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Two-Party Political Competition: A Geometric Study of the Nash Equilibrium in a Weighted Case

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Abstract

This paper investigates an abstract game of political competition between two parties. All political positions are represented by points in a plane, and the parties choose positions that are as close as possible to the greatest number of voters, that are divided into a finite number of types. To adapt the problem to various political landscapes (different countries, for example), one simply assumes that the distribution of voters is not uniform. This complexity can be represented by simply assigning an appropriate weight to each position in the policy plane.

The existence of Nash equilibria in the game is studied by a geometric argument. This approach, in addition to representing the voting population as a finite distribution of weights, represents the innovation of the present work.

An algorithm has been developed in order to search for the equilibrium position of a given population.

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1 Introduction.

Most of the works studying political competition and elections are based on the Spatial Theory of Voting, initially developed by Black [6] and Downs [7] with later contributions of Hinich and Pollard [11], Shepsle and Weingstag [18], Enlow and Hinich [8] or Hinich and Munger [10] among others.

The Nash equilibrium is studied in general models of competition. It was stated by first time by John Forbes Nash in his dissertation Non-cooperative games [12], as a way to obtain an optimum strategy for games with two or more players.

This paper presents a political competition two-party game where the existence of Nash equilibria is studied. The game is defined on a two-dimensional plane where each point represents a different political position. (For example, one axis might represent the spectrum of fiscal policy, while the other represents the spectrum of foreign policy.) The two players of the game represent political parties, who choose their positions in the plane to attract the largest possible number of voters. The voters are represented by n fixed points on the plane.

Each player is considered to capture the points which lie closer to its position than that of the other player. The perpendicular bisector of the players locations thus partitions the plane into two different voting regions. Each player wins the points in its own half-plane, and the winner will be the player whose region contains more points [17],[19], [4], [13].

To apply this game to a political landscape, we imagine that the players are two parties denoted p and q. Their locations in the plane are denoted $t^1, t^2 \in T = \mathbb{R}^2$, determined by the policies they offer (the set of points T is called the policy space). All the political positions appearing in the voter population are represented by a finite set of types $H = \{v_1, \ldots, v_n\} \subset \mathbb{R}^2$ [16], [1].

In order to adapt this model to a particular political reality, we stipulate that the voter positions v_i are not evenly distributed. That is to say, certain positions in the policy space will be supported by more voters. Extreme positions (with respect to the majority of political actions) usually have fewer supporters than moderate positions, for example. It thus seems more reasonable to consider a weighted distribution of voters.

In order to describe the political preferences of the voters, we work with a utility function based on the Euclidean distance between a voter position and a party position [2], [15], [20]. In this game, we choose to define the utility function γ of position v_i as

$$\gamma(t, v_i) = \frac{1}{d(t, v_i) + 1}, \tag{1}$$

where $d(t, v_i)$ stands for the Euclidean distance between policy t and position v_i. We consider a distribution of the different types according to a measure of probability of the form $F(\{v_i\}) = k_i$, where $k_1 + k_2 + \cdots + k_n = 1$ and all $k_i > 0$. Keeping these considerations in mind, we can model the problem in this way:

We trace the perpendicular bisector between t^1, t^2 (we assume $t^1 \neq t^2$), and consider the two half-planes so defined. We define $\Omega(t^1, t^2)$ as the subset of positions that prefer t^1 to t^2, i.e., those belonging to the half-plane containing t^1. The fraction ρ of voters that choose policy t^1 over t^2 is thus

$$\rho(t^1, t^2) = F(\Omega(t^1, t^2)) = \sum_{j=1}^{n_{t^1}} k_{i_j} \quad (t^1 \neq t^2) \tag{2}$$

We further assume that points which are equidistant between the two policies (those located on the bisector) prefer the policy t^1. Hence, in this game there are no indifferent voters. There are other ways to model this problem, see for example Persson & Tabellini [14] and Roemer [16].

The players payoff functions Π can thus be defined as

$$\left.\begin{aligned} \Pi^1(t^1, t^2) &= n \sum_{j=1}^{n_{t^1}} k_{i_j} \\ \Pi^2(t^1, t^2) &= n - \Pi^1(t^1, t^2) \end{aligned}\right\} \quad \text{if } t^1 \neq t^2, \tag{3}$$
$$\Pi^1(t^1, t^2) = \Pi^2(t^1, t^2) = \frac{n}{2} \quad \text{if } t^1 = t^2$$

If we define the weight of position v_i as weight $v_i = nk_i$, then the payoff of policy t^1 will be the sum of the weights of all positions located in the same half-plane as t^1, including the points on the bisector. Here we assume that the policies are distinct ($t^1 \neq t^2$). The second policy, t^2, follows the same pattern, except for positions on the bisector.

We note that the total payoff is equal to the number of voters:

$$\sum_{i=1}^{n} \text{weight}\, v_i = \sum_{i=1}^{n} nk_i = n$$

This game is a discrete version of the Downs game [16], [9], [3].

2 Nash equilibrium

2.1 Necessary conditions

Let us see the following necessary condition, that is very easy to prove:

Lemma 2.1 *In a game with complementary payoffs, if for any position t of the first player the second player can always obtain a payoff of $n/2$ (half of the whole payoff), then the equilibrium position (t^1, t^2) must satisfy $\Pi^1(t^1, t^2) = \Pi^2(t^1, t^2)$.*

2.2 Necessary and suficient conditions

Definition 2.2 *We define the weight of a set $\{v_{i_1}, \ldots, v_{i_k}\}$ as $\sum_{j=1}^{k} \text{weight } v_{i_j}$.*

Definition 2.3 *A* minimal subset *of the set $\{v_1, \ldots, v_n\}$ is a subset of points whose weight is greater than $n/2$, and that itself contains no other subset with weight greater than $n/2$.*

Theorem 2.4 *Consider all the possible minimal subsets of $\{v_1, \ldots, v_n\}$. There exist Nash equilibria in the game if and only if the intersection of the convex hulls [5] of these subsets is not the empty set. Furthermore, the equilibria positions are the positions (t^1, t^2) such that t^1, t^2 are in the intersection.*

Proof If the intersection of the convex hulls is not empty and t^1, t^2 are in the intersection, then neither of the parties can win more than $n/2$ voters by moving to a different point.

If $\Pi^1(t', t^2) > n/2$, for example, then there must exist a minimal subset in the half-plane not containing t^2. If this is true, however, then the convex hull doesn't contain t^2, a contradiction.

If $\Pi^1(t^1, t^2) < n/2$, on the other hand, since the playoffs are complementary we must have $\Pi^2(t^1, t^2) > n/2$. In this case there exists a minimal subset whose convex hull doesn't contain t^1 (because the points are in the half-plane to which t^2 belongs). This is also a contradiction. We thus have both $\Pi^1(t^1, t^2) \geq n/2$ and $\Pi^2(t^1, t^2) \geq n/2$, so the only possible solution for complementary payoffs is $\Pi^1(t^1, t^2) = \Pi^2(t^1, t^2) = n/2$.

To recapitulate, we have $\Pi(t', t^2) \leq n/2 = \Pi(t^1, t^2)$ for both Π^1 and Π^2. The point (t^1, t^2) is therefore a position of equilibrium.

Indeed, these are the only positions of equilibrium. If t^1 did not belong to the intersection of the convex hulls of minimal subsets, then the second party could choose a position that separated t^1 from the convex hull of a set of points with weight greater than $n/2$. The second party could then gain a payoff greater than $n/2$, which is a contradiction since (t^1, t^2) is a position of equilibrium .

If the intersection of the convex hulls is empty, on the other hand, we may consider any position (t^1, t^2). If $\Pi^1(t^1, t^2) \leq n/2$, then the first party can choose a position t' that separates t^2 from a subset of points with weight greater than $n/2$ (i.e., a subset of points whose convex hull does not contain t^2).

Then we have $\Pi^1(t', t^2) > n/2 \geq \Pi^1(t^1, t^2)$, so (t^1, t^2) is not a position of equilibrium. □

2.3 Uniqueness

We will now show that the intersection of the convex hulls of minimal subsets described in the previous section contains at most a single point, unless the n voter positions all lie on a single line. We will also look at a particular case where the intersection must belong to the set of positions $\{v_1, \ldots, v_n\}$.

Lemma 2.5 *If there is no combination of points from the set* $\{v_1, \ldots, v_n\}$ *with weight* $n/2$*, then the intersection of the convex hulls of minimal subsets is at most in one point of the set:* $\{v_i\}$.

Proof We can always obtain a straight line R containing only one point from the set $\{v_1, \ldots, v_n\}$ such that the positions in the closed half-plane below R have total weight greater than $n/2$ and such that no line parallel to R and below R can meet the same condition. R thus defines a minimal subset of points whose convex hull is contained in the closed half-plane below R.

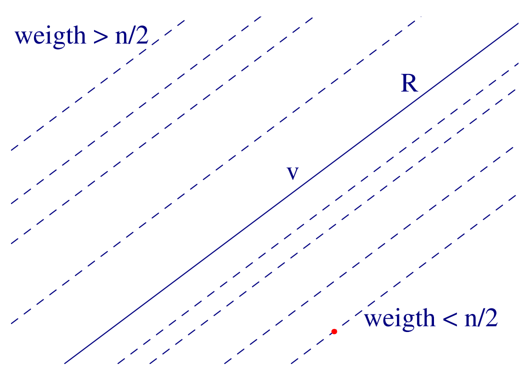

Figure 1: Case with no combination of points with weight $n/2$.

On the other hand, we next prove that the subset of points in the closed half-plane above R has a weight greater than $n/2$:

If the weight of the subset of points in the closed half-plane above R were less than $n/2$, then the subset of points in the open half-plane below R would have a weight greater than $n/2$. In this case there would also be a parallel line R' below R such that the points in the closed half-plane below R' have a weight greater than $n/2$, a contradiction. (The subset of points in the closed half-plane above R cannot have weight $n/2$, since we assume that there is no combination of points with a weight of exactly $n/2$). There is therefore a minimal subset of points whose convex hull is contained in the closed half-plane above R. The intersection of the two convex hulls must therefore be contained in R itself, which as we postulated earlier contains only one of the voter positions. The set of intersections of all the convex hulls, if it is not empty, must therefore be unitary and in a voter position (Figure 1). □

Now we can get to a more general result:

Lemma 2.6 *If the n voter positions are not contained in a single line, then the convex hulls of minimal subsets intersect is at most in one point.*

Proof Let v be a voter position on the boundary of the convex hull of the set of n points. We may choose v as the point with the lowest x-coordinate (or if the lowest x-coordinate is shared by more than one point, we may take v as the point with the lowest y-coordinate as well), then sort the remaining positions by their angle with respect to v: $\{v_1, \ldots, v_{n-1}\}$ (the angles can be measured from the vertical half-line below v, for example). Now consider the first point v_i such that weight$\{v_1, \ldots, v_{i-1}\} \geq n/2$ (or if such a point doesn't exist, define $v_i = v_{n-1}$).

If we have found $i < (n-1)$, then we have weight$\{v_1, \ldots, v_{i-1}, v\} > n/2$. The convex hull of a minimal subset of $\{v_1, \ldots, v_{i-1}, v\}$ is therefore contained in the closed half-plane below R connecting v and v_{i-1}.

Now consider the set $\{v_{i-1}, \ldots, v_{n-1}, v\}$, which is the complementary set to $\{v_1, \ldots, v_{i-2}\}$. The latter set must have a weight less than $n/2$, according to the definition of v_i. We thus have weight$\{v_{i-1}, \ldots, v_{n-1}, v\} > n/2$, and it follows that the convex hull of a minimal subset of $\{v_{i-1}, \ldots, v_{n-1}, v\}$ is contained in the upper closed half-plane defined by R. The intersection of the two convex hulls is therefore contained in R.

If $v_i = v_{n-1}$, then each closed half-plane defined by the line joining v, v_{n-1} contains subsets whose weights are greater than $n/2$. We thus have two convex hulls of minimal subsets with weight greater than $n/2$ whose intersection is included in the line joining v and v_{n-1}. Even in this case, we find that there exists a point v_j such that the intersection of the convex hulls is contained in the line joining v, v_j.

Since the n voter positions are not arranged in a line, however, we can choose a different point v' to begin the above argument; one that is on the boundary of the convex hull but not in R. By repeating the process, we will find another two minimal subsets whose convex hulls intersect somewhere along the line R' connecting v' to some other point v'_j. The intersection of all four convex hulls we have considered, if it is not emply, must therefore be the single point contained in the intersection of R and R'. So the intersection of all the convex hulls of minimal sets if it is not empty, must be in a single point. □

In the degenerate case that all the points are in a single line and there exists a combination of points with weight $n/2$, the intersection of the convex hulls is an infinite set.

This analysis leads to the following conclusion:

Theorem 2.7 *The equilibrium in the present game, if it exists, is the unique point (t, t) for some $t \in \mathbb{R}^2$. In other words, both parties will choose to offer the same policy, except in cases where the voter positions lie along a single line.*

3 The algorithm.

In this section, we develop an algorithm to find the equilibrium position if it exists. The algorithm is based on the technique described in Proposition 2.6, and on the following result:

Lemma 3.1 *If t belongs to the intersection of convex hulls of minimal subsets, and R is a line connecting t and some point of the initial position set, then the subset of positions contained in the set A defined by the union of the open half-line of R with t as its origin and the open half-plane defined by R, has a weight that is less than or equal than $n/2$.*

Proof If the positions in A had a weight greater than $n/2$, then the convex hull of the minimal subset of these points would also be contained in A, and then t would not belong to the convex hull, this is a contradiction. □

3.1 Development of the algorithm

Given an initial set of n points, we want to find the intersection of the convex hulls of minimal subsets.

Input: A set of n points in the plane not lying along a single line. Each of these points has an assigned weight.

- Step 1 (localization of the "candidate" point):

  Find the lines R and R' connecting v, v_j and v', v'_j by the method described in Lemma 2.6, such that the weights of the subsets included in each of the four closed half-planes so defined are greater than $n/2$. Find the intersection p of these two lines. Note that R and R' cannot be parallel.

- Step 2 (initialization of the weights):

  Trace the vertical line containing p, and determine the weight l of the points in the union of the open half-line below p and the open left half-plane (L). Then find the weight k of the points contained in the union of the open upper half-line and the open right half-plane (K). Note that if p belongs to the initial set of points, then $k+l+\text{weight}\, p = n$; otherwise $k+l=n$.

- Step 3 (Sort the other points in the set according to the "candidate"):

  Sort the points of the subset in L with respect to their angle from the half-line below p. Also sort by angle any points of the subset in K, taking as origin the half-line above p. The result is a sorted set $\{v_1, \ldots, v_{n-1}\}$ containing the points which are distinct from p (v_1 is the point with the smallest angle, v_2 is the point with the second smallest angle, and so on).

- Step 4 (calculation of the weights):

  For each point v_i ($i = 1, \ldots, n-1$), trace the line Q_i connecting (v_i, p) and define:

  - k_i as the weight of all points in the union of the open half-line to the left of p and the open half-plane below Q_i.
  - l_i as the weight of all points in the union of the open half-line to the right of p and the open half-plane above Q_i.

  Remark If v_i lies on the vertical line through p, then the sets that determine k_i and l_i are K and L respectively. In this case $k_i = k$ and $l_i = l$. We thus always have $k_i + l_i = k + l$.

  We also note that the weights k_i, l_i can be obtained recursively from k_{i-1}, l_{i-1} as follows:

  For each point, the weight k_i is equal to k_{i-1} *plus* the weight of the points in the open half-line (of Q_i) to the left of p and *minus* the weight of the points in the open half-line (of Q_i) to the right of p.

  Similarly, l_i is equal to l_{i-1} *plus* the weight of the points in the open half-line (of Q_i) to the right of p and *minus* the weight of the points in the open half-line (of Q_i) to the left of p.

  To begin the recursive calculation, we define $k_0 = k$, $l_0 = l$.

OUTPUT: If $k_i, l_i < n/2$ for every $i = i, \ldots, n-1$, then the unique position of equilibrium will be (p, p). Otherwise, as Lemma 3.1 shows, the intersection is an empty set and there is no equilibrium in the game.

3.2 Remarks on the algorithm

We now discuss the complexity of the above algorithm [5]:

To accomplish the first step of the algorithm, we may arrange the points by angle and define the recursive function

$$m_1 = \text{weight}\, v_1 + \text{weight}\, p, \tag{4}$$

$$m_i = m_{i-1} + \text{weight}\, v_i, \qquad i > 1. \tag{5}$$

v_j is the first point found for which $m_j > n/2$. The same approach is used to find v'_j.

The complexity of this task is linear with respect to n, because in the worst case we would have to calculate only the $n-1$ values $m_1, \ldots, m_{n-1}$.

The complexity of sorting n points is $O(n \log n)$. This step is necessary to find the initial point v and calculate the values k, l. The complexity of computing each value k_i, l_i calculated in the fourth step is $O(1)$. Since in the worst case we must calculate $n-1$ values for both k_i and l_i, the complexity of the fourth step is linear.

As sorting is the most complex task, the total complexity of the algorithm is $O(n \log n)$.

4 Conclusions

This work determined the Nash equilibrium of a competitive political game. The scenario presented is a discrete version of the Voronoi game in computational geometry, and also a discrete version of the Downs model in political economics. Furthermore, we have stated the conditions that must be satisfied by possible equilibrium positions.

To associate the model with a real political situation, we can divide the voter population into a finite number of types represented by specific points on a plane. Each of these positions is then assigned a weight representing the proportion of voters. Each party chooses a point on the plane representing their offered policy, and receives the maximum payoff when it minimizes the Euclidean distance to as many voters as possible. This treatment, when coupled with a geometric algorithm for determining the equilibrium position, represents a new insight into the solution of such games. Despite this simplification of the voter population, we obtain results similar to those presented by works where voter types are represented by a continuum [14], [16].

Except for the particular case where all voters are aligned along a single line of the plane, an equilibrium, if it exists, is attained only when both parties choose to offer the same policy to their voters. That is to say, the two parties will converge to essentially the same political program in order to maximize the number of voters.

Although in this paper we worked with a simplified (two-party) model, nowadays this treatment is adequate for the majority of countries. In most democracies there are two parties that represent the vast majority of voters, and we can observe that in general their policy offerings become more similar over time (they tend towards the equilibrium position). Our model succeeds in representing this fact.

To conclude, we note that the geometric scope in the treatment of the model allows us to obtain results related with geometric structures as the convex hulls and their intersections, that can have theoretical interest.

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References

[1] M. Abellanas, I. Lillo, M. López, and J. Rodrigo. Electoral strategies in a dynamical democratic system. Geometric models. European Journal of Operational Research, 175 (2006) 870–878.

[2] D. Achabal, W.L. Gurr, and V. Mahajan. Multiloc: A multiple store location decision model. Journal of Retailing, 58 (1982) 5–25.

[3] H.K. Ahn, S.W. Cheng, O. Cheong, M. Golin, and R. Van Oostrum. Competitive facility location: The Voronoi game. Theorical Comp. Sci., 310 (2004) 457–647.

[4] R. Aurenhammer and R. Klein. Voronoi diagrams. In J. R. Sack and Urrutia, editors, Handbook of Computational Geometry. Elsevier Science Publishers B. V. North-Holland, Amsterdam, 2000.

[5] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf. Computational Geometry-Algorithms and Applications. Springer, New York, second edition, 1997.

[6] D. Black. The Theory of Committees and Elections. Cambridge University Press, 1958.

[7] A. Downs. An Economic Theory of democracy. Harper and Row, 1957.

[8] J. Enlow and M. Hinich. Ideology, issues and the spatial theory of elections. Americal Political Science Review, 76 (1982) 493–501.

[9] S. Fekete and H. Meijer. The one-round Voronoi game replayed. Springer Lecture Notes in Computer Science, 2748 (2003) 150–161.

[10] M. Hinich and M. Munger. Ideology and Theory of political Choice. The University of Michigan Press, 1994.

[11] M. Hinich and W. Pollard. A new approach to the spatial theory of electoral competition. American Journal of Political Science, 25 (1981) 323–341.

[12] J. Nash. Non-cooperative games. The Annals of Mathematics, 1951 286–295.

[13] A. Okabe, B. Boots, K. Sugihara, and S. Chiu. Spatial Tessellations Concepts and Applications of Voronoi diagrams. John Wiley & Sons, Chichester, 2000.

[14] T. Persson and G. Tabellini. Political economics and public finance. In J. R. Sack and Urrutia, editors, NBER Working Papers Series, Handbook of Public Economics III. Elsevier Science, Amsterdam, 1999.

[15] C. Revelle. The maximum capture on "sphere of influence" location problem: Hotelling revisited on a network. Journal of Regional science, 26 n2 (1986) 343–358.

[16] J. Roemer. Political Competition. Harvard University Press, 2001.

[17] D. Serra and C. Revelle. Market capture by two competitors: The preemptive location problem. Journal of Regional science, 34 n4 (1994) 549–561.

[18] K. Shepsle and B. Weingast. Structure inducted equilibrium and legislative choice. Public Choice, 37 (1981) 503–519.

[19] M. Smid. Closest point problems in computational geometry. In J. R. Sack and Urrutia, editors, Handbook of Computational Geometry. Elsevier Science, Amsterdam, 1997.

[20] K. D. Vandell and C.C. Carter. Retail store location and market analysis: A review of the research. Journal of Real Estate Literature, 1 (1993) 13–45.

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