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Delayed Response Times

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Abstract

We compute the performance of delayed responses within stochastic decision models, and give examples when the underlying is a diffusion.

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1 Problem

A stochastic process $X = \{X_t = X_t(\omega), t \geq 0, \omega \in \Omega\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in $\mathcal{I} \subseteq (-\infty, +\infty)$ can be used to model stochastic decision models in the following way. The function $X_t(\cdot) : \Omega \to \mathcal{I}$ represents a potential response at time t, and the accumulated information available at time t is the sigma-algebra $\mathcal{F}_t$ generated by the random variables $\{X_s(\cdot), 0 \leq s \leq t\}$. Using the probabilistic notion of stopping times, we argued in [3] that the best response in a stochastic decision model is obtained by solving an optimal stopping problem, and explicit examples were given when the underlying process is a diffusion process.

In this note we consider the situation when there is a delay $\delta > 0$ in the flow in information available for the decision mechanism to respond. More precisely (cf. [2], [1]), there is a delay $\delta > 0$ from the decided response time, based on the complete current information available, to the time when the response is given. A function $\eta : \Omega \to [\delta, +\infty]$ is called δ-delayed stopping time for the stochastic process X if $\{\omega \in \Omega : \eta(\omega) \leq t\} \in \mathcal{F}_{t-\delta}$ for $t \geq \delta$. We denote by D^δ the set of all δ-delayed stopping times. Notice that $\eta(\omega)$ represents the response time and $\eta \in D^\delta$ if the decision whether or not to respond at or before time t is based on the information contained in $\mathcal{F}_{t-\delta}$. In particular, D^0 is exactly the set of all classical stopping times.

We consider in the sequel a standard Brownian motion $\{W_t(\cdot), t \geq 0\}$ on a complete probability space $(\Omega, \mathcal{F}, P)$ and assume that X is a diffusion process, homogeneous, continuous, with state space $\mathcal{I}$ and dynamics given by

$$dX_t(\cdot) = \mu(X_t(\cdot))dt + \sigma(X_t(\cdot))dW_t(\cdot), \tag{1}$$

for some Borel measurable functions $\mu : \mathcal{I} \to (-\infty, +\infty)$ and $\sigma : \mathcal{I} \to (0, +\infty)$.

Definition. *The best δ-delayed response in the stochastic model based on (1) is denoted by (Q^δ, η^*) and consists of the performance function Q^δ, solution to the stopping problem*

$$Q^\delta(x) = \sup_{\eta \in D^\delta} E^x(X_\eta), \tag{2}$$

where E^x denotes expectation with respect to the probability measure giving the law of X when $X_0 = x$, X_η denotes the process X stopped at η, that is, $X_\eta(\omega) := X_{\eta(\omega)}(\omega)$, and η^* is an optimal δ-delayed stopping time, for which the sup in formula (2) is attained.

Remark that $Q^0(x)$ is the performance function in the best response (non-delayed) problem from [3]. In the sequel we are interested in the relationship between the delayed and non-delayed performances. It is rather elementary to see that $\eta \in D^\delta$ if and only if $\tau := \eta - \delta \in D^0$, therefore the following formula holds:

$$Q^\delta(x) = \sup_{\tau \in D^0} E^x(X_{\tau+\delta}). \tag{3}$$

As such, the optimal stopping problem (2) is over classical stopping times, but with delayed effect of responding: if $\tau \in D^0$ is chosen, then the response is given at time $\tau + \delta$ (after a delay of δ). Note that $D^\delta \subseteq D^0$, hence formulas (2) and (3) imply that $Q^\delta(x) \leq Q^0(x)$, that is, the delayed performance Q^δ is weaker that the non-delayed performance Q^0 by the quantity $Q^0(x) - Q^\delta(x)$.

2 Examples

Example 1. Brownian motion with drift on $\mathcal{I} = [a, b]$ with $\mu(x) = \mu \leq 0$ and $\sigma(x) = 1$. We have $E^x(X_\eta) = E^x(W_\eta + \mu\eta) = x + \mu E^x(\eta)$, therefore $Q^\delta(x) = x + \mu\delta$ and $\eta^* = \delta+$ the first entrance of X in $[a, b]$. In particular, the performance in the delayed case is weaker than the non-delayed performance by the linear factor $-\mu\delta$.

Example 2. Ornstein-Uhlenbeck process on $\mathcal{I} = [a, b]$ with $\mu(x) = \mu x$ and $\sigma(x) = \sigma$ (with $\mu < 0, \sigma > 0$). Solving equation (1) we obtain

$$X_t = \exp(\mu t)\Big(x + \sigma \int_0^t \exp(-\mu s) dW_s\Big),$$

hence $E^x(X_\eta) = xE^x(\exp(\mu\eta))$, therefore $Q^\delta(x) = x \exp(\mu\delta)$ and $\eta^* = \delta+$ the first entrance of X in $[a, b]$. In particular, the performance in the delayed case is weaker than the non-delayed performance by the factor $x(1 - \exp(\mu\delta))$.

Example 3. Geometric Brownian motion on $\mathcal{I} = [a, b] \subseteq (0, +\infty)$ with $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$ (with $\mu \le 0, \sigma > 0$). Solving equation (1) we obtain

$$X_t = x \exp\Big[\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big].$$

Unfortunately, $E^x(X_\eta)$ cannot be computed directly, as in Examples 1-2 above; instead, we use formula (4) below, and obtain

$$\begin{aligned} E^x\Big(E^{X_\tau}(X_\delta)\Big) &= E^x\Big\{E^{X_\tau}\Big(x \exp\Big[\Big(\mu - \frac{\sigma^2}{2}\Big)\delta + \sigma W_\delta\Big]\Big)\Big\} \\ &= x \exp\Big[\Big(\mu - \frac{\sigma^2}{2}\Big)\delta\Big] E^x\Big(E^{X_\tau}(\exp(\sigma W_\delta))\Big), \end{aligned}$$

hence $Q^\delta(x) = x\exp(\mu\delta)$ and $\eta^* = \delta+$ the first entrance of X in $[a, b]$. In particular, the performance in the delayed case is weaker than the non-delayed performance by the factor $x(1 - \exp(\mu\delta))$.

All the above computations rely on the following

Theorem. (cf. [2], [1]) *With the above notations and hypotheses, we have:*

$$\sup_{\eta \in D^\delta} E^x(X_\eta) = \sup_{\tau \in D^0} E^x\Big(E^{X_\tau}(X_\delta)\Big), \tag{4}$$

where E^{X_τ} denotes expectation with respect to the probability measure giving the law of X starting at X_τ. In addition, η^* is optimal for the left-hand side in (4) if and only if $\tau^* := \eta^* - \delta$ is optimal for the right- hand side in (4).

Proof. Let $\eta \in D^\delta$ and put $\tau = \eta - \delta \in D^0$. We have

$$E^x(X_\tau) = E^x(X_{\tau+\delta}) = E^x\Big(E^x\theta_\tau(X_\delta)\Big),$$

where θ_τ is the shift operator $\theta_\tau(X_s) = X_{\tau+s}, s \ge 0$. By the strong Markov property of W (hence of X), the above identities also equal to

$$E^x\Big(E^x\Big[\theta_\tau(X_\delta)|\mathcal{F}_\tau\Big]\Big) = E^x\Big(E^{X_\tau}(X_\delta)\Big)$$

and the rest of the proof follows easily (we denoted by $\mathcal{F}_\tau$ the following sigma-algebra: $A \in \mathcal{F}_\tau$ if and only if $A \cap \{\omega \in \Omega : \tau(\omega) \le t\} \in \mathcal{F}_t$ for all $t > 0$).

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