Applied Mathematical Sciences, Vol. 1, 2007, no. 46, 2287 - 2292

On the Ishikawa Process for $(\alpha)-$Mappings Fixed Point by Approximated Sequences

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Abstract. In this paper, we deal with the Ishikawa process and we give a new convergence theorem for the Ishikawa's scheme iteration for the $(\alpha)-$mappings in Banach spaces.

Mathematics Subject Classification: 47H09, 49M05

Keywords: $(\alpha)-$mapping, Common fixed point, Ishikawa process, Opial's condition, uniformly convex, weak convergence

1. Introduction

Throughout this paper, X denotes a Banach space and C a nonempty closed convex subset of X.
A mapping on C into C is said to be (α)-mapping if there is two positive constants a_T and b_T such that $a_T + 2b_T = 1$ and

$$\| Tx - Ty \| \leq a_T \| x - y \| + b_T(\| Tx - y \| + \| Ty - x \|), \tag{1.1}$$

for all $x, y \in C$. That is a nonexpansive mapping is an $(\alpha)-$mapping but the converse is not true, see [2]. It is already known that the $(\alpha)-$mapping posses a fixed point when X is a Hilbert space and C is closed bounded convex, see

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[2] and [5]. A method to check for the existence of fixed points is the iteration procedures, the most popular fixed point iteration in use are those of Ishikawa [3] and Mann [4], this two iterations were originally developped to provide ways of computing fixed points for maps for which repeated function iteration failed to converge.
For a map T on X, the Mann iteration scheme is defined by

$$x_0 \in X,\ x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n, n \geq 1, \tag{1.2}$$

where $0 \leq \alpha_n \leq 1$.
While the Ishikawa iteration scheme is given by

$$\begin{aligned} x_0 \in X \quad , \quad x_{n+1} &= \alpha_n T y_n + (1-\alpha_n)x_n, \\ y_n &= \beta_n T x_n + (1-\beta_n)x_n, n \geq 1, \end{aligned} \tag{1.3}$$

where $0 \leq \alpha_n, \beta_n \leq 1$.
One can see immediately that the Ishikawa iteration is more stronger than the Mann's one, it suffices to set each $\beta_n = 0$.
An other more general iteration scheme which is in use is the following

$$\begin{aligned} x_0 \in X \quad , \quad x_{n+1} &= \alpha_n T y_n + (1-\alpha_n)S x_n \\ and \qquad y_n &= \beta_n T x_n + (1-\beta)x_n,\ n \leq 0, \end{aligned} \tag{1.4}$$

where S is a mapping on X, and $0 < \alpha \leq \alpha_n, \beta_n \leq \beta < 1$.
If we set $S = I$ where I is the identity map on X, then we obtain the Ishikawa process.

2. Preliminaries

Let X be a Banach space, C a nonempty closed convex subset of X and T an $(\alpha)-$mapping on C into itself. We denote by $F(T)$ the set of fixed points of the mapping T, in light of [2] we know that $F(T)$ is not empty in the case that X is a Hilbert space and C is a bounded closed and convex subset of X. Recall that a Banach space X is said to satisfy Opial's condition [6] if whenever a sequence $\{x_n\}$ in X converges weakly to x the following occurs

$$\limsup_{n\to\infty} \parallel x_n - x \parallel < \limsup_{n\to\infty} \parallel x_n - y \parallel,\ \forall y \neq x. \tag{2.1}$$

It is known that Hilbert spaces satisfy Opial's condition. Let $\{x_n\}$ be a bounded sequence of elements of X, then the asymptotic radius of $\{x_n\}$ at $x \in X$ is the number

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \parallel x_n - x \parallel \tag{2.2}$$

For fixed $\{x_n\}$, (2.2) defines a continuous convex nonnegative real values function of x.
For a given nonempty closed convex subset K of X, the asymptotic radius of $\{x_n\}$ in K is the number given by

$$r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}. \tag{2.3}$$

For K define the (possibly empty) set

$$A_K(\{x_n\}) = \{y \in K : \limsup_{n\to\infty} \| x_n - y \| = r(K, \{x_n\})\}. \tag{2.4}$$

We already know that if X is reflexive then $A_K(\{x_n\})$ is a nonempty bounded closed convex set and if X is uniformly convex then the set $A_K(\{x_n\})$ consists of a single point.

3. Main result

We shall make use of the following lemma.

Lemma 3.1. [7] *Let X be a uniformly convex Banach space, let $\{t_n\}$ be a sequence of positive real numbers such that $0 < \alpha \leq t_n \leq \beta < 1$, for all $n \in N$, and let $\lambda \geq 0$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\limsup_{n\to\infty} \| x_n \| \leq \lambda$ and $\limsup_{n\to\infty} \| y_n \| \leq \lambda$.*
If $\lim_{n\to\infty} \| t_n x_n + (1-t_n) y_n \| = \lambda$, then $\limsup_{n\to\infty} \| x_n - y_n \| = 0$.

Lemma 3.2. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X. Let S and T be two $(\alpha)-$mappings on C into itself. Then $F(S) \cap F(T)$ is nonempty if and only if, the sequence $\{x_n\}_{n\in N}$ given by (1.4) is bounded and*

$$\lim_{n\to\infty} (x_n - Tx_n) = \lim_{n\to\infty} (x_n - Sx_n) = 0.$$

Proof:
Suppose that $F(S) \cap F(T) \neq \emptyset$, let $z \in F(S) \cap F(T)$. By an easy computation we can check that $\| x_{n+1} - z \| \leq \| x_n - z \|$ and
$\| Ty_n - z \| \leq \| y_n - z \|$, set $\lim_{n\to\infty} \| x_n - z \| = l$, then we get that

$$\begin{aligned} \limsup_{n\to\infty} \| Ty_n - z \| &\leq \limsup_{n\to\infty} \| y_n - z \| \\ &\leq \limsup_{n\to\infty} \| x_n - z \| = l. \end{aligned} \tag{3.1}$$

Since

$$\begin{aligned} \limsup_{n\to\infty} \| x_{n+1} - z \| &= \lim_{n\to\infty} \| \alpha_n (Ty_n - z) + (1-\alpha_n)(Sx_n - z) \| \\ &= l, \end{aligned} \tag{3.2}$$

applying lemma 3.1 we deduce that $\lim_{n\to\infty} \| Ty_n - Sx_n \| = 0$.
Now, by

$$\begin{aligned} \| x_{n+1} - z \| &\leq \alpha_n \| Ty_n - z \| + (1-\alpha_n) \| x_n - z \| \\ &\leq \alpha_n \| y_n - z \| + (1-\alpha_n) \| x_n - z \|, \end{aligned} \tag{3.3}$$

we get

$$\frac{\| x_{n+1} - z \| - \| x_n - z \|}{\alpha_n} \leq \| y_n - z \| - \| x_n - z \| . \tag{3.4}$$

Since the sequence $\{\alpha_n\}$ is bounded away from zero, we deduce that

$$l \leq \liminf_{n\to\infty} \| y_n - z \| . \tag{3.5}$$

Since $\| y_n - z \| \leq \| x_n - z \| \ \forall n \geq 1$, we obtain

$$l = \lim_{n\to\infty} \| y_n - z \| = \lim_{n\to\infty} \| \beta_n(Tx_n - z) + (1-\beta_n)(x_n - z) \| . \tag{3.6}$$

Thus, by lemma 3.1, we get $\lim_{n\to\infty} \| Tx_n - x_n \| = 0$.
Now, since

$$\begin{aligned} \| x_n - Sx_n \| &\leq \| x_n - Tx_n \| + \| Tx_n - Ty_n \| + \| Ty_n - Sx_n \| \\ &\leq (1 + \beta_n + \frac{2b_T}{a_T + b_T}) \| x_n - Tx_n \| \\ &\quad + \| Ty_n - Sx_n \|, \end{aligned} \tag{3.7}$$

thus $\lim_{n\to\infty} (x_n - Sx_n) = 0$.
Now conversely, let us assume that the sequence $\{x_n\}$ is bounded, $\{x_n - Sx_n\}$ and $\{x_n - Tx_n\}$ converge to zero as n tends to ∞.
In light of [1], it is already known that the function given by

$$r(u, \{x_n\}) = \limsup_{n\to\infty} \| x_n - u \|, \ \forall u \in C,$$

is a nonnegative, continuous and convex function of u. Moreover, for $\| u_k \| \to \infty$, we have $r(u_k, \{x_n\}) \to \infty$. Then there exists an element $u_0 \in C$ such that

$$r(u_0, \{x_n\}) = r_0 = \min_{u\in C} r(u, \{x_n\}).$$

If we set $A_C(\{x_n\}) = \{u \in C : \limsup_{n\to\infty} \| x_n - u \| = r_0\}$, then $A_C(\{x_n\})$ is $T-$invariant, indeed, for $z \in A_C(\{x_n\})$ we have

$$\begin{aligned} \limsup_{n\to\infty} \| Tz - x_n \| &\leq \limsup_{n\to\infty} (\| Tz - Tx_n \| + \| Tx_n - x_n \|) \\ &\leq \limsup_{n\to\infty} \| x_n - z \| \\ &\quad + \frac{2b_T}{a_T + b_T} \limsup_{n\to\infty} \| Tx_n - x_n \| \\ &\leq \limsup_{n\to\infty} \| x_n - z \| \end{aligned} \tag{3.8}$$

Thus $Tz \in A_C(\{x_n\})$, by the same arguments we can prove that $A_C(\{x_n\})$ is $S-$invariant. Since X is uniformly convexe, $A_C(\{x_n\})$ consists of a single point z which is a common fixed point for both S and T. ∎

Theorem 3.1. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and let S and T be two $(\alpha)-$mappings on C into itself such that $F(S) \cap F(T) \neq \emptyset$.*
Then the sequences $\{x_n\}$ and $\{y_n\}$ given by (1.4) converge weakly to a common fixed point of S and T. Moreover $\{x_n\}$ and $\{y_n\}$ have the same weak limits.

Proof:
For given $z \in F(S) \cap F(T)$, we have $\lim_{n\to\infty} \parallel x_n - z \parallel$ exists. Let z_1 and z_2 be two weak limits of $\{x_n\}$, next we show that $z_1 = z_2$. For that we have to show that $z_1, z_2 \in F(S) \cap F(T)$. Indeed, if we suppose that $Tz_1 \neq z_1$, and $\{x_{n_p}\}$ (resp. $\{x_{n_q}\}$) an arbitrary subsequence of $\{x_n\}$ which converges weakly to z_1 (resp. z_2), then by the Opial's condition we must have

$$\limsup_{p\to\infty} \parallel x_{n_p} - z_1 \parallel < \limsup_{p\to\infty} \parallel x_{n_p} - Tz_1 \parallel .$$

But since T is $(\alpha)-$mapping, we get that

$$\begin{aligned}\limsup_{p\to\infty} \parallel Tx_{n_p} - Tz_1 \parallel \quad &\leq \quad \limsup_{p\to\infty} \parallel x_{n_p} - z_1 \parallel \\ &\quad + \limsup_{p\to\infty} \frac{2b_T}{a_T + B_T} \parallel Tx_{n_p} - x_{n_p} \parallel .\end{aligned} \tag{3.9}$$

Using lemma 3.2, we obtain

$$\limsup_{p\to\infty} \parallel Tx_{n_p} - Tz_1 \parallel \leq \limsup_{p\to\infty} \parallel x_{n_p} - z_1 \parallel . \tag{3.10}$$

This leads to contradiction, then we must have $Tz_1 = z_1$.
By symmetry of z_1 and z_2 we deduce that $Tz_2 = z_2$. Now it remains to show that $z_1 = z_2$.Assume the contrary, then by Opial's condition we obtain

$$\begin{aligned}\lim_{n\to\infty} \parallel x_n - z_1 \parallel \quad &= \quad \lim_{p\to\infty} \parallel x_{n_p} - z_1 \parallel \\ &< \quad \lim_{p\to\infty} \parallel x_{n_p} - z_2 \parallel \\ &= \quad \lim_{n\to\infty} \parallel x_n - z_2 \parallel \\ &= \quad \lim_{q\to\infty} \parallel x_{n_q} - z_2 \parallel \\ &< \quad \lim_{n\to\infty} \parallel x_{n_q} - z_1 \parallel \\ &= \quad \lim_{n\to\infty} \parallel x_n - z_1 \parallel .\end{aligned} \tag{3.11}$$

This is a contradiction, and thus $z_1 = z_2$. One can use the same arguments as above to show that z_1 are in $F(S)$. Since $\{x_{n_p}\}$ was taken arbitrarily we deduce that the sequence $\{x_n\}$ converges weakly to a common fixed point of S and T.
Let us denote by $W(\{u_n\})$ the set of weak limits of the sequence $\{u_n\}$. To complete the proof, we show that $W(\{x_n\}) = W(\{y_n\})$, in fact, we know already that $\lim_{n\to\infty} \parallel y_n - z \parallel$ exists, now by following the same steps as above, we show that $\{y_n\}$ converges weakly and $W(\{y_n\}) \subset F(S) \cap F(T)$.
Since $\parallel x_n - y_n \parallel \leq \beta_n \parallel Tx_n - x_n \parallel$, then $\lim_{n\to\infty} \beta_n \parallel Tx_n - x_n \parallel = 0$, and the result is completely established. ∎

Theorem 3.2. *Let X be a Hilbert space, C a nonempty closed subset of X and T an $(\alpha)-$mapping on C into itself. Assume that there exists $x \in C$ such that the sequence $\{T^i x\}_{i\in N}$ is bounded.Then for any fixed element $x_0 \in C$, the sequence $\{x_n\}$ given by (1.3), with $\{\alpha_n\}$ and $\{\beta_n\}$ are taken so that $\alpha_n \in [s,t]$ and $\beta_n \in [0,t]$ or $\alpha_n \in [s,1]$ and $\beta_n \in [s,t]$ for some $0 < s \leq t < 1$, converges strongly to a fixed point of T.*

Proof:
By the Corollary 6 of [2] and lemma 3.2 the sequence $\{x_n\}$ converges weakly to a fixed point z of T, since $\lim_{n\to\infty} \| x_n - z \|$ exists we deduce that $\{x_n\}$ converges strongly to z. ∎

Acknowledgements

Much of the work of the third author was done when he was in National school of Applied Sciences of Oujda. The second authors and third authors were grateful to Professor Larbi ROUBI Directeur of ENSA-Oujda for their supports.

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Received: April 4, 2007