

On Convergence Theorem for Nonself I - Nonexpansive Mapping in Banach Spaces

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Abstract

Suppose that E be a uniformly convex Banach space, let K be a nonempty convex subset of E with P as a nonexpansive retraction. Let $T : K \rightarrow E$ be a given nonself mapping. The modified Ishikawa iterative scheme $\{x_n\}$ is defined by (1.8). We establish the weak convergence of a sequence of a modified Ishikawa iteration of an nonself I -nonexpansive map in a Banach space which satisfies Opial's condition.

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1. Introduction and preliminaries

Let E be a Banach space, K a nonempty, convex subset of E , and T a self map of K . Three most popular iteration procedures for obtaining fixed points of T , if they exist, are Mann iteration [12], defined by

$$u_1 \in K, \quad u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n, \quad n \geq 1, \quad (1.1)$$

and Ishikawa iteration [8], defined by

$$x_1 \in K, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \quad (1.2)$$

for certain choices of $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$.

In the above taking $\beta_n = 0$ in (1.2) we obtain iteration (1.1).

Let K be a closed convex bounded subset of uniformly convex Banach space $E = (E, \|\cdot\|)$ and T self-mappings of E . Then T is called nonexpansive on K if

$$\|Tx - Ty\| \leq \|x - y\| \quad (1.3)$$

for all $x, y \in K$. Let $F(T) := \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T .

The first nonlinear ergodic theorem was proved by Baillon [4] for general nonexpansive mappings in Hilbert space H : if K is a closed and convex subset of H and T has a fixed point, then for every $x \in K$, $\{T^n x\}$ is weakly almost convergent, as $n \rightarrow \infty$, to a fixed point of T . It was also shown by Pazy [1] that if H is a real Hilbert space and $(\frac{1}{n}) \sum_{i=0}^{n-1} T^i x$ converges weakly, as $n \rightarrow \infty$, to $y \in K$, then $y \in F(T)$.

The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [5] and Dotson [11] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [10] in metric spaces which we adapt to a normed space as follows: T is called a quasi-nonexpansive mapping provided

$$\|Tx - f\| \leq \|x - f\| \quad (1.4)$$

for all $x \in K$ and $f \in F(T)$.

Recall that a Banach space E is said to satisfy Opial's condition [14] if, for each sequence $\{x_n\}$ in E , the condition $x_n \rightarrow x$ implies that

$$\overline{\lim}_{n \rightarrow \infty} \|x_n - x\| < \overline{\lim}_{n \rightarrow \infty} \|x_n - y\| \quad (1.5)$$

for all $y \in E$ with $y \neq x$. It is well known from [14] that all l_p spaces for $1 < p < \infty$ have this property. However, the L_p spaces do not, unless $p = 2$.

The following definitions and statements will be needed for the proof of our theorem.

Let K be a subset of a normed space $E = (E, \|\cdot\|)$ and T and I self-mappings of K . Then T is called I -nonexpansive on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\| \quad (1.6)$$

for all $x, y \in K$ [7].

T is called I -quasi-nonexpansive on K if

$$\|Tx - f\| \leq \|Ix - f\| \quad (1.7)$$

for all $x, y \in K$ and $f \in F(T) \cap F(I)$.

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \rightarrow K$ such that $Px = x$ for all $x \in K$. A map $P : E \rightarrow E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P . A set K is optimal if each point outside K can be moved to be closer to all points of K . Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Remark 1.1. From the above definitions it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive.

There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and

weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshyn and Williamson [13]. Their analysis was related to the convergence of Mann iterates studied by Dotson [11]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [6]. In [9], the weakly convergence theorem for I -asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [3], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In [2], Rhoades and Temir considered T and I self-mappings of K , where T is an I -nonexpansive mapping. They established the weak convergence of the sequence of Mann iterates to a common fixed point of T and I . More precisely, they proved the following theorems.

Theorem (Rhoades and Temir [2]): *Let K be a closed convex bounded subset of uniformly convex Banach space E , which satisfies Opial's condition, and let T, I self-mappings of K with T be an I -nonexpansive mapping, I a nonexpansive on K . Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified Ishikawa iterates converges weakly to common fixed point of $F(T) \cap F(I)$.*

In the above theorem, T remains self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space. If, however, the domain K of T is a proper subset of E and T maps K into E then, the iteration formula (1.1) may fail to be well defined. One method that has been used to overcome this in the case of single operator T is to introduce a retraction $P : E \rightarrow K$ in the recursion formula (1.1) as follows: $u_1 \in K$,

$$u_1 \in K, \quad u_{n+1} = (1 - \alpha_n)u_n + \alpha_n P T u_n, \quad n \geq 1.$$

In this paper, we consider T and I nonself mappings of K , where T is an I -nonexpansive mappings. We establish the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of T and I .

Let E be a uniformly convex Banach space, let K be a nonempty convex subset of E with P as a nonexpansive retraction. Let $T : K \rightarrow E$ be a given nonself mapping. The modified Ishikawa iterative scheme $\{x_n\}$ is defined by

$$\begin{aligned} x_1 \in K, \quad x_{n+1} &= P((1 - \alpha_n)x_n + \alpha_n T y_n) \\ y_n &= P((1 - \beta_n)x_n + \beta_n T x_n), \quad n \geq 1, \end{aligned} \tag{1.8}$$

for certain choices of $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Clearly, if T is self maps, then (1.8) reduces to an iteration scheme (1.2).

2. The main result

Theorem 2.1. *Let K be a closed convex bounded subset of uniformly convex Banach space E , which satisfies Opial's condition, and let T, I non-self mappings of K with T be an I -nonexpansive mapping, I a nonexpansive on K . Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified Ishikawa iterates converges weakly to common fixed point of $F(T) \cap F(I)$.*

Proof. If $F(T) \cap F(I)$ is nonempty and a singleton, then the proof is complete. We will assume that $F(T) \cap F(I)$ is nonempty and that $F(T) \cap F(I)$ is

not a singleton.

$$\begin{aligned}
 \|x_{n+1} - f\| &= \|P((1 - \alpha_n)x_n + \alpha_n T y_n) - f\| \\
 &\leq (1 - \alpha_n) \|x_n - f\| + \alpha_n \|TP((1 - \beta_n)x_n + \beta_n T x_n) - T f\| \\
 &\leq (1 - \alpha_n) \|x_n - f\| + \alpha_n \|(1 - \beta_n)x_n + \beta_n T x_n - f\| \\
 &\leq (1 - \alpha_n) \|x_n - f\| + \alpha_n \|(1 - \beta_n)x_n + \beta_n T x_n - f + \beta_n T f - \beta_n T f\| \\
 &\leq (1 - \alpha_n) \|x_n - f\| + \alpha_n (1 - \beta_n) \|x_n - f\| + \alpha_n \beta_n \|x_n - f\| \\
 &= \|x_n - f\|
 \end{aligned}
 \tag{2.1}$$

Thus, for $\alpha_n \neq 0$ and $\beta_n \neq 0$, $\{\|x_n - f\|\}$ is a nonincreasing sequence. Then, $\lim_{n \rightarrow \infty} \|x_n - f\|$ exists.

Now we show that $\{x_n\}$ converges weakly to a common fixed point of T and I . The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in K . Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converge weakly to f and q , respectively. We will show that $f = q$. Suppose that E satisfies Opial's condition and that $f \neq q$ is in weak limit set of the sequence $\{x_n\}$. Then $\{x_{n_k}\} \rightarrow f$ and $\{x_{m_k}\} \rightarrow q$, respectively. Since $\lim_{n \rightarrow \infty} \|x_n - f\|$ exists for any $f \in F(T) \cap F(I)$ by Opial's condition, we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - f\| = \lim_{k \rightarrow \infty} \|x_{n_k} - f\| < \lim_{k \rightarrow \infty} \|x_{n_k} - q\| < \lim_{j \rightarrow \infty} \|x_{m_j} - f\| = \lim_{n \rightarrow \infty} \|x_n - f\|.$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F(T) \cap F(I)$.

References

- [1] A. Pazy, On the asymptotic behavior of iterates of nonexpansive mappings in Hilbert space, *Israel Journal of Mathematics* 26 (1977), no. 2, 197-204.
- [2] B. E. Rhoades and S. Temir, Convergence theorems for I-nonexpansive mapping, to appear in *International Journal of Mathematics and Mathematical Sciences*.
- [3] H. Zhou, R. P. Agarwal, Y. J. Cho, and Y. S. Kim, Nonexpansive mappings and iterative methods in uniformly convex Banach spaces, *Georgian Mathematical Journal* 9 (2002), no. 3, 591-600.
- [4] J. B. Baillon, Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert, *Comptes Rendus de l'Academie des Sciences de Paris, Serie A* 280(1975), no. 22, 1511-1514.
- [5] J. B. Diaz and F. T. Metcalf, On the set of subsequential limit points of successive approximations, *Transactions of the American Mathematical Society*

ciety 135(1969), 459-485.

[6] M. K. Ghosh and L. Debnath, Convergence of Ishikawa iterates of quasi-nonexpansive mappings, *Journal of Mathematical Analysis and Applications* 207(1997), no. 1, 96-103.

[7] N. Shahzad, Generalized I-nonexpansive maps and best approximations in Banach spaces, *Demonstratio Mathematica* 37 (2004), no. 3, 597-600.

[8] S. Ishikawa, Fixed points by a new iteration method, *Proc. Am. Math. Soc.* 44 (1974) 147-150.

[9] S. Temir and O. Gul, Convergence theorem for I-asymptotically quasi-nonexpansive mapping in Hilbert space, *Journal of Mathematical Analysis and Applications* 329 (2007) 759-765.

[10] W. A. Kirk, Remarks on approximation and approximate fixed points in metric fixed point theory, *Annales Universitatis Mariae Curie-Sklodowska. Sectio A* 51(1997), no. 2, 167-178.

[11] W. G. Dotson Jr., On the Mann iterative process, *Transactions of the American Mathematical Society* 149(1970), no. 1, 65-73.

[12] W. R. Mann, Mean value methods in iteration, *Proc. Am. Math. Soc.* 4 (1953) 506-510.

[13] W. V. Petryshyn and T. E. Williamson Jr., Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings, *Journal of Mathematical Analysis and Applications* 43 (1973), 459-497.

[14] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bulletin of the American Mathematical Society* 73 (1967), 591-597.

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