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Sensitivity Analysis in Parametrized Convex Optimization

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Abstract

In this paper, we study the behavior of the optimal value function associated to a convex minimization problem subject possibly to a nonlinear convex constraint without assuming the existence of optimal solutions.

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1 Introduction.

Let X be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Let us consider the following convex minimization problem

$$(1.1) \quad \inf_{x\in X}\{f(x) : \ h(x) \leq_{Y_1} y_1, \ Ax = y_2\}$$

where $h(x) \leq_{Y_1} y_1$ represents possibly a nonlinear inequality in the preordered space (Y_1, Y_+^1) and $Ax = y_2$ is a linear constraint in the space Y_2. The concept sensitivity analysis plays a central role in various applied sciences such as financial applications, risk analysis, signal processing, neural networks and any area where models are developed and it has been thoroughly studied by

several authors.
In this paper, we are concerned with the sensitivity analysis of the optimal value function $p(y_1, y_2)$ associated to the convex program (1.1) without assuming the existence of optimal solutions. Let us point out that the case when the operator h is linear has been studied by M. Moussaoui and A. Seeger [4].

2 Notations, definitions and preliminaries.

Throughout this paper X and Y are two locally convex topological vector spaces whose topological dual spaces are X^* and Y^*. The spaces X and X^* (resp. Y and Y^*) are paired in duality by the bilinear form $(x^*, x) \in X^* \times X \to \langle x^*, x\rangle := x^*(x)$ (resp. $(y^*, y) \in Y^* \times Y \to \langle y^*, y\rangle := y^*(y)$). We assume that the space Y is endowed with a preorder induced by a convex cone Y_+ i.e.

$$y_1 \leq_Y y_2 \iff y_2 - y_1 \in Y_+$$

and an abstract maximal element $+\infty$ will be adjoined to Y. The positive polar cone Y_+^* associated to Y_+ is defined by

$$Y_+^* := \{y^* \in Y^* \ : \ \langle y^*, y\rangle \geq 0, \quad \forall\, y \in Y_+\}.$$

Let us recall that for a mapping $h : X \to Y \cup \{+\infty\}$ we denote by

$$\text{Epi } h : \ = \ \{(x, y) \in X \times Y : h(x) \leq_Y y\}$$

its epigraph and by

$$\text{dom } h : \ = \ \{x \in X : h(x) \in Y\}$$

its effective domain. When dom $h \neq \emptyset$, one says that h is proper. The mapping h is said to be Y_+-convex if for every x_1, x_2 in X and every $\alpha \in]0, 1[$ we have:

$$h(\alpha x_1 + (1 - \alpha)x_2) \leq_Y \alpha h(x_1) + (1 - \alpha)h(x_2).$$

Considering a function $g : Y \to \mathbb{R} \cup \{+\infty\}$, we define the composed function $(g \circ h) : X \to \mathbb{R} \cup \{+\infty\}$ by

$$x \to (g \circ h)(x) := \begin{cases} g(h(x)) & \text{if } \ x \in \text{dom } h \\ \\ \sup\limits_{y \in Y} g(y) & \text{otherwise.} \end{cases}$$

When g is further assumed to be convex, this amounts to taking $(g \circ h)(x) = +\infty$ for $x \notin$ dom h whenever g is not constant over all the space Y. For a constant function $g \equiv c$, obviously one gets $(g \circ h)(x) = c$ for all $x \in X$.

To each function $f : X \to \mathbb{R} \cup \{+\infty\}$ we denote by $f^* : X^* \to \mathbb{R} \cup \{-\infty, +\infty\}$ its conjugate function defined for any $x^* \in X^*$ by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}.$$

Let $\bar{x} \in \text{dom } f$ and $\epsilon \geq 0$, the ϵ-subdifferential of f at $\bar{x}$ is the set

$$\partial_\epsilon f(\bar{x}) : \;=\; \{x^* \in X^* \;:\; f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \epsilon,\ \forall x \in X\}.$$

The exact subdifferential of f at $\bar{x} \in \text{dom } f$ is defined as

$$\partial f(\bar{x}) := \bigcap_{\epsilon > 0} \partial_\epsilon f(\bar{x})$$

Let C be a nonempty subset of X. The cone that it generates is

$$\mathbb{R}_+ C : = \bigcup_{\lambda \geq 0} \lambda C,$$

its indicator function is

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C \\ \\ +\infty & \text{otherwise} \end{cases}$$

and its support function δ^*_C defined on the dual space X^* is

$$\delta^*_C(x^*) = \sup_{x \in C} \langle x^*, x \rangle.$$

For any $\epsilon \geq 0$, the set $N_\epsilon(\bar{x}, C)$ of ϵ-normals to C at $\bar{x}$ is defined as the ϵ-subdifferential of the indicator function δ_C at $\bar{x}$ i.e.

$$N_\epsilon(\bar{x}, C) :=\ \partial_\epsilon \delta_C(\bar{x}) = \{x^* \in X^* :\ \langle x^*, x - \bar{x} \rangle \leq \epsilon,\ \forall x \in C\}.$$

In the sequel of this paper, we will need the inf-convolution $f \square g$ of two functions f and g on X, defined for all $x \in X$ by

$$\begin{aligned} (f \square g)(x) : \;&=\; \inf\{f(u) + g(x - u) : u \in X\} \\ \\ &=\; \inf\{f(u) + g(v) : u + v = x\}. \end{aligned}$$

For stating our main results, we will need below a formula due to [2]. For this, let us consider the following conditions

$$(C.Q.M.R) \begin{cases} X \text{ and } Y \text{ are locally convex spaces} \\ f : X \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ is convex and proper} \\ g : X \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ is convex and proper} \\ f \text{ is finite and continuous at some point of dom } g. \end{cases}$$

$$(C.Q.A.B) \begin{cases} X \text{ and } Y \text{ are Banach spaces} \\ f : X \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ is convex, proper and lower semicontinuous} \\ g : X \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ is convex, proper and lower semicontinuous} \\ \mathbb{R}_+[\text{dom } f - \text{dom } g] \text{ is a closed vector subspace of } X. \end{cases}$$

Hence we have

Theorem 2.1 *[2] Under one of each condition* $(C.Q.A.B)$ *or* $(C.Q.M.R)$ *we have for every* $x \in \operatorname{dom} f \cap \operatorname{dom} g$

$$\partial_\epsilon(f+g)(x) = \bigsqcup_{\substack{\epsilon_1 \geq 0, \epsilon_2 \geq 0 \\ \epsilon_1+\epsilon_2=\epsilon}} [\partial_{\epsilon_1} f(x) + \partial_{\epsilon_2} g(x)],$$

Proposition 2.2 *[4] Let* $f : X \times Z \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a convex proper function (*Z *is a locally convex real topological vector space). Suppose that the marginal function* $\varphi : Z \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ *defined on* Z *by*

$$\varphi(z) = \inf_{x \in X} f(x,z)$$

is finite at $\bar{z} \in Z$*. Then we have for all* $\epsilon \geq 0$

$$z^* \in \partial_\epsilon \varphi(\bar{z}) \Leftrightarrow \forall\ \eta > 0,\ \exists\ x \in X :\ (0, z^*) \in \partial_{\epsilon+\eta} f(x, \bar{z}).$$

3 Convex programs with possibly nonlinear inequality constraint.

In this section, we are concerned with the sensitivity analysis of the optimal value function $y \mapsto p(y)$ associated to the following convex program

$$(P): \qquad \text{Minimize } f(x) \text{ subject to } h(x) \leq_Y y,$$

where $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and proper function and $h : X \longrightarrow Y \cup \{+\infty\}$ is a proper and Y_+-convex mapping. In the sequel we will assume that Y_+ is a closed convex cone. Before evaluating the approximate subdifferential of p at any point $\bar{y} \in \operatorname{dom} p$, let us consider the following auxiliary functions defined on $X \times Y$ by

$$(x,y) \rightarrow F(x,y) \quad := \quad f(x)$$

$$(x,y) \rightarrow G(x,y) \quad := \quad \delta_{\text{Epi}h}(x,y).$$

Hence the optimal value function p can be written equivalently as

$$y \in Y \mapsto p(y) = \inf_{x \in X} \{F(x,y) + G(x,y)\}.$$

It is easy to see that F and G are convex and proper on $X \times Y$ and their associated conjugate functions are given, for any $(x^*, y^*) \in X^* \times Y^*$, by

$$\begin{aligned} F^*(x^*, y^*) &= f^*(x^*) + \delta_{\{0\}}(y^*) \\ G^*(x^*, y^*) &= (-y^* \circ h)^*(x^*) + \delta_{Y_+^*}(-y^*). \end{aligned}$$

For evaluating the approximate subdifferential of the optimal value function $y \mapsto p(y) := \inf\{f(x) : h(x) \leq_Y y\}$, we will need at first the expressions of the approximate subdifferentials of the functions F and G given by

Lemma 3.1

1) For any $\epsilon \geq 0$ and $(x, y) \in X \times Y$ we have

$$(x^*, y^*) \in \partial_\epsilon G(x, y) \Leftrightarrow \begin{cases} \exists\ \epsilon_1 \geq 0,\ \epsilon_2 \geq 0 \text{ with } \epsilon_1 + \epsilon_2 = \epsilon \\ (x, y) \in \text{Epi } h \\ x^* \in \partial_{\epsilon_1}(-y^* \circ h)(x) \\ y^* \in N_{\epsilon_2}(y - h(x), Y_+). \end{cases}$$

2) For any $\epsilon \geq 0$ and $(x, y) \in X \times Y$ such that f is finite at x we have

$$\partial_\epsilon F(x, y) = \partial_\epsilon f(x) \times \{0\}.$$

Proof. 1) We have

$$(x^*, y^*) \in \partial_\epsilon G(x, y) \Leftrightarrow G^*(x^*, y^*) + G(x, y) - \langle x^*, x\rangle - \langle y^*, y\rangle \leq \epsilon,$$

i.e.

$$(-y^* \circ h)^*(x^*) + \delta_{Y_+^*}(-y^*) + \delta_{\text{Epi}h}(x, y) - \langle x^*, x\rangle - \langle y^*, y\rangle \leq \epsilon, \tag{3.1}$$

By taking $z = y - h(x) \in Y_+$, we may rewrite (3.1) as:

$$[(-y^* \circ h)^*(x^*) + (-y^* \circ h)(x) - \langle x^*, x\rangle] + [\delta^*_{Y_+}(y^*) + \delta_{Y_+}(z) - \langle y^*, z\rangle] \leq \epsilon.$$

According to Fenchel's inequality, it follows that

$$\begin{cases} (-y^* \circ h)^*(x^*) + (-y^* \circ h)(x) - \langle x^*, x\rangle \geq 0 \\ \\ \delta^*_{Y_+}(y^*) + \delta_{Y_+}(z) - \langle y^*, z\rangle \geq 0 \end{cases}$$

and hence, there exist some $\epsilon_1 \geq 0$ and $\epsilon_2 \geq 0$ satisfying $\epsilon = \epsilon_1 + \epsilon_2$ and

$$\begin{cases} (-y^* \circ h)^*(x^*) + (-y^* \circ h)(x) - \langle x^*, z\rangle \leq \epsilon_1 \\ \\ \delta^*_{Y_+}(y^*) + \delta_{Y_+}(z) - \langle y^*, z\rangle \leq \epsilon_2, \end{cases}$$

i.e.

$$\begin{cases} x^* \in \partial_{\epsilon_1}(-y^* \circ h)(x) \\ (x,y) \in \text{Epi } h \\ y^* \in N_{\epsilon_2}(y - h(x), Y_+). \end{cases}$$

2) By applying the same arguments as above we obtain easily

$$\partial_\epsilon F(x,y) = \partial_\epsilon f(x) \times \{0\}.$$

□

Before stating our main results, we will need in what follows that the inf-convolution function $(x^*, y^*) \to (F^* \Box G^*)(x^*, y^*)$ is lower semicontinuous.

Let us consider the following conditions

$$(C.Q.M.R)_1 \begin{cases} X \text{ and } Y \text{ are locally convex spaces} \\ f : X \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ is convex and proper} \\ h : X \longrightarrow Y \cup \{+\infty\} \text{ is } Y_+\text{-convex and proper} \\ \exists\, a \in \text{dom } f \cap \text{dom } h \text{ such that } f \text{ is finite and continuous at } a. \end{cases}$$

$$(C.Q.A.B)_1 \begin{cases} X \text{ and } Y \text{ are Banach spaces} \\ f : X \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ is convex proper and lower semicontinuous} \\ h : X \longrightarrow Y \cup \{+\infty\} \text{ is } Y_+\text{-convex and proper} \\ \text{Epi } h \text{ is closed} \\ \mathbb{R}_+[\text{dom } f \times Y - \text{Epi } h] \text{ is a closed vector subspace of } X \times Y. \end{cases}$$

Lemma 3.2 *If one of each conditions $(C.Q.M.R)_1$ or $(C.Q.A.B)_1$ is satisfied then we have*

$$(F+G)^*(x^*, y^*) = (F^* \Box G^*)(x^*, y^*),\ \forall (x^*, y^*) \in X^* \times Y^*$$

and moreover the inf-convolution is exact.

Proof. 1) It is easy to see that from the condition $(C.Q.M.R)_1$ F is finite and continuous at $(a, h(a)) \in \text{dom } F \cap \text{Epi } h$ and hence from the Moreau-Rockafellar's condition in locally convex spaces (see [3] and [5]) we obtain the desired result.

2) Since dom $F =$ dom $f \times Y$ and dom $G =$ Epi h, hence by vertue of Attouch-Brézis's result [1] we have the lower semicontinuity and the exactness of the inf-convolution $F^* \Box G^*$ if the following condition is satisfied

$$\mathbb{R}_+[\text{dom } F - \text{dom } G] \text{ is a closed vector subspace of } X \times Y,$$

i.e.

$$\mathbb{R}_+[\text{dom } f \times Y - \text{Epi } h] \text{ is a closed vector subspace of } X \times Y.$$

□

Now, we state our main results.

Theorem 3.3 *Let us consider the marginal function* $p : Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ *associated to problem* (P) *defined by*

$$p(y) := \inf\{f(x) : h(x) \leq_Y y\}, \quad \forall y \in Y.$$

We assume that p *is finite at* $\bar{y} \in Y$ *and one of the conditions* $(C.Q.M.R)_1$ *or* $(C.Q.A.B)_1$ *is satisfied. Then we have for any* $\epsilon \geq 0$

$$y^* \in \partial_\epsilon p(\bar{y}) \Leftrightarrow \begin{cases} \forall\ \eta > 0,\ \exists\ x \in X,\ \exists\ \epsilon_1, \epsilon_2, \epsilon_3 \geq 0 : \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon + \eta, \\ 0 \in \partial_{\epsilon_1} f(x) + \ \partial_{\epsilon_2}(-y^* \circ h)(x) \\ y^* \in -Y_+^* \\ \langle y^*, h(x) - \bar{y} \rangle \leq \epsilon_3 \\ \bar{y} - h(x) \in Y_+. \end{cases}$$

Proof. Let us fix $\epsilon \geq 0$. From Proposition 2.2 we have

$$\partial_\epsilon p(\bar{y}) = \bigcap_{\eta > 0} \bigcup_{x \in X} \{y^* \in Y^* :\ (0, y^*) \in \partial_{\epsilon+\eta}(F + G)(x, \bar{y})\}$$

and because of the lower semicontinuity and the exactness of $F^* \square G^*$ which are guaranteed under each one of the conditions $(C.Q.M.R)_1$ or $(C.Q.A.B)_1$ (see Lemma 3.2) we have, for any $\eta > 0$ and $x \in X$, that

$$\partial_{\epsilon+\eta}(F + G)(x, \bar{y}) = \bigsqcup_{\substack{\epsilon_1, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 = \epsilon + \eta}} (\partial_{\epsilon_1} F(x, \bar{y}) + \partial_{\epsilon_2} G(x, \bar{y})). \tag{3.2}$$

Let us take $\eta > 0$, $x \in X$ and $y^* \in Y^*$ such that

$$(0, y^*) \in \partial_{\epsilon+\eta}(F + G)(x, \bar{y}),$$

from (3.2) there exist some $\epsilon_1, \beta \geq 0$ such that $\epsilon_1 + \beta = \epsilon + \eta$ and

$$(0, y^*) \in \partial_{\epsilon_1} F(x, \bar{y}) + \partial_\beta G(x, \bar{y}),$$

which means that there exist $x^* \in X^*$ and $y_1^*, y_2^* \in Y^*$ such that

$$\begin{cases} (x^*, y_1^*) \in \partial_{\epsilon_1} F(x, \bar{y}) \\ \\ (-x^*, y_2^*) \in \partial_\beta G(x, \bar{y}) \\ \\ y_1^* + y_2^* = y^*. \end{cases}$$

By Lemma 3.1 we have $y_1^* = 0$, which means that $y_2^* = y^*$ and $x^* \in \partial_{\epsilon_1} f(x)$, and from the same lemma we get also

$$(-x^*, y^*) \in \partial_\beta G(x, \bar{y}) \Leftrightarrow \begin{cases} \exists\ \epsilon_2 \geq 0,\ \epsilon_3 \geq 0\ :\ \epsilon_2 + \epsilon_3 = \beta \\ \bar{y} - h(x) \in Y_+ \\ -x^* \in \partial_{\epsilon_2}(-y^* \circ h)(x) \\ y^* \in N_{\epsilon_3}(\bar{y} - h(x), Y_+). \end{cases}$$

Hence, $(0, y^*) \in \partial_{\epsilon_1} F(x, \bar{y}) + \partial_\beta G(x, \bar{y})$ is equivalent to

$$\begin{cases} \exists\ x^* \in X^*,\ \exists\ \epsilon_2 \geq 0,\ \epsilon_3 \geq 0 \\ \epsilon_2 + \epsilon_3 = \beta \\ \bar{y} - h(x) \in Y_+ \\ x^* \in \partial_{\epsilon_1} f(x) \\ -x^* \in \partial_{\epsilon_2}(-y^* \circ h)(x) \\ y^* \in N_{\epsilon_3}(\bar{y} - h(x), Y_+). \end{cases}$$

The desired result is obtained by observing that

$$y^* \in N_{\epsilon_3}(\bar{y} - h(x), Y_+) \ \Leftrightarrow\ \begin{cases} y^* \in -Y_+^* \\ \\ \langle y^*, h(x) - \bar{y} \rangle \leq \epsilon_3. \end{cases}$$

□

The exact subdifferential of the optimal value function p is obtained by taking $\epsilon = 0$ in Theorem 3.3.

Corollary 3.4 *Assume that either* $(C.Q.M.R)_1$ *or* $(C.Q.A.B)_1$ *holds and that* p *is finite at* $\bar{y}$*, then we have*

$$y^* \in \partial p(\bar{y}) \Leftrightarrow \begin{cases} \forall\ \eta > 0,\ \exists\ x \in X,\ \exists\ \epsilon_1, \epsilon_2,\ \epsilon_3 \geq 0\ : \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = \eta, \\ 0 \in \partial_{\epsilon_1} f(x) +\ \partial_{\epsilon_2}(-y^* \circ h)(x) \\ y^* \in -Y_+^* \\ \langle y^*, h(x) - \bar{y} \rangle \leq \epsilon_3 \\ \bar{y} - h(x) \in Y_+ \end{cases}$$

Consider now the case of linear constraints. Let $A : X \to Y$ be a linear operator and consider the following convex programming problem with linear inequality

$$(H): \qquad \text{Minimize } f(x) \text{ subject to } A(x) \leq_Y y.$$

So applying Theorem 3.3 one gets the following result

Corollary 3.5 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex and proper function which is finite and continuous at some point of its domain, $A : X \to Y$ is a continuous linear operator and the marginal function $y \to p(y) := \inf\{f(x) : Ax \leq_Y y\}$ is finite at $\bar{y}$. Then we have for any $\epsilon \geq 0$*

$$y^* \in \partial_\epsilon p(\bar{y}) \Leftrightarrow \begin{cases} \forall\ \eta > 0,\ \exists\ x \in X,\ \exists\ \epsilon_1, \epsilon_2 \geq 0\ : \\ \epsilon_1 + \epsilon_2 = \epsilon + \eta \\ 0 \in \partial_{\epsilon_1} f(x) - A^* y^* \\ \bar{y} - Ax \in Y_+ \\ y^* \in -Y_+^* \\ \langle y^*, Ax - \bar{y} \rangle \leq \epsilon_2, \end{cases}$$

where $A^* : Y^* \to X^*$ stands for the adjoint operator of $A : X \to Y$.

Proof. From Lemma 3.1, it follows that for every $\alpha \geq 0$ we have

$$(x^*, y^*) \in \partial_\alpha G(x, \bar{y}) \quad \Leftrightarrow \quad \begin{cases} x^* = -A^* \circ y^* \\ \bar{y} - Ax \in Y_+ \\ y^* \in -Y_+^* \\ \langle y^*, Ax - \bar{y} \rangle \leq \alpha. \end{cases}$$

By means of the continuity of the operator A and the fact that f is finite and continuous at some point of its domain, the condition $(C.Q.M.R)_1$ is fulfilled and by applying Theorem 3.3 we obtain the desired result. □

By taking $Y_+ = \{0_Y\}$ in the above corollary we get

Corollary 3.6 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex and proper function which is finite and continuous at some point of its domain, $A : X \to Y$ is a continuous linear operator and the marginal function $y \to p(y) := \inf\{f(x) : Ax = y\}$ is finite at $\bar{y}$. Then we have for any $\epsilon \geq 0$*

$$y^* \in \partial_\epsilon p(\bar{y}) \Leftrightarrow \begin{cases} \forall\ \eta > 0,\ \exists\ x \in X : \\ Ax = \bar{y} \\ A^* y^* \in \partial_{\epsilon+\eta} f(x). \end{cases}$$

4 Application to Convex programs with possibly nonlinear inequality and linear equality constraints.

Now, let us consider the general convex parametric program

$$(H): \qquad \text{Minimize } f(x) \text{ subject to } \{h(x) \leq_{Y_1} y_1,\ A(x) = y_2\},$$

where $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ is a convex and proper function and Y_1, Y_2 are two locally convex real topological vector spaces. We suppose that the space Y_1 is equipped with a preorder induced by a convex closed cone Y_+. We denote by $Y := Y_1 \times Y_2$ the linear vector product space endowed with the partial order induced by the closed convex cone $Y_+ \times \{0_{Y_2}\}$ where 0_{Y_2} stands for the origin in Y_2 and by $k : X \longrightarrow Y \cup \{+\infty\}$ the mapping defined by $k(x) := (h(x), A(x))$ for any $x \in \text{dom } h$. It is easy to see that the epigraph of k is given by

$$\text{Epi } k : \;=\; \{(x, h(x) + y, A(x)),\; x \in \text{dom } h,\; y \in Y_+\}.$$

Hence, the problem (H) takes the form of problem (P) and therefore we can write the optimal value function associated to problem (H) in the form

$$(y_1, y_2) \in Y_1 \times Y_2 \mapsto p(y_1, y_2) := \inf_{x\in X}\{f(x) :\; k(x) \leq_Y (y_1, y_2)\}.$$

By translating the conditions $(C.Q.M.R)_1$ and $(C.Q.A.B)_1$ by means of the mappings h and A and the classical duality between $Y_1 \times Y_2$ and $Y_1^* \times Y_2^*$ we obtain

$$(C.Q.M.R)_2 \begin{cases} X,\; Y_1 \text{ and } Y_2 \text{ are locally convex spaces} \\ f : X \longrightarrow \mathbb{R}\cup\{+\infty\} \text{ is convex and proper} \\ h : X \longrightarrow Y_1\cup\{+\infty\} \text{ is proper and } Y_+\text{-convex} \\ A : X \longrightarrow Y_2\cup\{+\infty\} \text{ is a linear continuous operator} \\ \exists\, a \in \text{dom } f \cap \text{dom } h \text{ such that } f \text{ is finite and continuous at } a. \end{cases}$$

$$(C.Q.A.B)_2 \begin{cases} X,\; Y_1 \text{ and } Y_2 \text{ are Banach spaces} \\ f : X \longrightarrow \mathbb{R}\cup\{+\infty\} \text{ is convex, proper and lower semicontinuous} \\ h : X \longrightarrow Y_1\cup\{+\infty\} \text{ is proper and } Y_+\text{-convex} \\ \text{Epi } h \text{ is closed} \\ A : X \longrightarrow Y_2\cup\{+\infty\} \text{ is a linear continuous operator} \\ \mathbb{R}_+[\text{dom } f \times Y_1 - \text{Epi } h] \text{ is a closed vector subspace of } X \times Y_1. \\ \mathbb{R}_+[\text{dom } f \times Y_2 - \text{Gr } A] \text{ is a closed vector subspace of } X \times Y_2. \end{cases}$$

Let us observe that the positive polar cone of the convex cone $Y_+ \times \{0_{Y_2}\}$ is $Y_+^* \times Y_2^*$. Now, by means of the above conditions and Theorem 3.3 we obtain

Theorem 4.1 *If we assume that one of the conditions* $(C.Q.M.R)_2$ *or* $(C.Q.A.B)_2$ *is satisfied and that* p *is finite at* $(\bar{y}_1, \bar{y}_2)$*. Then, we have for any* $\epsilon \geq 0$

$$(y_1^*, y_2^*) \in \partial_\epsilon p(\bar{y}_1, \bar{y}_2) \Leftrightarrow \begin{cases} \forall\, \eta > 0,\; \exists\, x \in X,\; \exists\, \epsilon_1, \epsilon_2,\; \epsilon_3 \geq 0 \;: \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon + \eta, \\ 0 \in \partial_{\epsilon_1} f(x) + \;\partial_{\epsilon_2}(-y_1^* \circ h - A^* y_2^*)(x) \\ y_1^* \in -Y_+^* \\ \langle y_1^*, h(x) - \bar{y}_1\rangle \leq \epsilon_3, \\ \bar{y}_1 - h(x) \in Y_+ \\ A(x) = \bar{y}_2. \end{cases}$$

Corollary 4.2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex and proper function which is finite and continuous at some point of its domain, $A_1 : X \to Y_1$ and $A_2 : X \to Y_2$ are continuous linear operators. We suppose that the marginal function $(y_1, y_2) \to p(y_1, y_2) := \inf\{f(x) : A_1 x \leq_{Y_1} y_1;\ A_2 x = y_2\}$ is finite at $(\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2$. Then we have for any $\epsilon \geq 0$*

$$(y_1^*, y_2^*) \in \partial_\epsilon p(\bar{y}_1, \bar{y}_2) \Leftrightarrow \begin{cases} \forall\ \eta > 0,\ \exists\ x \in X,\ \exists\ \epsilon_1, \epsilon_2 \geq 0\ : \\ \epsilon_1 + \epsilon_2 = \epsilon + \eta, \\ 0 \in \partial_{\epsilon_1} f(x) - A_1^* y_1^* - A_2^* y_2^* \\ \bar{y}_1 - A_1 x \in Y_+,\ A_2 x = \bar{y}_2 \\ y_1^* \in -Y_+^*, \\ \langle y_1^*, A_1 x - \bar{y}_1 \rangle \leq \epsilon_2. \end{cases}$$

Proof. We use the same arguments as in Corollary 3.5. □

Remark 4.1 Let us point out that Corollaries 3.5, 3.6 and 4.2 have been studied recently by M. Moussaoui and A. Seeger in [4] without supposing that f is finite and continuous at some point of its domain.

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