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The Convergence of an Explicit Iteration Sequence for Strictly Asymptotically Pseudo-Contractive Mappings[1]

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Abstract. C be a closed convex subset of a real Hilbert space H and assume that T_i is Strictly asymptotically pseudo-contractive mappings of Browder-petyshyn type on C, given an initial point $x_0 \in C$ and given a sequence $\{\alpha_n\}$ in (0,1), the modified Mann's algorithm, $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_[n]^k x_n$, $n \geq 0$, and $T_n^k = T_n(modN) = T_i^k$, $n = (k-1)N + i, i \in I = \{0, 1, \cdots, N-1\}$. It is proved that if the sequence satisfied some assumptions, strong convergence of the algorithm $\{x_n\}$ is proved. The results improve and extend the results of T.H.Kim and H.K.Xu[1, Nonlinear Analysis(2007)] and some others.

Keywords: An Explicit Iteration; Strictly Asymptotically Pseudo-contractive Mappings; Semi-compactness

1. Introduction and Preliminaries

Let H be a real *Hlbert* space, and C a nonempty closed convex subset of H, and $T : C \to C$ a self-mapping. Recall that T is non-expansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T, that is,

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$Fix(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a non-expansive mapping such that $Fix(T) \neq \emptyset$.

Defination 1(see 2,3,4)

1. T is called a pseudo-contractive, if

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$$

for every $x, y \in D(T)$.

2. T is called a asymptotically pseudo-contractive, if there exists a sequence $\{k_n\} \subset [1, +\infty)$, and $\lim_{n\to\infty} k_n = 1$

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2$$

for every $x, y \in D(T)$.

3. T is called a κ-strictly pseudo-contractive in the terminology of Browder and Petyshon, if there exists $\lambda > 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \kappa\|(I-T)x - (I-T)y\|^2 \tag{1}$$

for every $x, y \in D(T)$. Without loss of generality, we assume $\lambda \in (0, \frac{1}{2})$, In the Hilbert space, (1) is equivalent to the following:

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \kappa\|(I-T)x - (I-T)y\|^2 \tag{2}$$

where $\kappa = (1-2\lambda) < 1$ and $\kappa \in [0, 1)$. As we all know λ-strictly pseudo-contractive is Lipschitz mapping.(indeed, if I is idendtity operator, (1) can be written as

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda\|x - Tx - (y - Ty)\|$$

from the above, we can have

$$\|x-y\| \geq \lambda\|x - Tx - (y-Ty)\| \geq \lambda\|Tx - Ty\| - \lambda\|x-y\|$$

so $\|Tx - Ty\| \leq L\|x-y\|, \forall x, y \in D$, where $L = \frac{1+\lambda}{\lambda}$, so$\lambda$-strictly pseudo-contractive mapping is Lipschitz mapping.)

4. T is called a κ-strictly asymptotically pseudo-contractive in the terminology of Browder and Petyshon, if there exists $\kappa \in [0, 1)$ and $u_m \geq 0, \lim_{n\to\infty} u_m = 0$ such that

$$\|Tx - Ty\|^2 \leq (1 + u_m)\|x-y\|^2 + \kappa\|(I-T^n)x - (I-T^n)y\|^2 \tag{3}$$

for every $x, y \in D(T)$.

Iterative methods are often used to solve the fixed point equation $Tx = x$, for example Picard' method and Mann's method, the Mann's method is prevails to Picard's, but the Mann's method in general has not the strongly convergence [5] for either non-expansive mappings or strict pseudo-contractions. So in order to have the strongly convergence, we have to modify the iteration method. In [1,Tae-Hwa, Xu] have prove the the strongly convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \tag{4}$$

where the initial point $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\} \in (0,1), \kappa + \delta \leq \alpha_n \leq 1-\delta$ for some $\delta \in (0,1)$, and T is a λ-strictly asymptotically pseudo-contractive mapping, $\lim_{n\to\infty} \kappa_n < \infty$, then the sequence $\{x_n\}$ convergent weakly to a fixed point of the mapping T.

Recently, Xu and Ori[6] introduce implicit iteration process for a finite family of non-expansive mappings, following by

$$x_n = t_n x_{n-1} + (1-t_n)T_n x_n$$

where T_i are N non-expansive mappings, and $i \in \{1, 2, \cdot, \cdot, \cdot, N\}$, assume $F := \cap_{i=1}^{N} F(T_i) \neq \emptyset$, $\{t_n\} \in (0,1), x_0 \in D$, and $T_k = T_{kmodN}$. They proved the sequence $\{x_n\}$ convergent weakly to a common fixed point in Hilbert space.

The purpose of this paper is to introduce an explicit iteration scheme for approximating a common fixed point of a finite family of asymptotically pseudo-contractive mappings in a Hilbert spaces. We will proved the strong convergence of the iteration scheme $\{x_n\}$ defined by:

$$x_1 = \alpha_0 x_0 + (1-\alpha_0)T_{[0]}x_0$$
$$x_2 = \alpha_1 x_1 + (1-\alpha_1)T_{[1]}x_2$$
$$\cdots\cdots$$
$$x_N = \alpha_{N-1}x_{N-1} + (1-\alpha_N)T_{[N-1]}x_{N-1}$$
$$x_{N+1} = \alpha_N x_N + (1-\alpha_N)T^2_{[0]}x_0$$
$$\cdots\cdots..$$
$$x_{2N} = \alpha_{2N}x_{2N-1} + (1-\alpha_{2N-1})T^2_{[N-1]}x_{N-1}$$
$$x_{2N+1} = \alpha_{2N}x_{2N-1} + (1-\alpha_N)T^3_{[0]}x_0$$
$$\cdots\cdots..$$

In general, $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n + (1-\alpha_n)T^k_{[n]}x_n \tag{5}$$

The results improve and extend results of Tae-Hwa Kim and Xu[1], G and XU[6].

In order to prove our main results, we need the following definitions and Lemmas for the proof of our main results.

Lemma 1.1 In a Hilbert space H , there holds the inequality

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y)\rangle, \quad x, y \in E$

(ii)$\|tx-(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, $t \in (0,1), \forall x, y \in H$.

(iii) if $\{x_n\}$ is a sequence in H weakly convergent to z, then

$$\limsup_{n\to\infty} \|x_n - y_n\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2, \ \forall y \in H$$

Lemma 1.2 (see[7]) Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition

$$\alpha_{n+1} \leq (1+\gamma_n)\alpha_n + b_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty}\gamma_n \leq \infty$. $\sum_{n=1}^{\infty} b_n \leq \infty$
(i) $\lim_{n\to\infty}\alpha_n$ exists, (ii) if exists $\{x_{nj}\} \subset \{x_n\}, and x_{n_j} \to 0$, then $\alpha_n \to 0$

Defination 2 T is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in D such that $\|x_n - Tx_n\| \to 0 (n \to \infty)$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\} \to x^* \in D$.

remark $\rightharpoonup$ stands for weak convergence, $\to$ stands for strong convergence.

2. **Main Results**

Theorem Let C be a closed convex subset of a Hilbert space H, and let $T_i : C \to C$ be uniformly Lipschitz asymptotically κ-strict pseudo-contraction for some $0 \geq \kappa_i < 1$ and $0 \leq i \leq N-1$, such that for each T_i exists $\{u_{in}\}$, $u_{in} \geq 0$, and $\sum_{n\to\infty} u_{in} < \infty$, have $\kappa_i \in (0,1)$

$$\|T_i x - T_i y\|^2 \leq (1+u_{in})\|x-y\|^2 + \kappa_i\|(I-T)x-(I-T)y\|^2$$

Let γ=max$\{\gamma_i,: 0 \leq i \leq N-1\}$. Assume the common fixed point set $\bigcap_{i=0}^{N-1} F(T_i)$ of $\{T_i\}_{i=0}^{N-1}$ is nonempty. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ defined by (5). Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\kappa+\varepsilon \leq \alpha \leq 1-\varepsilon$ for all n and some $\varepsilon \in (0,1)$. Assume that exists $T \in \{T_i\}$ is semi-compact. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}$.

Proof: We take a point $p \in \bigcap_{i=0}^{N-1} F(T_i)$, noting that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)T_{[n]}^k x_n - p\|^2 \\
&= \|\alpha_n(x_n-p) + \alpha_n p + (1-\alpha_n)T_{[n]}^k x_n - p\|^2 \\
&= \|\alpha_n(x_n-p) + (1-\alpha_n)(T_{[n]}^k x_n - p)\|^2 \\
&\leq \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|T_{[n]}^k x_n - p\| - \alpha_n(1-\alpha_n)\|x_n - T_{[n]}^k x_n\|^2 \\
&= (1+(1-\alpha_n)u_{in})\|x_n-p\|^2 - (\alpha_n-\kappa)(1-\alpha_n)\|x_n - T_{[n]}^k x_n\|^2
\end{aligned} \quad (2.1)$$

Since $\kappa+\varepsilon \leq \alpha \leq 1-\varepsilon$ for all n, from (2.1) we have

$$\|x_{n+1}-p\|^2 \leq (1+u_{in})\|x_n-p\|^2 - \varepsilon^2\|x_n - T_{[n]}^k x_n\|^2 \quad (2.2)$$

Now (2.2) implies that

$$\|x_{n+1}-p\|^2 \leq (1+u_{in})\|x_n-p\|^2 \quad (2.3)$$

By Lemma 1.2, and the assumption $\sum_{n=0}^{\infty} u_{in} < \infty$, we have $\lim_{n\to\infty}\|x_n-p\|$ exists, so $\{x_n\}$ is bounded. r Now from (2.2), we can have

$$\|x_n - T_{[n]}^k x_n\|^2 \leq \frac{1}{\varepsilon^2}(\|x_n-p\|^2 - \|x_{n+1}-p\|^2) + \frac{u_{in}}{\varepsilon}\|x_n-p\|^2 \quad (2.4)$$

Since $\{x_n\}$ is bounded, and $u_{in} \to 0$. So we have

$$\lim_{n\to\infty} \|x_n - T^k_{[n]}x_n\| = 0 \tag{2.5}$$

From the difination of $\{x_n\}$, we have

$$\|x_{n+1} - x_n\| = (1-\alpha_n)\|x_n - T^k_{[n]}x_n\| \to 0$$

So, $\|x_n - x_{n+l}\| \to 0,\ l \in N$.

When $n \geq N$, since T is uniformly Lipschtizian with Lipschitz constant $L \geq 1$, so we can have

$$\begin{aligned}
\|x_n - T_{[n]}x_n\| &\leq \|x_n - T^k_{[n]}x_n\| + \|T^k_{[n]}x_n - T_{[n]}x_n\| \\
&\leq \|x_n - T^k_{[n]}x_n\| + L\|T^{k-1}_{[n]}x_n - T_{[n]}x_n\| \\
&\leq \|x_n - T^k_{[n]}x_n\| + L\|T^{k-1}_{[n]}x_n - T^{k-1}_{[n-N]}x_{n-N}\| + \|T^{k-1}_{[n-N]}x_{n-N} - x_{n-N}\| \\
&\quad + \|x_{n-N} - x_n\|
\end{aligned} \tag{2.6}$$

Noting that $n = n - N(modN)$ and $T_n = T_{n-N}$. So we have from (2.6)

$$\begin{aligned}
\|x_n - T_{[n]}x_n\| &\leq \|x_n - T^k_{[n]}x_n\| \\
&\leq L^2\|x_n - x_{n-N}\| + L\|T^{k-1}_{[n-N]}x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|
\end{aligned} \tag{2.7}$$

From (2.5), we have $\lim_{n\to\infty} \|x_n - T^k_{[n]}x_n\| = \lim_{n\to\infty} \|T^{k-1}_{[n-N]}x_{n-N} - x_{n-N}\| = 0$

$$\lim_{n\to\infty} \|x_{n-N} - x_n\| = \lim_{n\to\infty} \|x_{n-N} - x_n\| = 0$$

So, $\lim_{n\to\infty} \|x_n - T_{[n]}x_n\| = 0$.

Additional, $\forall l \in I$,

$$\begin{aligned}
\|x_n - T_{[n+l]}x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{[n+l]}x_{n+l}\| + \|T_{[n+l]}x_{n+l} - T_{[n+l]}x_n\| \\
&\leq (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{[n+l]}x_{n+l}\|
\end{aligned}$$

So, we have $\lim_{n\to\infty} \|x_n - T_{[n+l]}x_n\| = 0, or \lim_{n\to\infty} \|x_n - T[l]x_n\| = 0,\ \forall l \in I$.

From the assumption there exists $T \in \{T_i\}_{i=0}^{N-1}$ is semi-compactness, without loss the generality, let T_1 is semi-compactness. Since $\{x_n\}$ is bounded, so there exists subsequence $\{x_{n_j}\} \subset \{x_n\}$, $x_{n_j} \to x^*$,

$$\|x_n - x^*\| = \lim_{n\to\infty} \|x_{n_j} - T_{[l]}x_{n_j}\| = 0,\ l \in I$$

so there exists $x^* \in F(T[i])$ and $x^* = p$. By the Lemma 2.1(ii), we have $\lim_{n\to\infty} \|x_n - p\| = 0$ with $x^* = p$. So the proof is complete.

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