordingly, put

$$\psi=H(\zeta)\cdot F(\eta) \tag{12}$$

Hansen [8] has recommended that "Substitution of the product form of the dependent variable into the equation generally leads to an equation in which the functions of one variable cannot be isolated on the two sides of the equation unless certain parameters are specified". Keeping this in view we proceed choosing simply $H(\zeta) = \zeta$, which reduces (12) to

$$\psi = \zeta \cdot F(\eta) \tag{13}$$

Substituting (13) in (11), we obtain

$$\zeta \cdot F'(\eta) \cdot g(\zeta) \left[g(\zeta) \cdot F'(\eta) + g'(\zeta) \cdot \zeta \cdot F'(\eta) + g(\zeta) \cdot \eta \frac{d \ln g}{d\zeta} \cdot \zeta \cdot F''(\eta) \right] -$$

$$- \left(F(\eta) + \eta \frac{d \ln g}{d\zeta} \cdot \zeta \cdot F'(\eta) \right) \cdot g^2(\zeta) \cdot F''(\eta) \cdot \zeta = \nu g^3(\zeta) \cdot \zeta \cdot F'''(\eta) -$$

$$\kappa \left[\left\{ [g(\zeta) \cdot F'(\eta) + g'(\zeta) \cdot \zeta \cdot F'(\eta)] + g(\zeta) \cdot \eta \frac{d \ln g}{d\zeta} \cdot \zeta \cdot F''(\eta) \right\} \cdot g^3(\zeta) \cdot \zeta \cdot F'''(\eta) + \right.$$

$$+ g(\zeta) \cdot \zeta \cdot F'(\eta) \cdot \left[3g^2(\zeta) \cdot g'(\zeta) \cdot \zeta \cdot F'''(\eta) + g^3(\zeta) \cdot F'''(\eta) + g^2(\zeta) \cdot g'(\zeta) \cdot \eta \cdot \zeta \cdot F^{iv}(\eta) \right] -$$

$$- g^2(\zeta) \cdot \zeta \cdot F''(\eta) \left[2g(\zeta) \cdot g'(\zeta) \cdot F''(\eta) + g^2(\zeta) \cdot F''(\eta) + g^2(\zeta) \cdot \eta \frac{d \ln g}{d\zeta} \cdot \zeta \cdot F''(\eta) \right] -$$

$$\left. - \left[F(\eta) + \eta \frac{d \ln g}{d\zeta} \cdot \zeta \cdot F'(\eta) \right] \cdot \left[g^4(\zeta) \cdot \zeta \cdot F^{iv}(\eta) \right] \right] \tag{14}$$

On inspection, we can see that equation (14) is not yet easily separable. In view of the relations (7) and the transformations (9), it can be easily shown that the continuity equation (4) is satisfied if $g(\zeta) = 1$. Substituting $g(\zeta) = 1$ in equation (14), we obtain

$$\zeta \cdot F'(\eta) \cdot F'(\eta) - F''(\eta) \cdot F(\eta) \cdot \zeta = \nu \cdot \zeta \cdot F'''(\eta) -$$

$$-\kappa\left[\zeta\cdot F'''(\eta)\cdot F'(\eta)-F''(\eta)\cdot\zeta\cdot F''(\eta)+\zeta\cdot F'(\eta)\cdot F'''(\eta)-\zeta\cdot F^{iv}(\eta)\cdot F(\eta)\right]$$

or

$$\nu\cdot F'''(\eta)-F'^{2}(\eta)+F(\eta)\cdot F''(\eta)=$$

$$=\kappa\left[2F'(\eta)\cdot F'''(\eta)-F''^{2}(\eta)-F(\eta)\cdot F^{iv}(\eta)\right] \tag{15}$$

The boundary conditions (6) are transformed to

$$F'(\eta)=a,\quad F(\eta)=0\quad \text{as}\quad \eta=0$$

$$F'(\eta)=0\quad \text{as}\quad \eta\to\infty \tag{16}$$

Equation (15) with the boundary conditions (16) indicate that the free parameter method and the separation of variables technique are alike.

An exact solution of equation (15), satisfying the boundary conditions (16), is given by

$$F=\frac{1}{\gamma}\left(1-e^{-\alpha\gamma\eta}\right), \tag{17}$$

where

$$\gamma=\alpha^{-\frac{1}{2}}\left(\nu-\alpha\kappa\right)^{-\frac{1}{2}},\qquad 0\leq\frac{\alpha\kappa}{\nu}<1$$

Now taking (13), (17) and (9) with $g(\zeta)=1$ into account in (7), we determine u and υ as

$$u=axe^{-\alpha^{\frac{1}{2}}(\nu-\alpha\kappa)^{-\frac{1}{2}}y} \tag{18}$$

$$\upsilon=-\alpha^{\frac{1}{2}}\left(\nu-\alpha\kappa\right)^{\frac{1}{2}}\left[1-e^{-\alpha^{\frac{1}{2}}(\nu-\alpha\kappa)^{-\frac{1}{2}}y}\right] \tag{19}$$

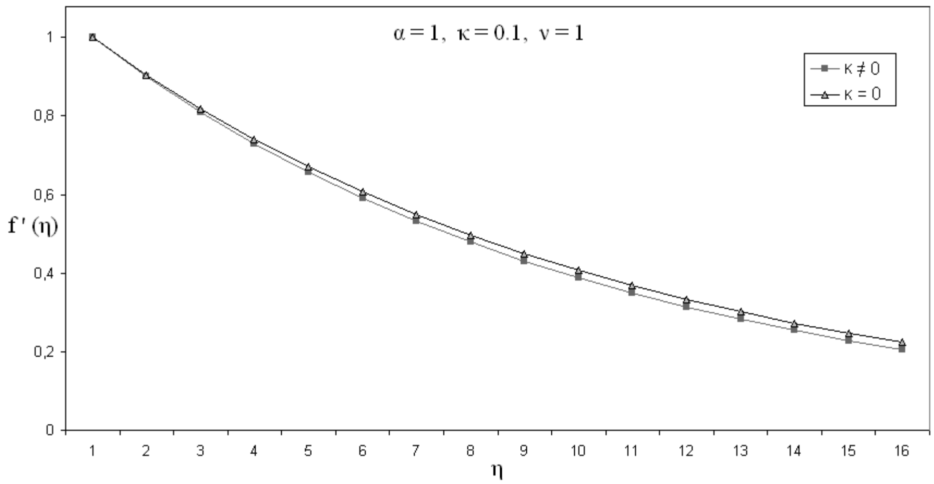

Figure 1: The velocity F' for α=1, κ=0.1 and ν=1

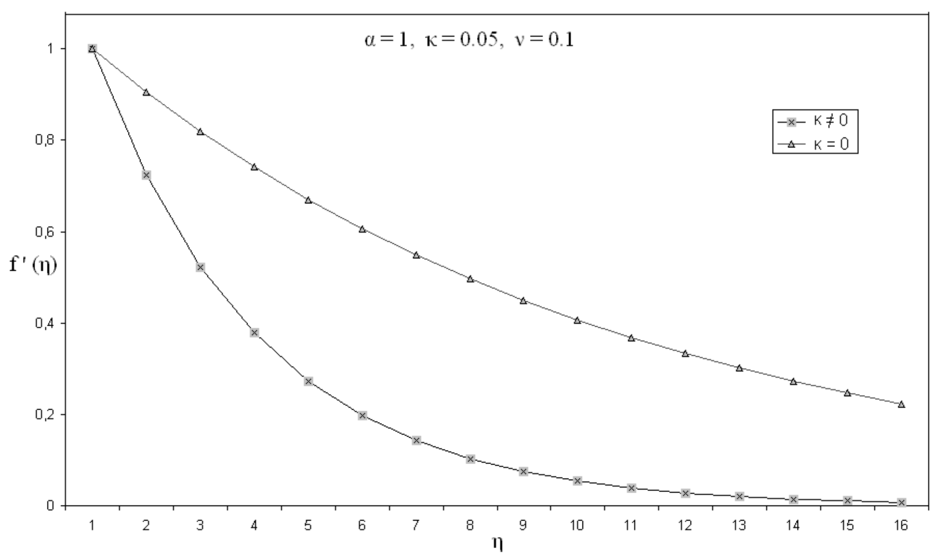

Figure 2: The velocity F' for α=1, κ=0.05 and ν=0.1

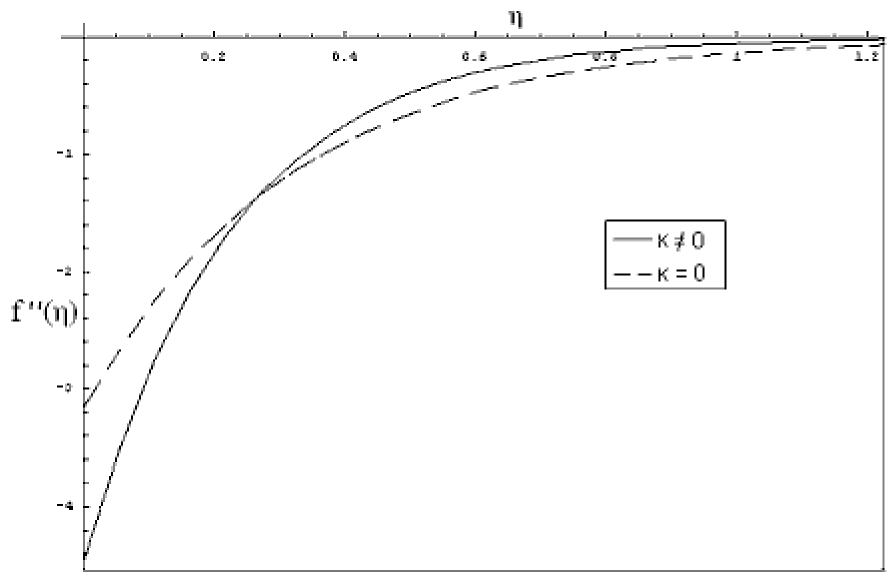

Figure 3: The wall-friction F"

4 Diagrams and Discussion

In the applications of the free - parameter method and the 'separation of variables' method to two-dimensional boundary layer flows, stream function ψ is introduced and subsequently a non-linear partial differential equation in ψ is derived. Transformations of the independent variables are then sought. The resulting equation in ψ is next subjected to the method of 'separation of variables'. Equation (15) is derived utilising the above concept and imposing some necessary restrictions namely, H(ζ)=ζ and g(ζ)=1. It is noticed that equation (15) is an ordinary differential equation in the similarity parameter η. Further, equation (15) is a fourth-order differential equation but we have three boundary conditions in (16). In the viscous Newtonian case ($\kappa = 0$), (15) however reduces to a third order equation. The suitability of present approach has been recommended by Hansen [8] for the present type of problems. Some authors have solved the problem numerically.

The exact solution (17), obtained here is in terms of dimensional quantities and agrees with that of Rajagopal et al. [12] and Siddapa and Abel [13]. Correct expressions for u and υ, as given by (18) and (19), respectively have been obtained as they satisfy the boundary conditions and the continuity equation (4). It is to mentioned in this context that the expressions derived by Dandapat and Gupta [6] for u and υ from the exact solution do not satisfy the continuity equation.

By simple calculations, for α=1, κ=0.1 and ν=1 in figure 1 and α=1,

κ=0.05 and ν=0.1 in figure 2, it can be seen that the velocity $F'(\kappa \neq 0)$ is reduced in comparison to $F'(\kappa = 0)$ and the wall-friction F'' for the case $(\kappa \neq 0)$ increases in comparison to that of the viscous Newtonian case $(\kappa = 0)$ figure 3.These effects are due to viscoelasticity of the fluid [14].

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