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A False Sense of Wealth

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Abstract

We consider a lognormal diffusion market and prove that the average portfolio does not provide accurate dynamics of the real portfolio, and the expected functionals of the average portfolio do not approximate efficiently the option pricing dynamics on the real portfolio.

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1 Introduction and results

Our aim is to analyze one intuitive approximation of the random evolution of a portfolio on the basis of the dynamics of its assets. The model for the asset prices that we consider here is based on a $(m+1)$-dimensional lognormal diffusion process consisting of one riskless asset or bond, m risky assets or stocks, and m driving processes or sources of randomness. The asset price processes S_t^i $(i = 0, 1, \ldots, m)$ over a finite horizon $t \in [0, T]$, $T < +\infty$, are given on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ by

$$dS_t^0 = r_t S_t^0 dt, dS_t^i = S_t^i \Big(b_t^i dt + \sum_{j=1}^{m} \sigma_t^{ij} dW_t^j \Big) \text{ for } i = 1, 2, \ldots, m, \tag{1.1}$$

where $S_0^0 = 1$ and $S_0^i > 0$ $(i = 1, 2, \ldots, m)$ are constant. We may and shall suppose that $\mathcal{F}_t$ is the minimal σ-algebra generated by the m-dimensional standard Brownian motion $(W_t^1, \ldots, W_t^m)$ augmented by the P-null sets. The appreciation rate b_t^i, white noise volatility σ_t^{ij} and interest rate r_t are deterministic and continuous on $[0, T]$.

Let us consider an investor holding g_t^i shares of the ith asset $(i = 0, 1, \ldots, m)$ at time $t \in [0, T]$. We assume (see [1]) that the functions g_t^i are strictly positive and continuous. Think for instance to the case of financial indices, that can be regarded as portfolios with time-constant strategy. Then the value of the investor's portfolio at time $t \in [0,T]$ is given by

$$V_t := \sum_{i=0}^{m} g_t^i S_t^i. \tag{1.2}$$

The product $g_t^i S_t^i$ can therefore be interpreted as the value of investment in the ith asset at time t, where the asset price S_t^i follows (1.1). In the sequel we assume the so-called self-financing constraint that at trading times the value of the portfolio does not change, that is, between trading times the number of invested shares g_t^i remains constant (cf. [1], [2]). As a consequence of (1.2) we have that V_0 is constant and

$$dV_t = \sum_{i=0}^{m} g_t^i dS_t^i \text{ for } t \in (0,T]. \tag{1.3}$$

Due to the nature of the component asset dynamics in (1.1), it is not obvious to identify in a simple way the dynamics of the appreciation rate and volatility of V from the coefficients of the underlying assets. Indeed, by (1.3) and Itô's formula for mutidimensional diffusion processes (see [2]), we can write for the portfolio V the following stochastic differential equation

$$dV_t = V_t\Big(b_t dt + \sum_{j=1}^{m} \sigma_t^j dW_t^j\Big), \tag{1.4}$$

where the appreciation rate b_t and noise volatility σ_t^j are given by

$$b_t = \frac{1}{V_t}\Big(r_t g_t^0 S_t^0 + \sum_{i=1}^{m} b_t^i g_t^i S_t^i\Big), \sigma_t^j = \frac{1}{V_t}\sum_{i=1}^{m} g_t^i S_t^i \sigma_t^{ij} \text{ for } j = 1, 2, \ldots, m.$$

Obviously the latter expressions are difficult to handle, as the price processes S_t^i (therefore the noises $(W_t^1, \ldots, W_t^m)$) are involved therein.

Instead, we shall consider that the portfolio dynamics is approximated by a process with average appreciation rate and average volatility that do not depend on the Brownian noise in the market, but only on the coefficients r_t, b_t^i and σ_t^{ij}. More precisely, we consider the *average portfolio process* $\bar{V}_t$ defined by the diffusion stochastic differential equation

$$d\bar{V}_t = \bar{V}_t\Big(\bar{b}_t dt + \sum_{j=1}^{m} \bar{\sigma}_t^j dW_t^j\Big), \tag{1.5}$$

with $\bar{V}_0 = V_0$, and where the appreciation rate $\bar{b}_t$ and noise volatility $\bar{\sigma}_t^j$ are given by

$$\bar{b}_t = \frac{1}{m+1}\Big(r_t + \sum_{i=1}^{m} b_t^i\Big), \bar{\sigma}_t^j = \frac{1}{m+1}\sum_{i=1}^{m} \sigma_t^{ij} \text{ for } j = 1, 2, \ldots, m.$$

For each time $t \in [0,T]$ we consider the proportions between the value that is held in the ith stock ($i = 1, 2, \dots, m$) and the value that is held in the bond ($i = 0$), namely:

$$g_t^i S_t^i = (1 + \varepsilon_t^i) g_t^0 S_t^0 \text{ for } i = 1, 2, \dots, m, \tag{1.6}$$

for some deterministic and continuous functions $\varepsilon_t^i > -1$ ($i = 1, 2, \dots, m$) on $[0,T]$. The next theoretical result (1.8)-(1.9) is a standard application of Itô stochastic calculus; although "déjà vu", we included a proof in the appendix.

Consider the financial model (1.1) and denote for $t \in [0,T]$

$$\Lambda_t := \frac{\left|\sum_{i=1}^m \varepsilon_t^i\right| + \sum_{i=1}^m \left|-\varepsilon_t^1 - \dots - \varepsilon_t^{i-1} + m\varepsilon_t^i - \varepsilon_t^{i+1} - \dots - \varepsilon_t^m\right|}{m + 1 + \sum_{i=1}^m \varepsilon_t^i}. \tag{1.7}$$

With the definitions and notations (1.4), (1.5) and (1.6), there exists a positive finite constant C depending on the model (1.1) but not on ε_t^i ($i = 1, 2, \dots, m$), and such that

$$E|V_T - \bar{V}_T| \le C \cdot \left(\int_0^T \Lambda_t dt\right)^{1/2}, \tag{1.8}$$

If f is twice continuously differentiable on $\mathbf{R}$ and, together with its derivatives up to and including the order two, have at most polynomial growth, then there exists a positive finite constant C depending on the model (1.1) and f but not on ε_t^i ($i = 1, 2, \dots, m$), and such that

$$|E[f(V_T) - f(\bar{V}_T)]| \le C \cdot \int_0^T \Lambda_t dt. \tag{1.9}$$

Observe that the function Λ_t in (1.7) approaches 0 if ε_t^i ($i = 1, 2, \dots, m$) are all approaching 0 for almost all $t \in [0,T]$, therefore, the intuition beyond (1.8)-(1.9) is the following. When the values of investment in stocks are close to each other at any time $t \in [0,T]$, the average portfolio matches rather closely the evolution of the real portfolio, and option price based on the average portfolio reasonably fits to the option price on the real portfolio. However, we performed the following simulations and obtained rather contradictory results! More precisely, we considered the case $m = 2$, choose the initial values $S_0^i = 1$ ($i = 1, 2$), $V_0 = 1$, terminal value $T = 1$, interest rate $r = 0.01$, appreciation rates $b_t^1 = 0.01, b_t^2 = 0.015$, volatilities $\sigma_t^{11} = \sigma_t^{12} = 0.1, \sigma_t^{21} = \sigma_t^{22} = 0.15$, and proportions of investment value $\varepsilon_t^1 = -0.2, \varepsilon_t^2 = -0.1$. Using (1.5), we computed directly the mean $E(\bar{V}_1) = 1.13$ and the variance $Var(\bar{V}_1) = 0.063$ of the average portfolio. By Monte-Carlo simulation with $10,000$ realizations we obtained for the difference $E(V_1) - E(\bar{V}_1)$ the 99% confidence interval $[-0.034, 0.0061]$, and for the difference of variations $Var(V_1) - Var(\bar{V}_1)$ the

99% confidence interval $[0.85, 0.93]$. The first confidence interval shows that the average portfolio matches rather closely the first moment of the real portfolio, whereas the second confidence interval says that the second moment of the real portfolio may be extremely far from the second moment of the average portfolio. We went further and, using the same data as above, we priced a European call option with strike price $K = 1$ and maturity $T = 1$. By the well-known analytical formula for European call options (see [2]), its price equals 0.072 on the average portfolio $\bar{V}$. On the other hand, by Monte-Carlo simulation with $10,000$ realizations, we obtain the corresponding 99% confidence interval for the option price on the real portfolio, namely $[0.969, 0.974]$. As the latter is way too large, we have no match between the two prices. The latter confidence interval becomes even more precise, namely $[0.971, 0.973]$, when the proportions of investment value are $\varepsilon_t^1 = \varepsilon_t^2 = -0.05$, yet the"no match" conclusion persists.

The explanation of the above paradox resides in the value of the constant C in (1.8)-(1.9). Although Λ_t in (1.7) approaches 0 if ε_t^i $(i = 1, 2, \ldots, m)$ are all approaching 0, the constant C, depending on each individual model (and f), may become extremely large and give a rather poor approximation in formulas (1.8) and (1.9). It would be interesting to find out what models provide good approximations between the real and average portfolios (1.4)-(1.5).

2 Appendix

Proof of 1.8. With the notations from (1.6), the coefficients of (1.4) become

$$b_t = \Big(m + 1 + \sum_{i=1}^{m} \varepsilon_t^i\Big)^{-1} \Big[r_t + \sum_{i=1}^{m} (1 + \varepsilon_t^i) b_t^i\Big], \sigma_t^j = \Big(m + 1 + \sum_{i=1}^{m} \varepsilon_t^i\Big)^{-1} \sum_{i=1}^{m} (1 + \varepsilon_t^i) \sigma_t^{ij}$$

for $j = 1, 2, \ldots, m$, hence

$$|b_t - \bar{b}_t|, |\sigma_t^j - \bar{\sigma}_t^j| \leq C \cdot \Lambda_t \text{ for } t \in [0, T] \text{ and } j = 1, 2, \ldots, m. \tag{2.1}$$

Solving (1.4) and (1.5) we have

$$V_T = V_0 \exp\Big[\int_0^T \Big(b_t - \frac{1}{2}\sum_{i=1}^{m} (\sigma_t^j)^2\Big) dt + \sum_{i=1}^{m} \int_0^T \sigma_t^j dW_t^j\Big],$$
$$\bar{V}_T = V_0 \exp\Big[\int_0^T \Big(\bar{b}_t - \frac{1}{2}\sum_{i=1}^{m} (\bar{\sigma}_t^j)^2\Big) dt + \sum_{i=1}^{m} \int_0^T \bar{\sigma}_t^j dW_t^j\Big].$$

Conditions $S_0^i > 0$ and $\varepsilon_t^i > -1$ (or $g_t^i > 0$) $(i = 1, 2, \ldots, m)$ for $t \in [0, T]$ ensure that $S_t^i > 0$, $V_t > 0$ and $\bar{V}_t > 0$ for almost all $t \in [0, T]$ and $i = 1, 2, \ldots, m$. Consider the expression $(V_T - \bar{V}_T)^2$, with V_T and $\bar{V}_T$ given in explicit form as above; using (2.1), we obtain the estimate

$$E(V_T - \bar{V}_T)^2 \leq C \cdot \int_0^T \Lambda_t dt$$

for some constant C depending on the coefficients in (1.1) hence, by Hölder's inequality, the required inequality in (1.8) now follows.

Proof of 1.9. By the Markov property of the process (1.5), the function $u(t,x) := E[f(\bar{V}_T)|\bar{V}_t = x]$ solves the backward integro-differential problem

$$Lu(t,x) = 0 \text{ for } (t,x) \in [0,T) \times \mathbf{R}; u(T,x) = f(x) \text{ for each } x \in \mathbf{R}, \tag{2.2}$$

where

$$Lu(t,x) := \frac{\partial u}{\partial t} + \bar{b}_t x \frac{\partial u}{\partial x} + \frac{1}{2}\sum_{j=1}^{m}(\bar{\sigma}_t^j)^2 x^2 \frac{\partial^2 u}{\partial x^2}.$$

By Itô's formula we have

$$\begin{aligned} E[f(V_T) - f(\bar{V}_T)] = E[u(T,V_T) - u(0,V_0)] \\ = E\int_0^T \left\{\frac{\partial u}{\partial t}(t,V_t) + b_t V_t \frac{\partial u}{\partial x}(t,V_t) + \frac{1}{2}\sum_{j=1}^{m}(\sigma_t^j)^2 V_t^2 \frac{\partial^2 u}{\partial x^2}(t,V_t)\right\} \end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned} |E[f(V_T) - f(\bar{V}_T)]| = \\ \left| E\int_0^T \left[(b_t - \bar{b}_t)V_t \frac{\partial u}{\partial x}(t,V_t) + \frac{1}{2}\sum_{j=1}^{m}[(\sigma_t^j)^2 - (\bar{\sigma}_t^j)^2]V_t^2 \frac{\partial^2 u}{\partial x^2}(t,V_t)\right|. \end{aligned} \tag{2.4}$$

By the polynomial growth constraint on f (hence on u), formula (2.4) and estimates (2.1) give the required inequality (1.9), for some constant C depending on the coefficients in (1.1) and f.

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