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Variance Balanced Block Designs with Repeated Blocks

Bronisław Ceranka and Małgorzata Graczyk

Department of Mathematical and Statistical Methods
Agricultural University of Poznań
Wojska Polskiego 28, 60-637 Poznań, Poland
bronicer@au.poznan.pl, magra@au.poznan.pl

Abstract

Some construction methods of the variance balanced block designs with repeated blocks are given. They are based on the incidence matrices of the balanced incomplete block designs with repeated blocks.

1 Introduction

Block designs with repeated blocks with the equireplications and equiblock sizes are widely used in several fields of research and they are available in the literature, see Foody and Hedayat (1977), Hedayat and Li (1979), Hedayat and Hwang (1984). However from the practical point of view, it may be not possible to construct the design with equisize blocks accomodating the equireplication of each treatment in all the blocks. Here we consider a class of block designs called variance balanced block designs which can be made available in unequal block sizes and for varying replications.
Let us consider v treatments arranged in b blocks in a block design with the incidence matrix $\mathbf{N} = [n_{ij}]$, $i = 1, 2, ..., v$, $j = 1, 2, ..., b$, where n_{ij} denotes the number of experimental units in the jth block getting the ith treatment. When $n_{ij} = 1$ or 0 for all i and j, the design is said to be binary. Otherwise it is said to be nonbinary. In this paper we consider binary block design, only. The following notation is used $\mathbf{r} = [r_1, r_2, ..., r_v]'$ is the vector of treatment replications, $\mathbf{k} = [k_1, k_2, ..., k_b]'$ is the vector of block sizes, with this $\mathbf{N}\mathbf{1}_b = \mathbf{r}$ and $\mathbf{N}'\mathbf{1}_v = \mathbf{k}$, where $\mathbf{1}_a$ is the $a \times 1$ vector of ones.
The information matrix for treatment effects $\mathbf{C}$ defined below as

$$\mathbf{C} = \mathbf{R} - \mathbf{N}\mathbf{K}^{-1}\mathbf{N}', \tag{1.1}$$

where $\mathbf{R} = \text{diag}\,(r_1, r_2, .., r_v)\,,\ \mathbf{K} = \text{diag}\,(k_1, k_2, ..., k_b)\,,$ is very suitable in determining properties of block design.
For several reasons, in particular from the practical point of view, it is desirable to have repeated blocks in the design. For example, some treatment combinations may be preferable than the others and also the design implementation may cost differently according to the design structure admitting or not repeated blocks. The set of all distinct blocks in a block design is called the support of the design and the cardinality of the support is denoted by b^* and is referred to as the support size of the design.
Though there have been balanced designs in various sens (see Puri and Nigam (1977), Caliński (1977)), we will consider a balanced design of the following type. A block design is said to be balanced if every elementary contrast of treatment is estimated with the same variance (see Rao (1958)). In this sense this design is also called a variance balanced block design.
It is well known that block design is a variance balanced if and only if it has

$$\mathbf{C} = \eta \left[\mathbf{I}_v - \frac{1}{v} \mathbf{1}_v \mathbf{1}_v^{'} \right], \tag{1.2}$$

where η is the unique nonzero eigenvalue of the matrix $\mathbf{C}$ with the multiplicity $v-1$, $\mathbf{I}_v$ is the $v \times v$ identity matrix. For binary block design

$$\eta = \frac{\sum_{i=1}^{v} r_i - b}{v-1} \tag{1.3}$$

(see Kageyama and Tsuji (1979)).
In particular case when block design is a balanced incomplete block design then $\eta = \frac{vr-b}{v-1}$.

2 Construction for v treatments

Let $\mathbf{N}_i,\ i = 1, 2, ..., t$, be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters $v,\ b_i,\ r_i,\ k_i,\ \lambda_i,\ b_i^*$. Let $\mathbf{C}_i$ be the $\mathbf{C}-$matrix of this design defined by $\mathbf{N}_i,\ i = 1, 2, ..., t$. Now, we form the matrix $\mathbf{N}$ as

$$\mathbf{N} = [\mathbf{N}_1\ \mathbf{N}_2\ ...\ \mathbf{N}_t]\,. \tag{2.1}$$

Theorem 1 *Block design with the incidence matrix* $\mathbf{N}$ *of the form (2.1) is the variance balanced block design with repeated blocks with the parameters*

v, $b=\sum_{i=1}^{t}b_i$, $r=\sum_{i=1}^{t}r_i$, $\mathbf{k}=\left[k_1\mathbf{1}_{b_1}^{'} \quad k_2\mathbf{1}_{b_2}^{'} \ldots k_t\mathbf{1}_{b_t}^{'}\right]^{'}$, $\lambda=\sum_{i=1}^{t}\lambda_i$, $b^*=\sum_{i=1}^{t}b_i^*$.

Proof. The matrix $\mathbf{C}$ of the block design (2.1) is

$$\mathbf{C}=r\mathbf{I}_v-\sum_{i=1}^{t}\tfrac{1}{k_i}\mathbf{N}_i\mathbf{N}_i^{'}=\sum_{i=1}^{t}\left[\left(r_i-\tfrac{1}{k_i}(r_i-\lambda_i)\right)\mathbf{I}_v-\tfrac{\lambda_i}{k_i}\mathbf{1}_v\mathbf{1}_v^{'}\right]$$

$$=\sum_{i=1}^{t}\mathbf{C}_i=\sum_{i=1}^{t}\eta_i\left[\mathbf{I}_v-\tfrac{1}{v}\mathbf{1}_v\mathbf{1}_v^{'}\right]=\eta\left[\mathbf{I}_v-\tfrac{1}{v}\mathbf{1}_v\mathbf{1}_v^{'}\right],$$

where $\eta=\sum_{i=1}^{n}\eta_i$, η_i is the unique nonzero eigenvalue of the matrix $\mathbf{C}_i$, $i=1,2,...,t$. So, the theorem is proved.

Example 1 *Let us consider the balanced incomplete block design with the parameters* $v=7,\ b_1=21,\ r_1=6,\ k_1=2,\ \lambda_1=1,\ b_1^*=21$ *with the incidence matrix* $\mathbf{N}_1$ *given throught the blocks* $(1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (1,7),$ $(2,3),\ (2,4),\ (2,5),\ (2,6),\ (2,7),\ (3,4),\ (3,5),\ (3,6),\ (3,7),\ (4,5),\ (4,6),$ $(4,7),\ (5,6),\ (5,7),\ (6,7)$ *and the balanced incomplete block design with the parameters* $v=7,\ b_2=14,\ r_2=6,\ k_2=3,\ \lambda_2=2,\ b_2^*=7$ *with the incidence matrix* $\mathbf{N}_2$ *given throught the blocks* $(1,2,4),\ (2,3,5),\ (3,4,6),\ (4,5,7),$ $(1,5,6),\ (2,6,7),\ (1,3,7)$, *each block is repeated two times. Based on the matrices* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *for* $t=2$ *we form the incidence matix* $\mathbf{N}$ *in the form (2.1) of the variance balanced block design with repeated blocks with the parameters* $v=7,\ b=35,\ r=12,\ \mathbf{k}=\left[2\cdot\mathbf{1}_{21}^{'} \quad 3\cdot\mathbf{1}_{14}^{'}\right]^{'},\ \lambda=3,\ b^*=28.$ *Hence the matrix* $\mathbf{C}$ *is given as* $\mathbf{C}=\frac{49}{6}\left[\mathbf{I}_7-\frac{1}{7}\mathbf{1}_7\mathbf{1}_7^{'}\right].$

Example 2 *Let us consider the balanced incomplete block design with the parameters* $v=9,\ b_1=24,\ r_1=8,\ k_1=3,\ \lambda_1=2,\ b_1^*=12$ *with the incidence matrix* $\mathbf{N}_1$ *given throught the blocks* $(1,2,6),\ (1,3,7),\ (1,4,8),\ (1,5,9),$ $(2,3,8),\ (2,4,9),\ (2,5,7),\ (3,4,5),\ (3,6,9),\ (4,6,7),\ (5,6,8),\ (7,8,9)$, *each block is repeated two times. And we consider the balanced incomplete block design with the parameters* $v=9,\ b_2=36,\ r_2=16,\ k_2=4,\ \lambda_2=6,\ b_2^*=18$ *with the incidence matrix* $\mathbf{N}_2$ *given throught the blocks* $(1,2,3,4),\ (1,2,4,9),$ $(1,2,5,7),\ (1,3,6,8),\ (1,3,8,9),\ (1,4,6,7),\ (1,5,6,9),\ (1,5,7,8),\ (2,3,5,6),$ $(2,3,6,7),\ (2,4,5,8),\ (2,6,8,9),\ (2,7,8,9),\ (3,4,5,8),\ (3,4,7,9),\ (3,5,7,9),$ $(4,5,6,9),\ (4,6,7,8)$, *each block is repeated two times. Based on the matrices* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *for* $t=2$ *we form the incidence matix* $\mathbf{N}$ *in (2.1) of the variance balanced block design with repeated blocks with the parameters* $v=9,\ b=60,\ r=24,\ \mathbf{k}=\left[3\cdot\mathbf{1}_{24}^{'} \quad 4\cdot\mathbf{1}_{36}^{'}\right]^{'},\ \lambda=8,\ b^*=30.$ *Hence we have*

$\mathbf{C} = \frac{39}{2}\left[\mathbf{I}_9 - \frac{1}{9}\mathbf{1}_9\mathbf{1}_9'\right].$

In particular case when $t = 1$ we have the following corollaries.

Corollary *1* *Block design with the incidence matrix* $\mathbf{N}$ *of the form*

$$\mathbf{N} = [\mathbf{N}_1 \qquad \mathbf{I}_v] \tag{2.2}$$

is the variance balanced block design with repeated blocks with the parameters $v, \quad b = b_1 + v, \quad r = r_1 + 1, \quad \mathbf{k} = \left[k_1\mathbf{1}_{b_1}' \quad \mathbf{1}_v'\right]', \quad \lambda = \lambda_1, \quad b^* = b_1^* + v.$

Example 3 *Let us consider the balanced incomplete block design with the parameters* $v = 7,\ b_1 = 14,\ r_1 = 6,\ k_1 = 3,\ \lambda_1 = 2,\ b_1^* = 7$ *with the incidence matrix* $\mathbf{N}_1$ *given throught the blocks* $(1,2,4),\ (2,3,5),\ (3,4,6),\ (4,5,7),\ (1,5,6),\ (2,6,7),\ (1,3,7)$, *each block is repeated two times. Based on the matrix* $\mathbf{N}_1$ *we form the incidence matix* $\mathbf{N}$ *in the form (2.2) of the variance balanced block design with repeated blocks with the parameters* $v = 7,\ b = 21,\ r = 7,\ \mathbf{k} = \left[3 \cdot \mathbf{1}_{14}' \quad \mathbf{1}_7'\right]',\ \lambda = 2,\ b^* = 14$. *Hence we have* $\mathbf{C} = \frac{14}{3}\left[\mathbf{I}_7 - \frac{1}{7}\mathbf{1}_7\mathbf{1}_7'\right].$

Corollary *2* *Block design with the incidence matrix* $\mathbf{N}$ *of the form*

$$\mathbf{N} = [\mathbf{N}_1 \quad \mathbf{1}_v] \tag{2.3}$$

is the variance balanced block design with repeated blocks with the parameters $v, \quad b = b_1 + 1, \quad r = r_1 + 1, \quad \mathbf{k} = \left[k_1\mathbf{1}_{b_1}' \quad v\right]', \quad \lambda = \lambda_1 + 1, \quad b^* = b_1^* + 1.$

Example 4 *Let us consider the balanced incomplete block design with the parameters* $v = 9,\ b_1 = 24,\ r_1 = 8,\ k_1 = 3,\ \lambda_1 = 2,\ b_1^* = 12$ *with the incidence matrix* $\mathbf{N}_1$ *given throught the blocks* $(1,2,6),\ (1,3,7),\ (1,4,8),\ (1,5,9),\ (2,3,8),\ (2,4,9),\ (2,5,7),\ (3,4,5),\ (3,6,9),\ (4,6,7),\ (5,6,8),\ (7,8,9)$, *each block is repeated two times. Based on the matrices* $\mathbf{N}_1$ *we form the incidence matix* $\mathbf{N}$ *in the form (2.3) of the variance balanced block design with repeated blocks with the parameters* $v = 9,\ b = 25,\ r = 9,\ \mathbf{k} = \left[3 \cdot \mathbf{1}_{24}' \quad 9\right]',\ \lambda = 3,\ b^* = 13$. *Hence we have* $\mathbf{C} = 7\left[\mathbf{I}_9 - \frac{1}{9}\mathbf{1}_9\mathbf{1}_9'\right].$

3 Construction for $v+1$ treatments

Let $\mathbf{N}_i,\ i=1,2$, be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters $v,\ b_i,\ r_i,\ k_i,\ \lambda_i,\ b_i^*$. Now, we form the matrix $\mathbf{N}$ as

$$\mathbf{N}=\left[\begin{array}{cc}\mathbf{N}_1\otimes\mathbf{1}_t^{'} & \mathbf{N}_2\otimes\mathbf{1}_u^{'}\\ \mathbf{1}_{b_1}^{'}\otimes\mathbf{1}_t^{'} & \mathbf{0}_{b_2}^{'}\otimes\mathbf{1}_u^{'}\end{array}\right]. \tag{3.1}$$

Theorem *2 Block design with the incidence matrix* $\mathbf{N}$ *of the form (3.1) is the variance balanced block design with repeated blocks with the parameters* $v{+}1,\ b=tb_1{+}ub_2,\quad \mathbf{r}=\left[(tr_1+ur_2)\,\mathbf{1}_v^{'}\quad tb_1\right]^{'},\quad \mathbf{k}=\left[(k_1+1)\,\mathbf{1}_{tb_1}^{'}\quad k_2\mathbf{1}_{ub_2}^{'}\right]^{'}$, $b^*=tb_1^*+ub_2^*$ *if and only if the constants* t *and* u *satisfy the equality*

$$t\,(r_1-\lambda_1)\,k_2=u\lambda_2\,(k_1+1)\,. \tag{3.2}$$

Proof. For the block design with the matrix $\mathbf{N}$ given in (3.1) we have
$\mathbf{C}=$

$$\left[\begin{array}{cc}\left[t\left(r_1-\frac{r_1-\lambda_1}{k_1+1}\right)+u\left(r_2-\frac{r_2-\lambda_2}{k_2}\right)\right]\mathbf{I}_v-\left[\frac{t\lambda_1}{k_1+1}+\frac{u\lambda_2}{k_2}\right]\mathbf{1}_v\mathbf{1}_v^{'} & -\frac{tr_1}{k_1+1}\mathbf{1}_v\\ -\frac{tr_1}{k_1+1}\mathbf{1}_v^{'} & \frac{tb_1k_1}{k_1+1}\end{array}\right].\tag{3.3}$$

Comparing the diagonal elements of the matrix $\mathbf{C}$ of the form (3.3) we have

$$\frac{t\,(r_1(k_1-1)+r_1)}{k_1+1}+\frac{ur_2(k_2-1)}{k_2}=\frac{tb_1k_1}{k_1+1}$$
$$\frac{t\,(\lambda_1(v-1)+r_1-vr_1)}{k_1+1}=-\frac{u\lambda_2(v-1)}{k_2}$$
$$t(r_1-\lambda_1)k_2=u\lambda_2(k_1+1).$$

Comparing the offdiagonal elements of the matrix $\mathbf{C}$ of the form (3.3) we have

$$\frac{t\lambda_1}{k_1+1}+\frac{u\lambda_2}{k_2}=\frac{tr_1}{k_1+1}$$

$$t(r_1-\lambda_1)k_2=u\lambda_2(k_1+1).$$

If the condition (3.2) holds then the matrix $\mathbf{C}=\eta\left[\mathbf{I}_{v+1}-\frac{1}{v+1}\mathbf{1}_{v+1}\mathbf{1}_{v+1}^{'}\right]$, where $\eta=\frac{tr_1(v+1)}{k_1+1}$. So, the Theorem is proved.

In particular case when $t=u=1$ we have

Corollary *3 Block design with the incidence matrix* $\mathbf{N}$ *of the form*

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{1}'_{b_1} & \mathbf{0}'_{b_2} \end{bmatrix} \tag{3.4}$$

is the variance balanced block design with repeated blocks with the parameters $v+1, \quad b = b_1 + b_2, \qquad \mathbf{r} = \left[(r_1+r_2)\mathbf{1}'_v \qquad b_1\right]', \quad \mathbf{k} = \left[(k_1+1)\mathbf{1}'_{b_1} \quad k_2\mathbf{1}'_{b_2}\right]',$ $b^* = b_1^* + b_2^*$ *if and only if*

$$(r_1 - \lambda_1)k_2 = \lambda_2(k_1 + 1).$$

Example 5 *Let us consider the balanced incomplete block design with the parameters* $v = 7,\ b_1 = 21,\ r_1 = 6,\ k_1 = 2,\ \lambda_1 = 1,\ b_1^* = 21$ *with the incidence matrix* $\mathbf{N}_1$ *given throught the blocks* (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (2,3), (2,4), (2,5), (2,6), (2,7), (3,4), (3,5), (3,6), (3,7), (4,5), (4,6), (4,7), (5,6), (5,7), (6,7) *and the balanced incomplete block design with the parameters* $v = 7,\ b_2 = 35,\ r_2 = 15,\ k_2 = 3,\ \lambda_2 = 5,\ b_2^* = 7$ *with the incidence matrix* $\mathbf{N}_2$ *given throught the blocks* (1,2,4), (2,3,5), (3,4,6), (4,5,7), (1,5,6), (2,6,7), (1,3,7), *each block is repeated five times. Based on the matrices* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *for* $t = u = 1$ *we construct the incidence matix* $\mathbf{N}$ *in the form (3.4) of the variance balanced block design with repeated blocks with the parameters* $v = 8,\ b = 56,\ r = 21,\ k = 3,\ b^* = 28$. *Hence we have* $\mathbf{C} = 16\left[\mathbf{I}_8 - \frac{1}{8}\mathbf{1}_8\mathbf{1}'_8\right]$.

Example 6 *Let us consider the balanced incomplete block design with the parameters* $v = 9,\ b_1 = 24,\ r_1 = 8,\ k_1 = 3,\ \lambda_1 = 2,\ b_1^* = 12$ *with the incidence matrix* $\mathbf{N}_1$ *given throught the blocks* (1,2,6), (1,3,7), (1,4,8), (1,5,9), (2,3,8), (2,4,9), (2,5,7), (3,4,5), (3,6,9), (4,6,7), (5,6,8), (7,8,9), *each block is repeated two times. And we consider the balanced incomplete block design with the parameters* $v = 9,\ b_2 = 36,\ r_2 = 16,\ k_2 = 4,\ \lambda_2 = 6,\ b_2^* = 18$ *with the incidence matrix* $\mathbf{N}_2$ *given throught the blocks* (1,2,3,4), (1,2,4,9), (1,2,5,7), (1,3,6,8), (1,3,8,9), (1,4,6,7), (1,5,6,9), (1,5,7,8), (2,3,5,6), (2,3,6,7), (2,4,5,8), (2,6,8,9), (2,7,8,9), (3,4,5,8), (3,4,7,9), (3,5,7,9), (4,5,6,9), (4,6,7,8), *each block is repeated two times. Based on the matrices* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *for* $t = u = 1$ *we construct the incidence matix* $\mathbf{N}$ *in the form (3.4) of the variance balanced block design with repeated blocks with the parameters* $v = 10,\ b = 60,\ r = 24,\ k = 4,\ b^* = 30$. *Hence we have* $\mathbf{C} = 20\left[\mathbf{I}_{10} - \frac{1}{10}\mathbf{1}_{10}\mathbf{1}'_{10}\right]$.

If $\mathbf{N}_1 = \mathbf{I}_v$ then

Corollary 4 *Block design with the incidence matrix* $\mathbf{N}$ *of the form*

$$\mathbf{N} = \begin{bmatrix} \mathbf{I}_v & \mathbf{N}_2 \\ \mathbf{1}_v^{'} & \mathbf{0}_{b_2}^{'} \end{bmatrix} \tag{3.5}$$

is the variance balanced block design with repeated blocks with the parameters $v{+}1$, $b = v{+}b_2$, $\mathbf{r} = \left[(r_2+1)\mathbf{1}_v^{'} \quad v\right]^{'}$, $\mathbf{k} = \left[2\mathbf{1}_v^{'} \quad k_2\mathbf{1}_{b_2}^{'}\right]^{'}$, $b^* = b_2^*{+}v$ *if and only if*

$$k_2 = 2\lambda_2.$$

Example 7 *Let us consider the balanced incomplete block design with the parameters* $v = 13$, $b_2 = 26$, $r_2 = 8$, $k_2 = 4$, $\lambda_2 = 2$, $b_1^* = 13$ *with the incidence matrix* $\mathbf{N}_2$ *given throught the blocks* $(1,2,4,10)$, $(2,3,5,11)$, $(3,4,6,12)$, $(4,5,7,13)$, $(1,5,6,8)$, $(2,6,7,9)$, $(3,7,8,10)$, $(4,8,9,11)$, $(5,9,10,12)$, $(6,10,11,13)$, $(1,7,11,12)$, $(2,8,12,13)$, $(1,3,9,13)$, *each block is repeated two times. Based on the matrix* $\mathbf{N}_2$ *we construct the incidence matix* $\mathbf{N}$ *in the form (3.5) of the variance balanced block design with repeated blocks with the parameters* $v = 14$, $b = 39$, $\mathbf{r} = \left[9\mathbf{1}_{13}^{'} \quad 13\right]^{'}$, $\mathbf{k} = \left[2\mathbf{1}_{13}^{'} \quad 4\mathbf{1}_{26}^{'}\right]^{'}$, $b^* = 26$. *Hence we have* $\mathbf{C} = 7\left[\mathbf{I}_{14} - \frac{1}{14}\mathbf{1}_{14}\mathbf{1}_{14}^{'}\right]$.

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