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Performance Under A Priori Response Knowledge

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Abstract

We consider a decision-response model based on a Brownian bridge process, and evaluate the response performance by knowing in advance the response at the end of the time period; as a consequence we deduce the relative efficiency of responses.

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1 Model description

The models that give the most natural account of accuracy and response time in simple decision tasks are sequential sampling models based on stochastic processes, because their statistics depend on the stimulus encoding mechanisms and instantaneous fluctuations in noise within the observer. The most common model used to fit the empirical response time and accuracy data was the Brownian motion model (cf. [3] and the references therein). However, to take into account that the mean of the decision process may decay after stimulus offset, or the information may decay as it is being accumulated, more general models have been employed, such as diffusion models (cf. [5]). The best response problem was addressed in [6] in the form of an optimal stopping problem for such diffusion decision models; furthermore, in [7] we formulated and solved the problem of improving the performance through an optimal stopping problem with two rights to choose.

In this note we consider a related natural problem, namely of obtaining a better performance *knowing in advance the response at the end of the time period*. In mathematical terms, such a stochastic process must be conditioned, or "pinned down", by the future. Therefore, on a complete probability space $(\Omega, \mathcal{F}, P)$ we consider a stochastic decision-response model represented by a Brownian motion process $\{W_t = W_t(\omega), t \geq 0, \omega \in \Omega\}$ starting at 0, that is, $W_0 = 0$. At time $T > 0$ the process is observed to have reached the value $W_T = b$ almost surely, for some fixed $b > 0$; this represents the response at the end of time interval. Given this prior knowledge of the process, it is possible to simulate what happens during the time interval $(0, T)$, by considering the *Brownian bridge* (or pinned down Brownian motion), denoted by W_t^*, and defined as

$$W_t^* = W_t + \int_0^t \frac{b - W_s^*}{T - s} ds \text{ for } 0 < t \leq T, \text{ and } W_0^* = 0. \tag{1}$$

It is easy to see that W_t^* is a Gaussian process with mean and variance given by

$$E(W_t^*) = \frac{bt}{T}, \quad Var(W_t^*) = t\Big(1 - \frac{t}{T}\Big), \tag{2}$$

respectively. It is now possible to use the information from the observations of the original Brownian motion at the endpoints of the time interval to simulate new observations of the same Brownian motion in the interval $(0, T)$: simply generate a normal variable with mean and variance given in formula (2). In addition, according to a theorem of Doob (cf. [2]), one can use the following relationship between Brownian motion and Brownian bridge:

$$W_t^* = \frac{bt}{T} + \Big(1 - \frac{t}{T}\Big) W_{t/(1-t/T)} \text{ for } 0 \leq t < T, \text{ and } W_T^* = b. \tag{3}$$

One can see from equations (1) and (3) that W_t^* contains a drift towards the known value b at time T. Moreover, if t is close to T but W_t^* is far from b, the drift towards b must be stronger than if t were further away from T.

2 Response Performance

An interesting feature of the Brownian bridge is that *it crosses any level larger than b with strictly positive probability.* More precisely (see [2]), for fixed $\lambda > b$, one has

$$P(\tau_\lambda \leq T) = \exp\Big[-\frac{2}{T}\lambda(\lambda - b)\Big], \text{ where } \tau_\lambda := \inf\{t : W_t^* \geq \lambda\}. \tag{4}$$

This fact suggests the following optimal stopping problem:

Determine the best performance based on the Brownian bridge decision model, that is, the numerical value

$$\sup \int_{\Omega} W^*_{\tau(\omega)}(\omega) dP(\omega), \tag{5}$$

where the supremum is taken over all stopping times $\tau \leq T$, together with an optimal stopping time τ^*, that is, a stopping time for which the above supremum is attained.

In other words, the best performance based on the Brownian bridge is the highest expectation (that is, integral with respect to P) of the stopped process $W^*_\tau := W^*_{\tau(\omega)}(\omega)$ at all stopping times $\tau \leq T$. If we would have considered in (5) the Brownian motion process instead of the Brownian bridge, the corresponding performance would be equal to zero (the expectation of W_τ is zero for all stopping times τ, see [2]). According to the previous discussion, we expect a value in (5) larger than b (therefore strictly positive), and the existence of an optimal stopping time smaller than T.

This is indeed the case (cf. [4]): under the above assumptions and notations, the value in (5) equals

$$b + T(1-\alpha^2) \int_0^\infty \exp(-bx - Tx^2/2) dx, \tag{6}$$

where $\alpha = 0.83992...$ is the unique solution of the equation

$$\frac{\alpha}{1-\alpha^2} = \int_0^\infty \exp(\alpha x - x^2/2) dx,$$

and the optimal stopping time is given by

$$\tau^* := \inf\{t : W^*_t \geq b + \alpha\sqrt{T-t}\}. \tag{7}$$

(one can easily see that $\tau^* < T$ almost surely).

The best performance in formula (6) can be easily computed using the tables for the standard normal cumulative distribution function $\phi(x)$, as the expression in (6) can be written as

$$b + (1-\alpha^2)\sqrt{2\pi T} \exp\Big(\frac{b^2}{2T}\Big)\Big[1 - \phi\Big(\frac{b}{\sqrt{T}}\Big)\Big].$$

However, the explicit formula (7) for the optimal stopping time is not practical because of the term $\sqrt{T-t}$. Instead, for small time intervals, we can provide a very good approximation for the true value of τ^*, as follows. Let f be a continuous function with Lipschitz continuous derivative and $f(0) > b$. Define

$$\tau_f := \inf\{t : W^*_t \geq f(t)\}. \tag{8}$$

Then (cf. [1]), for small values of T, one has the following extension of formula (4):

$$P(\tau_f \le T) \sim \exp\Big[-2f'(0)f(0) - \frac{2}{T}f^2(0) + \frac{2b}{T}f(0)\Big]. \tag{9}$$

In the sequel we consider the following approximation valid for small values of t and T:

$$\sqrt{T-t} \approx \sqrt{T} - \frac{t}{\sqrt{T}}.$$

We can now approximate τ^* in (7) by the following stopping time:

$$\tau^{**} = \inf\Big\{t : W_t^* \ge b + \alpha\Big(\sqrt{T} - \frac{t}{\sqrt{T}}\Big)\Big\} \tag{10}$$

Indeed, taking $f(t) = b + \alpha(\sqrt{T} - t/\sqrt{T})$ in (8) and (9) we obtain that $\tau^{**} \le T$ with probability one. Moreover, the advantage of working with the stopping time (10) is the following. Using formula (3), we obtain the following equivalent form:

$$\tau^{**} = \inf\{t : W_t \ge b + \alpha\sqrt{T}\},$$

that is, τ^{**} is the first crossing of W_t through a constant level (independent of t), and therefore it can be easily recovered from the Gaussian simulations of the Brownian motion process.

3 Response Efficiency

We can determine *the relative efficiency of responses under a priori knowledge.* To this aim, we consider the following optimal stopping problem: determine the numerical value

$$\sup \int_\Omega \frac{W_{\tau(\omega)}(\omega) - b}{T + \tau(\omega)} dP(\omega), \tag{11}$$

where the supremum is taken over all stopping times for which the expectation above makes sense, together with an optimal stopping time (where the supremum is attained). The relative efficiency should be the ratio between response and time; however, as we know in advance the performance at the end of time interval, it makes sense to divide in (11) by $T + \tau$ (and not by τ) and to subtract b (the known value of performance) from W_τ. According to [4] and [8], the value in (11) is given by

$$(1-\alpha^2)\int_0^\infty \exp(-bx - Tx^2/2)dx = (1-\alpha^2)\sqrt{\frac{2\pi}{T}}\exp\Big(\frac{b^2}{2T}\Big)\Big[1-\phi\Big(\frac{b}{\sqrt{T}}\Big)\Big], \tag{12}$$

where ϕ is the standard normal cumulative distribution function, $\alpha = 0.83992...$ is described above, and the optimal stopping time is given by

$$\inf\{t : W_t \ge b + \alpha\sqrt{T+t}\}.$$

It is interesting to remark that, if we drop the restriction that τ must be a stopping time, consider usual times t, and allow a priori knowledge of the future, the counterpart of formula (11) becomes

$$\int_{\Omega} \sup_{0\leq t\leq T} \frac{W_t(\omega)-b}{T+t} dP(\omega).$$

According to [2], the numerical value of the latter expression is

$$\int_0^{\infty} \exp(-2bx-2Tx^2)dx = \sqrt{\frac{\pi}{2T}}\exp\Big(\frac{b^2}{2T}\Big)\Big[1-\phi\Big(\frac{b}{2\sqrt{T}}\Big)\Big],$$

which is larger than the value in (12), as expected.

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