A Regional Asymptotic Analysis of the Compensation Problem in Disturbed Systems

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Abstract

In this work, we present a regional asymptotic analysis of the problem of space compensation for a class of disturbed systems. It is an extension of previous works developed on the finite time remediability problem. We define and we give characterization results of the notion of regional asymptotic remediability and we study its relationship with the regional asymptotic notions of controllability, stability and stabilizability.

Then we show how to find the optimal control ensuring the asymptotic compensation, in the considered region, of a known or unknown disturbance. A characterization of the set of asymptotically remediable disturbances in this region is also presented.

The cases of multi-actuators and multi-sensors are examined and applications are given. We particularly show that in the regional asymptotic case, a system may be remediable without being controllable, stable or stabilizable. Other situations are also examined and illustrative examples are given.

Keywords: Asymptotic regional analysis, remediability, controllability, stability, disturbed systems, region, actuators, sensors

1   Introduction

In this paper, we consider a class of linear disturbed systems and we study, with respect to the output (observation), the possibility of regional asymptotic compensation of a known or unknown disturbance.

This work is an extension to the regional asymptotic case of previous works on the finite time remediability problem for linear parabolic, hyperbolic, continuous or discrete systems [1,2,3].

Concerning the asymptotic aspect, one know the great importance of the asymptotic analysis, particularly the notions of stability and stabilizability, in control theory and their direct relation with the spectral analysis in the case of linear systems.

A natural question is to consider the asymptotic version of the compensation problem and to study a possible extension of the developed approaches as well as the results obtained in the finite time case. Hence, by analogy with the relation between the remediability and the controllability examined in the finite time case, it is then natural to study the relationship between the asymptotic remediability and the asymptotic notions of controllability, stability and stabilizability. Let us note that for finite dimension systems, it is well known that if a linear system is controllable, then it is stabilizable. For distributed systems, the situation is not obvious and needs more precautions.

The regional aspect [3,7,9 ...] of the considered problem is motivated by the fact that a system can be asymptotically remediable in a region $\omega \subset \Omega$ but not on the whole domain Ω, and even if it is asymptotically remediable in Ω, the cost is reduced if we are interested only in a subregion $\omega \subset \Omega$.

This paper is organized as follows: We recall in paragraph 2, the notions of exact and weak remediability in the finite time case.

In paragraph 3 and under convenient hypothesis, we introduce and we characterize the regional asymptotic version of these notions first in the general case and then in the case of multi-actuators and multi-sensors. Regionally asymptotic efficient actuators ensuring the regional weak asymptotic compensation of any disturbance are particularly examined.

The problem of regional asymptotic remediability with minimum energy is studied in paragraph 4. We show how to find the optimal control ensuring the regional asymptotic compensation of a disturbance. Then we characterize the set of disturbances which are asymptotically remediable on a region ω of Ω.

In the fifth paragraph, we define and we characterize the notions of weak and exact regional asymptotic controllability and regionally asymptotic strategic actuators. Then we study their relationship with the weak and exact regional asymptotic remediability and regionally asymptotic efficient actuators. Indeed, we show that also in the asymptotic case, regionally strategic actuators are regionally efficient and that the converse is not true.

In the last paragraph, we study the nature of the relation between regional asymptotic remediability and the notions of stability and stabilizability. We particularly show that a system can be asymptotically remediable without being stable or stabilizable, but this relation depend, as it will be shown, depend on the choice of the sensors and the actuators.

Applications and illustrative examples are given.

2   Preliminaries

Let Ω be an open and bounded subset of R^n with a sufficiently regular boundary $\Gamma = \partial\Omega$, and let ω be a subregion of Ω. We consider without loss of generality, a class of linear distributed systems described by the following state equation:

$$(S) \begin{cases} \dot{z}(t) = Az(t) + Bu(t) + f(t) \\ z(0) = z_0 \end{cases} \tag{1}$$

where A generates a strongly continuous semi-group (s.c.s.g.) $(S(t))_{t \geq 0}$; $B \in \mathcal{L}(U, Z)$; U is the control space and $Z = L^2(\Omega)$ is the state space, U is supposed to be a Hilbert space. The state of the system at time t is given by:

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds + \int_o^t S(t-s)f(s)ds \tag{2}$$

The system (1) is augmented by the following regional output equation:

$$(E^\omega) \ \ y^\omega(t) = Ci_\omega p_\omega z(t) \tag{3}$$

where p_ω be is the restriction operator defined by:

$$p_\omega \ : \ \begin{array}{ccc} L^2(\Omega) & & L^2(\omega) \\ z & \longrightarrow & p_\omega z = z_{|\omega} \end{array} \tag{4}$$

i_ω is the adjoint operator of p_ω, it is defined by: .

$$i_\omega \ : \ \begin{array}{ccc} L^2(\omega) & & L^2(\Omega) \\ z & \longrightarrow & i_\omega z = \begin{cases} z \text{ in } \omega \\ 0 \text{ otherwise} \end{cases} \end{array}$$

and $C \in \mathcal{L}(Z, Y)$, Y is the observation space, a Hilbert space. The observation is given by:

$$
\begin{aligned}
y^\omega(t) \;=\;& C i_\omega p_\omega S(t) z_0 \;+ \int_0^t C i_\omega p_\omega S(t-s) B u(s) ds \\
& + \int_0^t C i_\omega p_\omega S(t-s) f(s) ds
\end{aligned}
\tag{5}
$$

Obviously in (5), if $u = 0$ and $f = 0$, $y^\omega(\cdot) = C i_\omega p_\omega S(\cdot) z_0$, the observation is then normal. But if $f \neq 0$ and $u \neq 0$, generally $y^\omega(\cdot) \neq C i_\omega p_\omega S(\cdot) z_0$.

The finite time problem consists to study the existence of an input operator B (actuators) with respect to the output operator C (sensors), ensuring the compensation at the final time T, of any disturbance, i.e.

For $f \in L^2(0, T; Z)$, there exists $u \in L^2(0, T; U)$ such that:

$$
y^\omega(T) = C i_\omega p_\omega S(T) z_0
$$

or equivalently

$$
C H_T^\omega u + R_T^\omega f = 0
\tag{6}
$$

where

$$
\begin{aligned}
H_T \;:\; L^2(0, T; U) \;&\to\; Z \\
u \;&\longrightarrow\; H_T u = \int_0^T S(T - t) B u(t) dt
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\overline{H}_T \;:\; L^2(0, T; Z) \;&\longrightarrow\; Z \\
f \;&\longrightarrow\; \int_0^T S(T - t) f(t) dt
\end{aligned}
\tag{8}
$$

and $R_T^\omega = C i_\omega p_\omega \overline{H}_T$, $H_T^\omega = i_\omega p_\omega H_T$. The definitions can be formulated as follows:

Definition 2.1
(i) We say that (S) augmented by (E^ω), (or $(S) + (E^\omega)$) is $\omega-$ exactly remediable on $[0, T]$, if for every $f \in L^2(0, T; Z)$, there exists $u \in L^2(0, T; U)$ such that

$$
C H_T^\omega u + R_T^\omega f = 0
$$

(ii) We say that $(S) + (E^\omega)$ is $\omega-$ weakly remediable on $[0, T]$, if for every $f \in L^2(0, T; Z)$ and $\varepsilon > 0$, there exists $u \in L^2(0, T; U)$ such that:

$$
\| C H_T^\omega u + R_T^\omega f \| < \varepsilon
$$

Characterization results on the exact and weak remediability in finite time are developed in [1,2,3]. It is also shown that the regional remediability is a weaker notion than the regional controllability. The cases where the input and the output are respectively given by actuators and sensors, are considered and illustrating examples are presented.

3 Regional asymptotic compensation

3.1 Problem statement

In this case, we consider the system (1) augmented by the output equation (3) with $f \in L^2(0, +\infty; Z)$ and $u \in L^2(0, +\infty; U)$. Let us consider consider the following operators

$$H_\omega^\infty = i_\omega p_\omega H^\infty \tag{9}$$

$$R_\omega^\infty f = C i_\omega p_\omega \overline{H^\infty} f \tag{10}$$

where

$$
\begin{aligned}
H^\infty \;:\; L^2(0, +\infty; U) &\longrightarrow Z \\
u &\longrightarrow H^\infty u = \int_0^{+\infty} S(t) B u(t) dt
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
\overline{H^\infty} \;:\; L^2(0, +\infty; Z) &\longrightarrow Z \\
f &\longrightarrow \int_0^{+\infty} S(t) f(t) dt
\end{aligned}
\tag{12}
$$

The considered problem, so called regional asymptotic remediability problem, consists to study, with respect to the output operator C and the region ω, the existence of an input one B ensuring regionally the asymptotic compensation of any disturbance, i.e.

For every $f \in L^2(0, +\infty; Z)$, there exists $u \in L^2(0, +\infty; U)$ such that:

$$CH_\omega^\infty u + R_\omega^\infty f = 0 \tag{13}$$

Let us note that if

$$\exists M_\omega(.) \in L^2(0, +\infty; R^+) \text{ such that } \|p_\omega S(t)\| \le M_\omega(t) \ ; \forall t \ge 0 \qquad (14)$$

the operators

$$p_\omega H^\infty u \equiv \int_0^{+\infty} p_\omega S(t) B u(t) dt$$

and

$$p_\omega H^\infty f \equiv \int_0^{+\infty} p_\omega S(t) f(t) dt$$

are well defined, then CH_ω^∞ and R_ω^∞ are also well defined.

Remark 3.1
1- If $(S(t))_{t\ge 0}$ is $\omega-$exponentially stable [4,5,8], i.e.

$$\exists M_\omega > 0 \text{ and } \alpha_\omega > 0 \text{ such that } \|p_\omega S(t)\| \le M_\omega e^{-\alpha_\omega t} \ ; \forall t \ge 0 \qquad (15)$$

we have (14), consequently CH_ω^∞ and R_ω^∞ are well defined.

2- In fact, we are concerned by the operators $K_{C,\omega}^\infty$ and $R_{C,\omega}^\infty$ defined by:

$$K_{C,\omega}^\infty u = \int_0^{+\infty} C i_\omega p_\omega S(t) B u(t) dt \text{ and } R_{C,\omega}^\infty f = \int_0^{+\infty} C i_\omega p_\omega S(t) f(t) dt$$

then we need a weaker hypothesis than (14). Indeed, we suppose that there exists a function $k_\omega(.) \in L^2(0, +\infty; R^+)$ such that:

$$\|C i_\omega p_\omega S(t)\| \le k_\omega(t) \ ; \forall t \ge 0 \qquad (16)$$

In this case, $K_{C,\omega}^\infty$ and $R_{C,\omega}^\infty$ are well defined and (13) becomes:

$$K_{C,\omega}^\infty u + R_{C,\omega}^\infty f = 0 \qquad (17)$$

and as it will be shown later in this paper, the regional exponential stability is not necessary.

Under hypothesis (16), the notions of exact and weak regional asymptotic remediability can be formulated as follows:

Definition 3.2 *We say that:*

1- $(S) + (E^\omega)$ is exactly $\omega-$remediable asymptotically, if for every $f \in L^2(0, +\infty; Z)$, there exists $u \in L^2(0, +\infty; U)$ such that:

$$K_{C,\omega}^\infty u + R_{C,\omega}^\infty f = 0$$

2- $(S) + (E^\omega)$ is weakly $\omega-$remediable asymptotically, if for every $f \in L^2(0, +\infty; Z)$ and every $\varepsilon > 0$, there exists $u \in L^2(0, +\infty; U)$ such that

$$\left\| K_{C,\omega}^\infty u + R_{C,\omega}^\infty f \right\| < \varepsilon$$

Let us note that for $T > 0$; $f \in L^2(0, +\infty; Z)$ and $u \in L^2(0, +\infty; U)$, the operators H^T and R^T defined by:

$$H^T u_T \equiv \int_0^T S(t) B u(t) dt$$

and

$$R_\omega^T f_T \equiv \int_0^T C i_\omega p_\omega S(t) f(t) dt$$

verifies

$$H^T u_T = H_T v_T \text{ and } R_\omega^T f_T = R_T^\omega g_T$$

where u_T and f_T are respectively the restrictions of the functions f and u to the interval $[0, T]$; $v_T(t) = u_T(T - t)$ and $g_T(t) = f_T(T - t)$. Moreover, under hypothesis (16)

$$
\begin{aligned}
K_{C,\omega}^\infty u + R_{C,\omega}^\infty f &= C i_\omega p_\omega H^T u_T + R_\omega^T f_T \\
&\quad + \int_T^{+\infty} C i_\omega p_\omega S(t) B u(t) dt + \int_T^{+\infty} C i_\omega p_\omega S(t) f(t) dt \\
&= C H_T^\omega v_T + R_T^\omega g_T + [\varepsilon_1(T) + \varepsilon_2(T)]
\end{aligned}
$$

with $\varepsilon_1(T) + \varepsilon_2(T) \longrightarrow 0$ when $T \longrightarrow +\infty$, then for any $f \in L^2(0, +\infty; Z)$ and $u \in L^2(0, +\infty; U)$, we have

$$\lim_{T \to +\infty} (C H_T^\omega v_T + R_T^\omega g_T) = K_{C,\omega}^\infty u + R_{C,\omega}^\infty f$$

3.2 Characterization

Let us note that for $f = -Bu$, we have $R^\infty_{C,\omega} f = -K^\infty_{C,\omega} u$, consequently

$$Im(K^\infty_{C,\omega}) \subset Im(R^\infty_{C,\omega}) \tag{18}$$

We have the following characterization result:

Proposition 3.3 *Under hypothesis (16):*

(i) $(S) + (E^\omega)$ *is exactly* $\omega-$*remediable asymptotically if and only if*

$$Im(R^\infty_{C,\omega}) \subset Im(K^\infty_{C,\omega})$$

this is equivalent to:

(ii) $\exists \gamma_\omega > 0$ *such that* $\forall\, \theta \in Y'$, *we have*

$$\left\| S^*(.) i_\omega p_\omega C^* \theta \right\|_{L^2(0,+\infty;Z')} \le \gamma_\omega \left\| B^* S^*(.) i_\omega p_\omega C^* \theta \right\|_{L^2(0,+\infty;U')} \tag{19}$$

<u>Proof:</u>
(i) derives from the definition.
(i) $\iff$ (ii) derives from the following lemma [4,5]   □

Lemma 3.4 *Let* X, Y, Z *be Banach reflexive spaces,* $P \in \mathcal{L}(X,Z)$ *and* $Q \in \mathcal{L}(Y,Z)$. *There is equivalence between:*

$$Im(P) \subset Im(Q)$$

and

$$\exists \gamma > 0 \text{ such that for any } z^* \in Z', \text{ we have } \|P^* z^*\|_{X'} \le \gamma \|Q^* z^*\|_{Y'}$$

Concerning the weak regional asymptotic remediability, we have the following result.

Proposition 3.5 *Under hypothesis (16):*
(i) $(S) + (E^\omega)$ *is weakly* $\omega-$*remediable asymptotically if and only if*

$$Im(R^\infty_{C,\omega}) \subset \overline{Im(K^\infty_{C,\omega})} \tag{20}$$

or equivalently

(ii)

$$Ker[B^*(R_{C,\omega}^\infty)^*] = Ker[(R_{C,\omega}^\infty)^*] \tag{21}$$

Proof:
(i) derives from the definition.
(ii) $\Longleftrightarrow$ (iii) is proved by considering the orthogonal spaces, and using (18) and the fact that

$$(K_{C,\omega}^\infty)^* = B^*(R_{C,\omega}^\infty)^* \tag{22}$$

$\square$

We examine hereafter the case where the system is exited by actuators and the observation is given by sensors [6,7].

3.3   Case of actuators and sensors

In the case of p actuators $(\Omega_k, g_k)_{1 \leq k \leq p}$, we have $U = I\!R^p$, $Z = L^2(\Omega)$ and

$$\begin{array}{rcl} B: & I\!R^p & \longrightarrow \quad L^2(\Omega) \\ & u(t) & \longrightarrow \quad Bu(t) = \sum_{k=1}^p g_k u_k(t) \end{array} \tag{23}$$

where $u = (u_1, \cdots, u_p)^{tr} \in L^2(0, +\infty; I\!R^p)$ and $g_k \in L^2(\Omega)$; $\Omega_k = supp(g_k) \subset \Omega$, we have

$$B^* z = (\langle g_1, z \rangle, \cdots, \langle g_p, z \rangle)$$

$\langle ., . \rangle$ is the inner product in $L^2(\Omega)$. It is easy to show the following result:

Corollary 3.6 $(S) + (E^\omega)$ *is exactly* $\omega-$ *remediable asymptotically, if there exists* $\gamma_\omega > 0$ *such that:* $\forall \theta \in Y'$, *we have*

$$\int_0^{+\infty} \|S^*(t) i_\omega p_\omega C^* \theta\|_{Z'}^2 \, dt \leq \gamma_\omega \int_0^{+\infty} \sum_{k=1}^p (\langle g_k, S^*(t) i_\omega p_\omega C^* \theta \rangle)^2 dt \tag{24}$$

Now, if the output is given by q sensors $(D_l, h_l)_{1 \leq l \leq q}$; $h_l \in L^2(\Omega)$; $D_l = supp(h_l)$ and for $l \neq j$, $D_l \cap D_j = \emptyset$, $Y = I\!R^q$ and the operator C is defined by:

$$\begin{array}{rcl} C: & L^2(\Omega) & \longrightarrow \quad I\!R^q \\ & z & \longrightarrow \quad Cz = (\langle h_1, z \rangle, \cdots, \langle h_q, z \rangle)^{tr} \end{array} \tag{25}$$

we have

$$C^*\theta = \sum_{l=1}^{q} \theta_l h_l \text{ for } \theta = (\theta_1, \cdots, \theta_q) \in \mathbb{R}^q \tag{26}$$

We have the following characterization result:

Corollary 3.7 $(S)+(E^\omega)$ *is exactly* $\omega-$ *remediable asymptotically if and only if, there exists* $\gamma_\omega > 0$ *such that* $\forall \theta = (\theta_1, \cdots, \theta_q) \in \mathbb{R}^q$

$$\int_0^{+\infty} \left\| \sum_{l=1}^{q} S^*(t)i_\omega p_\omega \theta_l h_l \right\|^2 dt \leq \gamma_\omega \int_0^{+\infty} \sum_{k=1}^{p} (\sum_{l=1}^{q} \langle g_k, S^*(t)i_\omega p_\omega \theta_l h_l \rangle)^2 dt \tag{27}$$

3.4   Regional asymptotically efficient actuators

We introduce hereafter the notion of regional asymptotically efficient actuators [1,2,3] and we give characterization results in the case of a class of linear systems.

Definition 3.8 *Actuators* $(\Omega_k, g_k)_{1 \leq k \leq p}$ *are* ω *-efficient asymptotically (or just* $\omega-$ *efficient), if the corresponding system* $(S)+(E^\omega)$ *is weakly* $\omega-$ *remediable asymptotically.*

For the characterization, we consider without loss of generality, the system (S) with a dynamics A of the form

$$Az = \sum_{n=1}^{+\infty} \lambda_n \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj} \tag{28}$$

where $\lambda_1, \lambda_2, \ldots$ are reals such that $\lambda_1 > \lambda_2 > \lambda_3 > \ldots$, $\{\varphi_{nj}, n \geq 1; j = 1, r_n\}$, is an orthonormal basis of Z, r_n is the multiplicity of the eigenvalue λ_n.
It is well known that A generates a s.c.s.g. $(S(t))_{t \geq 0}$ given by

$$S(t)z = \sum_{n=1}^{+\infty} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj} \tag{29}$$

Obviously, if

$$\sup_{n \geq 1} \lambda_n = \lambda_1 < 0 \tag{30}$$

$(S(t))_{t\geq 0}$ is exponentially stable and then $\omega-$exponentially stable. The system (S) is augmented by the regional output equation

$$(E^\omega)\ \ y^\omega = Ci_\omega p_\omega z$$

and the operator B is given by (23) , i.e.

$$Bu(t) = \sum_{k=1}^{p} g_k u_k(t)$$

For $n \geq 1$, let M_n be the matrix defined by

$$M_n = (\langle g_k, \varphi_{nj}\rangle)_{1\leq k\leq p; 1\leq j\leq r_n} \tag{31}$$

We have the following characterization result.

Proposition 3.9 *The actuators* $(\Omega_k, g_k)_{k=1,p}$ *are* $\omega-$ *efficient if and only if*

$$Ker(p_\omega C^\star) = \bigcap_{n\geq 1} Ker(M_n f_n^\omega)$$

where f_n^ω *is defined by*

$$f_n^\omega :\ \ \theta \in Y^\star\ \ \longrightarrow\ \ f_n^\omega(\theta) = (< C^\star\theta, \varphi_{n1} >_\omega, ..., < C^\star\theta, \varphi_{nr_n} >_\omega)^{tr} \in I\!\!R^{r_n} \tag{32}$$

$Y^\star$ *is the dual of* Y *and* $< ., . >_\omega$ *is the inner product in* $L^2(\omega)$.

Proof: For such systems and using the analyticity property, the proof is similar to that established in the finite time case. Indeed, $(S)+(E^\omega)$ is weakly $\omega-$remediable asymptotically if and only if

$$Ker[B^\star(R_{C,\omega}^\infty)^\star] = Ker[(R_{C,\omega}^\infty)^\star]$$

For $\theta \in Y^\star$, we have

$$(R_{C,\omega}^\infty)^\star\theta = \sum_{n\geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} < i_\omega p_\omega C^\star\theta, \varphi_{nj} > \varphi_{nj}$$

by analyticity, we have

$$(R_{C,\omega}^\infty)^\star\theta = 0 \iff \sum_{j=1}^{r_n} < i_\omega p_\omega C^\star\theta, \varphi_{nj} > \varphi_{nj} = 0 \ ;\ \ \forall\, n \geq 1$$

$$\iff\ p_\omega C^\star\theta = 0$$

then

$$Ker[(R_{C,\omega}^{\infty})^{\star}] = Ker[p_{\omega}C^{\star}] \qquad (33)$$

On the other hand

$$[B^{\star}(R_{C,\omega}^{\infty})^{\star}\theta](t) = \left(\sum_{n \geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} < g_k, \varphi_{nj} >< i_{\omega}p_{\omega}C^{\star}\theta, \varphi_{nj} > \right)_{k=1,p}^{tr}$$

and also by analyticity, we obtain

$$B^{\star}(R_{C,\omega}^{\infty})^{\star}\theta = 0 \iff \sum_{j=1}^{r_n} < g_k, \varphi_{nj} >< C^{\star}\theta, \varphi_{nj} >_{\omega}= 0 \; ; \; \forall \, n \geq 1, \; \forall \, k = 1, p$$

$$\iff M_n f_n^{\omega}(\theta) = 0 \; ; \; \forall \, n \geq 1$$

then

$$Ker[B^{\star}(R_{C,\omega}^{\infty})^{\star}] = \bigcap_{n \geq 1} Ker(M_n f_n^{\omega})$$

and hence

$$Ker[p_{\omega}C^{\star}] = \bigcap_{n \geq 1} Ker(M_n f_n^{\omega})$$

$\square$

Now, we assume that the output is given by q zone sensors $(D_l, h_l)_{1 \leq l \leq q}$ with $h_l \in L^2(D_l)$, $D_l = supp(h_l) \subset \Omega$ and $measure(D_l \cap \omega) > 0$ for $1 \leq l \leq q$. In this case, the functions $(h_l)_{l=1,q}$ are linearly independent, because $D_l \cap D_j = \emptyset$ for $l \neq j$, and measure $(D_l \cap \omega) > 0$, then

$$Ker[p_{\omega}C^{\star}] = \{0\}$$

and using (33), we have

$$Ker[(R_{C,\omega}^{\infty})^{\star}] = \{0\}$$

On the other hand

$$B^{\star}(R_{C,\omega}^{\infty})^{\star}\theta = \left(\sum_{n \geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} < g_k, \varphi_{nj} > \sum_{l=1}^{q} \theta_l < h_l, \varphi_{nj} >_{\omega} \right)_{k=1,p}^{tr}$$

consequently

$$B^\star(R_{C,\omega}^\infty)^\star\theta = 0 \iff \sum_{j=1}^{r_n} <g_k, \varphi_{nj}> \sum_{l=1}^{q} \theta_l <h_l, \varphi_{nj}>_\omega = 0 \ ; \ \forall \, k = 1, p \ ; \ \forall \, n \geq 1$$

$$\iff M_n G_{n,\omega}^{tr}\theta = 0 \ ; \ \forall \, n \geq 1$$

where $G_{n,\omega}$ is the matrix defined by

$$G_{n,\omega} = \left(<h_l, \varphi_{nj}>_\omega\right)_{\substack{l=1,q \\ j=1,r_n}}$$

then

$$Ker[B^\star(R_{C,\omega}^\infty)^\star] = \bigcap_{n\geq 1} Ker(M_n G_{n,\omega}^{tr})$$

and using (21), we have the following characterization result.

Proposition 3.10 *The actuators $(\Omega_k, g_k)_{k=1,p}$ are $\omega-$ efficient if and only if*

$$\bigcap_{n\geq 1} Ker(M_n G_{n,\omega}^{tr}) = \{0\} \tag{34}$$

It is easy to deduce the following corollary.

Corollary 3.11 *If there exists $n_0 \geq 1$ such that*

$$rank(M_{n_0} G_{n_0,\omega}^{tr}) = q \tag{35}$$

or such that

$$rank(G_{n_0,\omega}^{tr}) = q \tag{36}$$

and

$$rank(M_{n_0}) = r_{n_0} \tag{37}$$

then the actuators $(\Omega_k, g_k)_{k=1,p}$ are $\omega-$efficient.

4   Regional asymptotic remediability with minimum energy

In this part and under hypothesis (16), we consider the following optimal control problem:

For $f \in L^2(0, +\infty; Z)$, does exists a control $u \in L^2(0, +\infty; U)$ such that:

$$K^\infty_{C,\omega} u + R^\infty_{C,\omega} f = 0 \ ?$$

If u exists, is it optimal ?
Let

$$D_\omega = \left\{ u \in L^2(0, +\infty; U) \text{ such that } K^\infty_{C,\omega} u + R^\infty_{C,\omega} f = 0 \right\}$$

D_ω is supposed to be a non empty set. We consider the function:

$$J_\omega(u) = \left\| K^\infty_{C,\omega} u + R^\infty_{C,\omega} f \right\|^2_Y + \|u\|^2_{L^2(0,+\infty;U)}$$

The problem becomes

$$Min_{u \in D_\omega} J_\omega(u)$$

and will be resolved using an extension of the Hilbert Uniqueness Method (H.U.M.). For $\theta \in Y' \equiv Y$, we consider

$$\|\theta\|_{F_\omega} = \Big(\int_0^{+\infty} \| B^* S^*(t) i_\omega p_\omega C^* \theta \|^2_{U'} \, dt \Big)^{\frac{1}{2}}$$

$\|.\|_{F_\omega}$ is a semi-norm. We suppose that it is a norm, this is equivalent to assume that $(S) + (E^\omega)$ is weakly $\omega-$remediable asymptotically. Let

$$F_\omega = \overline{Y}^{\|.\|_{F_\omega}}$$

F_ω is a Hilbert space with the inner product

$$\langle \theta, \tau \rangle_{F_\omega} = \int_0^{+\infty} \langle B^* S^*(t) i_\omega p_\omega C^* \theta, B^* S^*(t) i_\omega p_\omega C^* \tau \rangle \, dt; \forall \theta, \tau \in F_\omega$$

Let $\Lambda^\infty_{C,\omega}$ be the operator defined by

$$\Lambda^\infty_{C,\omega} = K^\infty_{C,\omega} (K^\infty_{C,\omega})^*$$

$\Lambda^\infty_{C,\omega}$ has a unique extension as an isomorphism $F_\omega \to F'_\omega$ such that:

$$\langle \Lambda^\infty_{C,\omega} \theta, \tau \rangle_Y = \langle \theta, \tau \rangle_{F_\omega} \ ; \ \ \forall \theta, \tau \in F_\omega$$

and

$$\left\|\Lambda_{C,\omega}^{\infty}\theta\right\|_{F_{\omega}'} = \|\theta\|_{F_{\omega}} \; ; \forall \theta \in F_{\omega}$$

We show hereafter how to find the optimal control ensuring the regional asymptotic compensation of a disturbance f.

Proposition 4.1
If $R_{C,\omega}^{\infty}f \in F_{\omega}'$, then there exists a unique θ_f in F_{ω} such that

$$\Lambda_{C,\omega}^{\infty}\theta_f = -R_{C,\omega}^{\infty}f$$

and the control $u_{\theta_f} = (K_{C,\omega}^{\infty})^{*}\theta_f$ verifies

$$K_{C,\omega}^{\infty}u_{\theta_f} + R_{C,\omega}^{\infty}f = 0$$

Moreover, u_{θ_f} is optimal with

$$\left\|u_{\theta_f}\right\|_{L^2(0,+\infty;U)} = \|\theta_f\|_{F_{\omega}}$$

Proof: We have

$$\Lambda_{C,\omega}^{\infty}\theta_f = \int_0^{+\infty} Ci_{\omega}p_{\omega}S(t)BB^{*}S^{*}(t)i_{\omega}p_{\omega}C^{*}\theta_f dt = K_{C,\omega}^{\infty}u_{\theta_f} = -R_{C,\omega}^{\infty}f$$

D_{ω} is closed, convex and non empty. For $u \in D_{\omega}$, we have

$$J_{\omega}(u) = \|u\|_{L^2(0,+\infty;U)}^2$$

J_{ω} is strictly convex on D_{ω}, then it admits a unique minimum in $u^{\star} \in D_{\omega}$ characterized by

$$\langle u^{\star}, v - u^{\star}\rangle \geq 0; \forall v \in D_{\omega}$$

If $v \in D_{\omega}$, we have

$$
\begin{aligned}
\langle u_{\theta_f}, v - u_{\theta_f}\rangle &= \langle (K_{C,\omega}^{\infty})^{*}\theta_f, v - (K_{C,\omega}^{\infty})^{*}\theta_f\rangle \\
&= \langle \theta_f, K_{C,\omega}^{\infty}v - K_{C,\omega}^{\infty}(K_{C,\omega}^{\infty})^{*}\theta_f\rangle = \langle \theta_f, K_{C,\omega}^{\infty}v - \Lambda_{C,\omega}^{\infty}\theta_f\rangle = 0
\end{aligned}
$$

Since $u^{\star}$ is unique, then $u^{\star} = u_{\theta_f}$ and u_{θ_f} is optimal with

$$\left\|u_{\theta_f}\right\|^2 = \left\|(K_{C,\omega}^{\infty})^{*}\theta_f\right\|^2 = \langle \theta_f, \Lambda_{C,\omega}^{\infty}\theta_f\rangle = \|\theta_f\|_{F_{\omega}}^2$$

Remark 4.2 *This result can be extended to the case where the observation is not exact, i.e. is as follows*

$$z^\omega(t) = y^\omega(t) + e^\omega(t)$$

where $y^\omega(t)$ is the exact observation given by (3) and $e^\omega(t)$ is a measurement error. The result is similar.

Now, we give hereafter a characterization of the set $\mathcal{E}_\omega$ of disturbances f which are exactly $\omega-$remediable asymptotically, i.e.

$$\mathcal{E}_\omega = \left\{ f \in L^2(0, +\infty; Z) : \exists u \in L^2(0, +\infty; U) \text{ such that } K^\infty_{C,\omega} u + R^\infty_{C,\omega} f = 0 \right\}$$

Proposition 4.3 $\mathcal{E}_\omega$ *is the inverse image of F'_ω by the operator $R^\infty_{C,\omega}$, i.e.*

$$R^\infty_{C,\omega} \mathcal{E}_\omega = F'_\omega$$

Proof:
For $y \in F'_\omega$, there exists a unique θ in F_ω such that $\Lambda^\infty_{C,\omega} \theta = y$, then

$$K^\infty_{C,\omega} (K^\infty_{C,\omega})^* \theta = y$$

Let u be the control defined by

$$u = (K^\infty_{C,\omega})^* \theta$$

we have $K^\infty_{C,\omega} u = y$, and for $f = -Bu \in L^2(0, +\infty; L^2(\Omega))$, we have $K^\infty_{C,\omega} u = -R^\infty_{C,\omega} f = y$, then $y \in R^\infty_{C,\omega} \mathcal{E}_\omega$.

Conversely, let $y \in R^\infty_{C,\omega} \mathcal{E}_\omega$, then there exists $f \in L^2(0, +\infty; L^2(\Omega))$ such that $y = R^\infty_{C,\omega} f$ and $K^\infty_{C,\omega} u + R^\infty_{C,\omega} f = 0$ with $u \in L^2(0, +\infty; U)$.

If we identify $K^\infty_{C,\omega} u$ with the linear mapping $L_\omega : \theta \in Y \longrightarrow \langle K^\infty_{C,\omega} u, \theta \rangle$, we have:

$$L_\omega(\theta) = \langle K^\infty_{C,\omega} u, \theta \rangle = \langle u, (K^\infty_{C,\omega})^* \theta \rangle$$

Then $|L_\omega(\theta)| \leq \|u\|_{L^2(0,+\infty;U)} \cdot \|\theta\|_{F_\omega}$ and consequently L_ω is continuous on Y for the topology of F_ω and can be extended continuously, in a unique way, to the space F_ω. Then $L_\omega \in F'$ and $K^\infty_{C,\omega} u = R^\infty_{C,\omega} f = y \in F'_\omega$. $\qquad \square$

5 Regional asymptotic remediability and regional asymptotic controllability

In the finite time case, it is shown that the controllability is stronger than the remediability. In the asymptotic one, this relation is not obvious and needs more mathematical precautions. Hence, as it will be seen in the next paragraph (example 2), a system can be asymptotically remediable even if the asymptotic controllability problem is not well defined or has no sense.

In this part, with a convenient choice of spaces and operators and under convenient hypothesis, we define and we characterize the notion of regional asymptotic controllability and we study its relationship with the regional asymptotic remediability. More precisely, we show that also in the regional asymptotic case, the controllability remain stronger than the remediability. The case of multi-actuators is examined and an application with various illustrative situations is presented.

5.1 Regional asymptotic controllability

We consider the system described by the following equation

$$(S_0) \begin{cases} \dot{z}(t) &= Az(t) + Bu(t) \ ; \ t > 0 \\ z(0) &= z_0 \end{cases}$$

We suppose that A generates a s.c.s.g.$(S(t))_{t \geq o}$ such that:

$$\exists M_\omega(.) \in L^2(0, +\infty; R^+) \text{ such that } \|p_\omega S(t)\| \leq M_\omega(t) \ ; \ \forall t \geq 0 \qquad (38)$$

In this case, the notion of regional asymptotic controllability introduced hereafter is well defined.

Definition 5.1 *The system (S_0) is said to be exactly (resp. weakly) $\omega-$controllable asymptotically if*

$$Im(p_\omega H^\infty) = L^2(\omega) \ (\ resp. \ \overline{Im(p_\omega H^\infty)} = L^2(\omega) \ )$$

Using Lemma 3.4, it is easy to show that the system (S_0) is exactly $\omega-$controllable asymptotically if and only if

$$\exists \gamma_\omega > 0 \text{ such that } \| z^\star \|_{Z'} \leq \gamma_\omega \| \ (p_\omega H^\infty)^\star z^\star \|_{L^2(0,+\infty;U')}; \ \forall z^\star \in L^2(\omega)$$

or equivalently

$$\| z^\star \|_{Z'} \le \gamma_\omega \| B^\star S^\star(.)i_\omega z^\star \|_{L^2(0,+\infty;U')}; \ \forall z^\star \in L^2(\omega)$$

Concerning the weak regional asymptotic controllability, the system (S_0) is weakly ω-controllable asymptotically, if and only if

$$Ker[(p_\omega H^\infty)^\star] = \{0\}$$

This is equivalent to

The operator $= p_\omega H^\infty (H^\infty)^\star i_\omega$ is positive definite

In the case of an operator A given by (28):

$$Az = \sum_{n \ge 1} \lambda_n \sum_{j=1}^{r_n} < z, \varphi_{nj} > \varphi_{nj} \tag{39}$$

and (S_0) excited by p zone actuators, i.e.

$$Bu(t) = \sum_{k=1}^{p} g_k u_k(t)$$

For $n \ge 1$, let M_n be the matrix defined by

$$M_n = (\langle g_k, \varphi_{nj} \rangle)_{1 \le k \le p; 1 \le j \le r_n} \tag{40}$$

and $\gamma_n(\omega)$ defined by

$$\gamma_n(\omega) = \begin{pmatrix} \gamma_{n,1}(\omega) \\ . \\ . \\ . \\ \gamma_{n,r_n}(\omega) \end{pmatrix}$$

with

$$\gamma_{n,j}(\omega) = (\gamma_{nj,km}(\omega)_{\{k \ge 1; m=1, r_n\}}$$

and

$$\gamma_{nj,km}(\omega) = \langle \varphi_{nj}, \varphi_{km} \rangle_\omega$$

Then, using the analyticity property, (S_0) is weakly $\omega-$ controllable asymptotically if and only if

$$\bigcap_{n \geq 1} Ker[M_n \gamma_n(\omega)] = \{0\} \tag{41}$$

Such actuators are said to be asymptotically $\omega-$ strategic.

We have the following result showing that equally in the regional asymptotic case, the controllability is stronger than the remediability.

Proposition 5.2 *If* (S_0) *is exactly (respectively weakly)* $\omega-$ *controllable asymptotically, then* $(S)+(E^\omega)$ *is exactly (respectively weakly)* $\omega-$ *remediable asymptotically.*

Proof:

1) For $\theta \in Y'$, we have

$$\parallel S^\star(.)i_\omega p_\omega C^\star\theta \parallel^2_{L^2(0,+\infty;L^2(\Omega))} = \int_0^{+\infty} \parallel S^\star(t)i_\omega p_\omega C^\star\theta \parallel^2_{L^2(\Omega)} dt$$

$$\leq \int_0^{+\infty} \parallel S^\star(t)i_\omega \parallel^2 dt \parallel p_\omega C^\star\theta \parallel^2_{L^2(\omega)}$$

$$\leq M_\omega \parallel p_\omega C^\star\theta \parallel^2_{L^2(\omega)} \text{ with } M_\omega > 0$$

Since (S_0) is exactly $\omega-$ controllable asymptotically, there exists $\gamma_\omega > 0$ such that

$$\parallel p_\omega C^\star\theta \parallel^2_{L^2(\omega)} \leq \gamma_\omega \parallel B^\star S^\star(.)i_\omega p_\omega C^\star\theta \parallel^2_{L^2(0,+\infty;U')}$$

then

$$\parallel S^\star(.)i_\omega p_\omega C^\star\theta \parallel^2_{L^2(0,+\infty;L^2(\Omega))} \leq \gamma \parallel B^\star S^\star(.)i_\omega p_\omega C^\star\theta \parallel^2_{L^2(0,+\infty;U')}$$

with $\gamma = M_\omega \gamma_\omega^2 > 0$. The result is then given by proposition 3.3.

2) $(S) + (E^\omega)$ weakly $\omega-$ remediable asymptotically is equivalent to

$$Ker[B^\star(R^\infty_{C,\omega})^\star] = Ker[(R^\infty_{C,\omega})^\star]$$

i.e.

$$Ker[B^\star(R^\infty_{C,\omega})^\star] \subset Ker[(R^\infty_{C,\omega})^\star)]$$

this is equivalent to

$$Ker[(K_{C,\omega}^\infty)^\star] \subset Ker[(R_{C,\omega}^\infty)^\star]$$

because $(K_{C,\omega}^\infty)^\star = B^\star(R_C^\infty)^\star$. For $\theta \in Ker[(K_{C,\omega}^\infty)^\star]$, we have

$(K_{C,\omega}^\infty)^\star\theta = 0$, then $(p_\omega C^\star)\theta = 0$ because $Ker[(p_\omega H^\infty)^\star] = \{0\}$, then

$\theta \in Ker[p_\omega C^\star]$. Since $Ker(p_\omega C^\star) \subset Ker[(R_{C,\omega}^\infty)^\star]$, we have the result. □

Remark 5.3

The converse is not true, this is illustrated in the following paragraph.

5.2 Example 1

We consider the following diffusion system with a Dirichlet boundary condition:

$$(S_1) \begin{cases} \frac{\partial z(x,t)}{\partial t} = \Delta z(x,t) + \sum_{k=1}^p g_k(x)u_k(t) + f(x,t) \text{ in } \Omega\times]0,+\infty[\\ z(x,0) = z_0(x) \text{ in } \Omega \\ z(x,t) = 0 \text{ in } \partial\Omega\times]0,+\infty[\end{cases} \tag{42}$$

augmented by the following output equation given by q zone sensors

$$(E_1^\omega) \ \ y^\omega = (\langle h_1, z\rangle_\omega, \cdots, \langle h_q, z\rangle_\omega)^{tr}$$

In the one dimension case with $\Omega =]0,1[$, the Laplacian operator Δ admits an orthonormal basis of eigenfunctions defined by

$$\varphi_n(\xi) = \sqrt{2}\sin(n\pi\xi); n \geq 1$$

the corresponding eigenvalues are simple and given by

$$\lambda_n = -n^2\pi^2; n \geq 1$$

Δ generates a self adjoint s.c.s.g. $(S(t))_{t\geq0}$ defined by:

$$S(t)z = \sum_{n=1}^{+\infty} e^{-n^2\pi^2 t}\langle z, \varphi_n\rangle\varphi_n \tag{43}$$

and which is **exponentially stable**. $(S(t))_{t\geq0}$ is then $\omega-$exponentially stable and the operators

$$H^\infty u = \sum_{k=1}^{p} \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{-n^2\pi^2 t} u_k(t)dt \langle g_k, \varphi_n \rangle \varphi_n$$

and

$$\overline{H}^\infty f = \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{-n^2\pi^2 t} \langle f(.,t), \varphi_n \rangle \varphi_n dt$$

are well defined and $(S_1) + (E_1^\omega)$ is $\omega-$exactly remediable asymptotically if and only if , there exists $\gamma_\omega > 0$ such that: $\forall\, \theta = (\theta_1, \cdots, \theta_q) \in \mathbb{R}^q$, we have

$$\sum_{n\geq 1} \frac{1}{2n^2\pi^2} (\sum_{l=1}^{q} \theta_l \langle h_l, \varphi_n \rangle_\omega)^2 \leq \gamma_\omega \sum_{k=1}^{p} \int_0^{+\infty} (\sum_{n\geq 1} e^{-n^2\pi^2 t} \langle g_k, \varphi_n \rangle \sum_{l=1}^{q} \theta_l \langle h_l, \varphi_n \rangle_\omega)^2 dt$$

In the case of one sensor and one actuator, this inequality becomes:

$$\sum_{n\geq 1} \frac{1}{2n^2\pi^2} [\theta \langle h, \varphi_n \rangle_\omega]^2 \leq \gamma_\omega \int_0^{+\infty} [\sum_{n\geq 1} e^{-n^2\pi^2 t} \langle g, \varphi_n \rangle \theta \langle h, \varphi_n \rangle_\omega]^2 dt;\ \forall\, \theta \in \mathbb{R}$$

or equivalently

$$\sum_{n\geq 1} \frac{1}{2n^2\pi^2} [\langle h, \varphi_n \rangle_\omega]^2 \leq \gamma_\omega \int_0^{+\infty} [\sum_{n\geq 1} e^{-n^2\pi^2 t} \langle g, \varphi_n \rangle \langle h, \varphi_n \rangle_\omega]^2 dt$$

this is verified for example for $\omega = \Omega$, $g = h = \varphi_{n_0}$ with $n_0 \geq 1$. But the corresponding system (S_1) is not exactly Ω -controllable asymptotically because it is not weakly $\Omega-$ controllable asymptotically.

Concerning the weak asymptotic regional remediability, the general characterization result is given in proposition 3.10. In the case of an actuator (Ω_1, g_1) and a sensor (D, h), we have $p = q = 1$.

Let $n_0 \geq 1$ such that $\langle h, \varphi_{n_0} \rangle \neq 0$, then $rank(G_{n_0,\omega}^{tr}) = 1$.

The actuator (Ω_1, g_1) is $\Omega-$efficient asymptotically if $\langle g_1, \varphi_{n_0} \rangle \neq 0$, i.e.

$$\int_{\Omega_1} g_1(\xi) sin(n\pi\xi)d\xi \neq 0$$

For example, if $g_1 = \varphi_{n_0}$, (Ω_1, g_1) is $\Omega-$efficient asymptotically but not $\Omega-$strategic asymptotically, because the condition

$$\int_0^1 sin(n\pi\xi) sin(n_0\pi\xi)d\xi \neq 0; n \geq 1$$

is not satisfied. On the other hand, for $\omega = \left]0, \frac{1}{2}\right[$, $h = \varphi_1$ and $g = \varphi_2$, the actuator (Ω, g) is not Ω−efficient asymptotically because

$$\langle g, \varphi_n \rangle \langle h, \varphi_n \rangle_\Omega = 0; \forall\ n \geq 1$$

But (Ω, g) is ω−efficient asymptotically, because for $n_0 = 2$, we have

$$\langle g, \varphi_2 \rangle \langle h, \varphi_2 \rangle_\omega = \langle \varphi_1, \varphi_2 \rangle_\omega \neq 0$$

Then the relation (35) in corollary 3.11 is verified. Hence, also in the asymptotic case, a system can be regionally remediable without being it on the whole domain.

6 Asymptotic remediability and stabilizability

In this section, we study regionally the relation between the asymptotic compensation and the notions of stability and stabilizability. We particularly show that the problem of regional asymptotic compensation may be well defined even if the system is not regionally stable and that a non stable system may be remediable without being stabilizable. The nature of this relation depend on the choice of the actuators, the sensors and the other parameters of the considered system. To show this, we consider without loss of generality, the case where $\omega = \Omega$. We assume that (S_0) is not exponentially stable and that the unstable part is a finite dimension subspace of the state space Z.

These properties and other situations are illustrated with more details by considering as second example, a diffusion system with a Neuman boundary condition.

6.1 Stabilizability and actuators

First, let us recall the notion of stabilizability.

Definition 6.1
The system (S_0) is said to be exponentially stabilizable if there exists a feedback control

$$u = -Fz$$

such that the operator $A - BF$ generates a s.c.s.g. $(S_F(t))_{t \geq o}$ which is exponentially stable.

With the same notations, we consider the case of actuators $(\Omega_i, g_i)_{i=1,p}$ and an operator A defined by (39) and generating a non stable s.c.s.g. $(S(t))_{t\geq o}$. We have the following characterization result showing the relation between the unstable part and the choice of actuators stabilizing the system [8,9]

Proposition 6.2
We assume that the system (S_0) is not exponentially stable and that there exist a finite number $J \geq 1$ of non negative eigenvalues noted $\lambda_1, ..., \lambda_J$. Then the system (S_0) excited by p actuators $(\Omega_i, g_i)_{i=1,p}$ is stabilizable if and only if

i) $p \geq \sup\limits_{1\leq n\leq J} r_n$

ii) $rank M_n = r_n$ for $1 \leq n \leq J$, with M_n defined in (31).

Proof:

Let

$$z = \begin{pmatrix} z_u \\ z_s \end{pmatrix}$$

where z_u and z_s are respectively the projections of the state z on the unstable and the stable parts. We have

$$\begin{cases} (S_u) & \dot{z}_u(t) & = & A_u z_u(t) + PBu(t) \\ (S_s) & \dot{z}_s(t) & = & A_s z_s(t) + (I - P)Bu(t) \end{cases} \tag{44}$$

P is the projection operator on the unstable part and A_u is the diagonal matrix given by

$$A_u = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_J & & \\ & & & & & \ddots & \\ & & & & & & \lambda_J \end{pmatrix} \tag{45}$$

where λ_i appears r_i times for $i = 1, J$. The order of A_u is then $(\sum\limits_{i=1}^{J} r_i, \sum\limits_{i=1}^{J} r_i)$.

On the other hand, we have

$$PB = \begin{pmatrix} M_1^T \\ M_2^T \\ \vdots \\ M_J^T \end{pmatrix} \tag{46}$$

M_i^T is the transposal matrix of M_i. Under the condition ii) in the proposition, the **finite dimension** system (S_u) is controllable and hence is stabilizable. Consequently, there exists a control

$$u = -Fz_u$$

such that

$$\| e^{(A_u - PBF)t} \| \le \beta e^{-\alpha t} \text{ with } \beta, \alpha > 0 \text{ and } t \ge 0$$

Then

$$\| z_u(t) \| \le \| Pz_0 \| \beta e^{-\alpha t}$$

and

$$\| u(t) \| \le \| F \| \| Pz_0 \| \beta e^{-\alpha t}$$

Concerning the system (S_s), the operator A_s generates a s.c.s.g. which is exponentially stable, then there exist $\overline{\beta}, \overline{\alpha} > 0$, with $\overline{\alpha} < \alpha$, such that

$$\| z_s(t) \| \le \overline{\beta} \| (I - P)z_0 \| e^{-\overline{\alpha} t} + \overline{\beta} \| (I - P)z_0 \| \int_0^t e^{-\overline{\alpha}(t-\tau)} u(\tau) d\tau$$

$$\le \overline{\beta} \| (I - P)z_0 \| e^{-\overline{\alpha} t} +$$

$$\beta \overline{\beta} \| F \| \| Pz_0 \| (I - P)z_0 \| \int_0^t e^{-\overline{\alpha} t} e^{-(\alpha - \overline{\alpha})\tau} d\tau$$

We then have the result.   □

If a non stable system is stabilizable, we have $rank M_n = r_n$ for $1 \le n \le J$, i.e.

$$ker M_n = \{0\} \ ; \ 1 \le n \le J$$

then for convenient sensors (satisfying (34) in proposition 3.10), the system is also weakly asymptotically remediable.

On the other hand, a non stable system may be asymptotically remediable but not stabilizable. This is illustrated by the following example.

6.2   Example 2

We consider, without loss of generality, the following one dimension system with $\Omega =]0, 1[$ and a Neumann boundary condition:

$$(S2) \begin{cases} \dfrac{\partial z(x,t)}{\partial t} = \Delta z(x,t) + f(x,t) + \sum_{i=1}^{p} g_i(x)u_i(t) \text{ in }]0,1[\times]0,+\infty[\\[2mm] z(x,0) = z_0(x) \text{ in }]0,1[\\[2mm] \dfrac{\partial z(0,t)}{\partial x} = \dfrac{\partial z(1,t)}{\partial x} = 0 \text{ in }]0,+\infty[\end{cases}$$

In this case, we have

$$S(t)z = \sum_{n \geq 0} e^{-n^2\pi^2 t} \langle z, \varphi_n \rangle \varphi_n$$

with

$$\varphi_n(\xi) = \sqrt{2}\cos(n\pi\xi); n \geq 1 \text{ and } \varphi_0 \equiv 1$$

The eigenvalues are given by

$$\lambda_n = -n^2\pi^2 \; ; \; n \geq 1 \text{ and } \lambda_0 = 0$$

$(S(t))_{t \geq 0}$ **is not exponentially stable** and the number of non negative eigenvalues is $J = 1$. The operators

$$H^\infty u = \sum_{i=1}^{p} \sum_{n \geq 0} \int_0^{+\infty} e^{-n^2\pi^2 t} u_i(t)dt \langle g_i, \varphi_n \rangle \varphi_n$$

$$\overline{H}^\infty f = \sum_{n \geq 0} \int_0^{+\infty} e^{-n^2\pi^2 t} \langle f, \varphi_n \rangle \varphi_n dt$$

and hence the asymptotic controllability problem **are not generally well defined**. The system (S_2) is augmented by the output equation:

$$(E_2) \;\; y = (\langle h_1, z \rangle, \cdots, \langle h_q, z \rangle)^{tr}$$

with $h_1, \cdots, h_q$ orthogonal to φ_0 , i.e. to the unstable part:

$$\langle h_i, \varphi_0 \rangle = 0; \; 1 \leq i \leq q$$

The operators K_C^∞ and R_C^∞ are then well defined and the characterization results are similar to those obtained in example 1 for a Dirichlet boundary condition.

On the other hand, concerning the stabilizability, in the case of one actuator, we have

$$p = 1 \geq r_0 = 1$$

and using proposition 6.2, the system is stabilizable if and only if

$$rank M_0 = r_0 = 1$$

i.e.

$$\langle g, \varphi_0 \rangle \neq 0$$

For $g = \varphi_{n_0}$ with $n_0 \geq 1$, we have $\langle g, \varphi_0 \rangle = 0$ and then the system is not stabilizable. But for example, if also $h = \varphi_{n_0}$, we have

$$\langle h, \varphi_0 \rangle = 0$$

The problem of asymptotic compensation is well posed. We have

$$\langle g, \varphi_{n_0} \rangle \langle h, \varphi_{n_0} \rangle = 1 \neq 0$$

The system is then remediable asymptotically.
This not means that the asymptotic remediability is weaker than the stabilizabilty. The relation between these two notions depend on the choice of the sensors and the actuators. Indeed:

If $g(x) = 2x$, we have $\langle g, \varphi_0 \rangle = 1$, the system is then stabilizable. On the other hand, for $h = \varphi_1$, we have

$$\langle h, \varphi_0 \rangle = 0$$

In this case, the problem of asymptotic compensation is well posed. Moreover, we have

$$\langle g, \varphi_1 \rangle \langle h, \varphi_1 \rangle \neq 0$$

and hence, the system is also asymptotically remediable.

Now, if $g = \varphi_0$, we have $\langle g, \varphi_0 \rangle \neq 0$ and then the system is stabilizable. But for $h = \varphi_{n_0}$ with $n_0 \geq 1$, we have

$$\langle g, \varphi_n \rangle \langle h, \varphi_n \rangle = 0 \; ; \; \forall n \geq 1$$

consequently, the system is not asymptotically remediable.

Obviously, with a non convenient choice of the sensors and the actuators, the system may be non remediable asymptotically and non stabilizable. Hence, for $g = \varphi_1$, we have $\langle g, \varphi_0 \rangle = 0$ and then the system is not stabilizable. Concerning the remediability, if $h = \varphi_2$, we have

$$\langle g, \varphi_n \rangle \langle h, \varphi_n \rangle = 0 \; ; \; \forall n \geq 1$$

then, the system is not also asymptotically remediable.

<u>Conclusion</u>

In this paper, we have presented a regional asymptotic analysis of the compensation problem. Indeed, under convenient hypothesis and a convenient choice of operators and spaces, we have first introduced and characterized the notions of weak and exact regional asymptotic remediability and regionally asymptotic efficient actuators. Then, we have introduced and characterized the notions of weak and exact regional asymptotic controllability and regionally asymptotic strategic actuators.

Using an extension of the Hilbert Uniqueness Method, we have equally shown how to find, with respect to the observation only, the optimal control ensuring regionally the asymptotic compensation of a known or unknown disturbance. We have also characterized the set of disturbances which are asymptotically remediable in a region ω of Ω.

We have shown that also in the asymptotic case, a system can be regionally remediable but not regionally controllable or without being remediable on the whole domain Ω.

The relation between the regional asymptotic compensation and the notions of stability and stabilizability is also examined. It is particularly shown that a system can be asymptotically remediable in a region without being stable or stabilizable in this region.

Applications to a diffusion system are given and various other situations are considered.

The results are developed essentially for a class of linear distributed systems and for zone sensors and actuators, but these results can be extended to other systems and, with a convenient choice of spaces, to the case where the input and output operators are not bounded (pointwise sensors and pointwise actuators).

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