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# Propagation of Shear Wave in Anisotropic Medium

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**Abstract**

In this paper, propagation of shear waves in an initially stressed anisotropic medium has been studied. The medium has also been considered as incompressible and non-homogeneous. The velocity equation has been obtained. From the numerical computation it has been shown that due to the presence of initial compressive stresses the velocity decreases and the tensile stress increases. The velocity of the wave increases as the non-homogeneity parameter increases. The increase of anisotropy factor also increases the velocity. It has been observed that the anisotropy, non-homogeneity, initial stress, direction of propagation and depth parameters have considerable effect in the shear wave propagation.

# 1 Introduction

Most materials behave as incompressible media and the velocities of longitudinal waves in them are very high. The varieties of hard rock present in the earth are also almost incompressible. Further, due to different factors such as the external pressure, slow process of creep, difference in temperature, manufacturing processes, nitriding, pointing etc. huge quantities of stresses (called initial stresses) are stored in the medium. Owing to the variation of elastic properties and due to the presence of these initial stresses the medium becomes isotropic as well. The propagation of surface waves in detail may be found in the precious book of Ewing, Jardetzky and Press [7], Bath [5], Achenback [2] and other eminent authors. The earth is supposed to be initially stressed medium. Biot [4] formulated the dynamical equations for pre-stressed elastic medium and discussed the influence of pre-stresses on the propagation of elastic waves in a body. Muranaghan [1], Kappus [6] have discussed in a series of investigations the problems of finite deformations of an elastic body and the effect of high initial stresses on wave propagation. In this paper an attempt has been made to show the effect of initial stresses, anisotropy and non-homogeneity on the propagation of shear wave.

# 2 Formulation of the problem

Consider an unbounded incompressible anisotropic medium under initial stresses $S_{11}$ and $S_{22}$ along the x, y directions respectively. Incase the medium is slightly disturbed (u,v), the incremental stress $s_{11}$ , $s_{12}$ and $s_{22}$ are developed and the equations of motion in the incremental state are [3]

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial w}{\partial y} = \rho\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial w}{\partial x} = \rho\frac{\partial^2 v}{\partial t^2} \qquad (1)$$

where $P = S_{22} - S_{11}$, $w = 1/2(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ and $\rho$ is the density of the medium.

The stress-strain relations for an incompressible medium may be taken as [4]

$$s_{11} - s = 2Ne_{xx}, s_{22} - s = 2Ne_{yy} and s_{12} = 2Qe_{xy}. \qquad (2)$$

where $s = (s_{11} + s_{22})/2$

The incompressibility condition $e_{xx} + e_{yy} = 0$ is satisfied by

$$u = -\frac{\partial \phi}{\partial y} and v = \frac{\partial \phi}{\partial x} \qquad (3)$$

Using (2) and (3) in (1) we get

$$\frac{\partial s}{\partial x}-\frac{\partial}{\partial y}[(2N-Q+\frac{P}{2})\frac{\partial^2\phi}{\partial x^2}+(Q+\frac{P}{2})\frac{\partial^2\phi}{\partial y^2}]=\rho\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial s}{\partial y}-\frac{\partial}{\partial x}[(2N-Q-\frac{P}{2})\frac{\partial^2\phi}{\partial y^2}+(Q-\frac{P}{2})\frac{\partial^2\phi}{\partial x^2}]=\rho\frac{\partial^2 v}{\partial t^2} \tag{4}$$

Assuming the non homogeneity parameters of the form

$$N=N_0(1+by)$$

$$Q=Q_0(1+ay) \tag{5}$$

where $N_0$ and $Q_0$ are rigidities at y = 0.

After eliminating s from (4),we have

$$[Q_0(1+ay)-P/2]\frac{\partial^4\phi}{\partial x^4}+[4N_0(1+by)-2Q_0(1+ay)]\frac{\partial^4\phi}{\partial x^2\partial y^2}+$$

$$+[Q_0(1+ay)+P/2]\frac{\partial^4\phi}{\partial y^4}+[2N_0b-Q_0a]\frac{\partial^3\phi}{\partial x^2\partial y}+Q_0a\frac{\partial^3\phi}{\partial y^3}=\rho[\frac{\partial^4\phi}{\partial x^2\partial t^2}+\frac{\partial^4\phi}{\partial y^2\partial t^2}] \tag{6}$$

# 3 solution

For propagation of sinusoidal waves in any arbitrary direction, the solution of equation (6) may be considered as

$$\phi(x,y,t)=Ae^{ik(p_1x+p_2y-ct)} \tag{7}$$

where $p_1$ and $p_2$ are cosine of the angles made by the direction of propagation with the x- and y- axis respectively, c and k are the velocity of propagation and wave number respectively.

Substituting (7) in (6) and equating imaginary and real parts to zero, we have

$$[2N_0b-Q_0a]\frac{1}{k}p_1^2p_2+Q_0a\frac{1}{k}p_2^2=0 \tag{8}$$

and

$$\frac{c^2}{\beta^2}=[1+ay-\frac{P}{2Q_0}]p_1^4+2[\frac{2N_0}{Q_0}(1+by)-(1+ay)]p_1^2p_2^2+$$

$$+[1+ay+\frac{P}{2Q_0}]p_2^4 \tag{9}$$

where $\beta = (\frac{Q_0}{\rho_0})^{\frac{1}{2}}$, is the shear wave velocity in homogeneous isotropic medium.

It is clear that the equation (8) is not concerned with the wave propagation due to absence of wave velocity, whereas the equation (9) gives the velocity of shear wave in the medium. This equation shows that the velocity $(\frac{c}{\beta})$ depends much on the anisotropy factors $N$, $Q$, initial stress factor $\frac{P}{2Q}$ and also the direction of propagation parameters $p_1$ and $p_2$ .

The following particular cases are considered here:

**Case I :** For non-homogeneity in N (a = 0)

$$\frac{c^2}{\beta^2}=[1-\frac{P}{2Q_0}]p_1^4+2[\frac{2N_0}{Q_0}(1+by)-1]p_1^2p_2^2+[1+\frac{P}{2Q_0}]p_2^4$$

The velocity of wave along x-direction $(p_2 = 0, p_1 = 1)$ is given by $c_1^2 = \beta^2(1-\frac{P}{2Q_0})$, which depends on the initial stress. In case the medium is free from initial stress then $c_1 = \beta$.

Similarly the velocity of propagation along y-direction $(p_1 = 0, p_2 = 1)$ is obtained as $c_2^2 = \beta^2(1+\frac{P}{2Q_0})$.

It is interesting to note that $\frac{c_2^2-c_1^2}{\beta^2}$, a function of non-dimensional initial stress only.

This may also be observed that if $P = S_{22} - S_{11} > 0$ the effect of initial stresses on the body is compressive along x direction and tensile along y direction. The compressive initial stress decreases the velocity of shear wave along x direction while tensile stress increases. Reverse effect is obtained along y direction.

**Case II:** Considering b = 0, i.e. for non-homogeneity in Q only, we get

$$\frac{c^2}{\beta^2}=[1+ay-\frac{P}{2Q_0}]p_1^4+2[\frac{2N_0}{Q_0}-(1+ay)]p_1^2p_2^2+[1+ay+\frac{P}{2Q_0}]p_2^4$$

The velocity along x- direction $(p_2 = 0, p_1 = 1)$ is given by $c_1^2 = \beta^2(1+ay-\frac{P}{2Q_0})$ , which depends on depth y. The velocity c increases as y increases and the wave is non-dispersive.
The velocity along y direction is $c_2^2 = \beta^2(1+ay+\frac{P}{2Q_0})$.
If $P > 0$, then the velocity along y direction may increase considerably at a distance y from free surface and the wave is dispersive.

**Case III:** For homogeneous medium (a = 0, b = 0) under initial stress

$$\frac{c^2}{\beta^2}=[1+-\frac{P}{2Q_0}]p_1^4+2[\frac{2N_0}{Q_0}-1]p_1^2p_2^2+[1+\frac{P}{2Q_0}]p_2^4$$

In the absence of initial stress the velocity equation is

$$\frac{c^2}{\beta^2}=1-4[1-\frac{2N_0}{Q_0}]p_1^2p_2^2$$

This shows that velocity $c=\beta$ in $x(p_2=0,p_1=1)$ direction and $y(p_2=1,p_1=0)$ direction. The velocity does not depend on anisotropy. However, in other direction the anisotropy has considerable effects on the velocity.

**Case IV:** In the absence of initial stress (P = 0) for non-homogeneous anisotropic medium the velocity equation is

$$\frac{c^2}{\beta^2}=1+4[\frac{N_0}{Q_0-1}]p_1^2p_2^2+ay(1-4p_1^2p_2^2)+\frac{4N_0}{Q_0}byp_1^2p_2^2$$

Now if $N_0=Q_0$, then the above equation takes the form

$$\frac{c^2}{\beta^2}=1+ay(1-4p_1^2p_2^2)+4byp_1^2p_2^2$$

In y-direction $(p_2=1,p_1=0)$, $\frac{c^2}{\beta^2}=1+ay$

Along x direction $(p_2=0,p_1=1)$, we have $\frac{c^2}{\beta^2}=1+ay$.

This means that the velocity increases in both the directions as y increases and wave is non-dispersive in both the cases.

**Numerical Calculation :** To get numerical information on the velocity of shear waves in the non-homogeneous initially stressed medium the equation (9) may be written in non-dimensional form as

$$\frac{c^2}{\beta^2}=[1+\frac{a}{b}by-\frac{P}{2Q_0}]p_1^4+2[\frac{2N_0}{Q_0}(1+by)-(1+\frac{a}{b}by)]p_1^2p_2^2+[1+\frac{a}{b}by+\frac{P}{2Q_0}]p_2^4$$

The numerical values of $\frac{c}{\beta}$ has been calculated for different values of a/b , b/k , $N_0/Q_0$ , $p_1, p_2$ and $P/2Q_0$ and results are presented fig 1-6. Fig-1 gives the variation in velocities of shear wave in the direction of $60^0$ with x-axis at different depths and different values of initial stress parameter when a/b=3, b/k = 0.01 and $N_0/Q_0=0.01$. The velocity of the wave increases as depth increases.

Fig. 2 is the velocity profile of shear wave at different direction at a depth y= 2/b for a/b = 2 , b/k = 0.01 , $N_0/Q_0=0.2$ and initial stress parameter $P/2Q_0=0.4$. It is observed that velocity becomes minimum at the direction of propagation $0^0$approximately from x- axis.

Fig.3 gives the information of variation of velocity for different values of a/b (non-homogeneity) and states that as the values of a/b increases the velocity of shear wave increases.

Fig.4 gives the information that as the anisotropic factor $(N_0/Q_0)$ increases the velocity of shear wave also increases.

Fig.5 gives the velocity of shear wave in an anisotropic initially stressed homogeneous medium for different values of initial stress parameter $P/2Q_0$ and $N/Q_0$ .

Fig.6 shows that as $P/2Q_0$ increases the velocity of propagation of shear wave of first kind decreases up to the direction of propagation from $0^0$ to $35^0$ with x-axis and second kind decreases up to the direction from $0^0$ to $40^0$but the velocity increases when the direction of propagation is more than $35^0$ and $40^0$ respectively.

**Conclusion :** It may be concluded that the anisotropy, non-homogeneity, the initial stresses, the direction of propagation and the depth (in case of non-homogeneous medium) have considerable effect in the velocity of propagation of shear wave and attracts the attention of earth scientists in their work.

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| Curve No. | P/2Q | a/b | b/k | N/Q | Theta (in degree) |
|---|---|---|---|---|---|
| 1 | −0.8 | 3 | 0.01 | 0.01 | 60 |
| 2 | 0 | 3 | 0.01 | 0.01 | 60 |
| 3 | 0.8 | 3 | 0.01 | 0.01 | 60 |

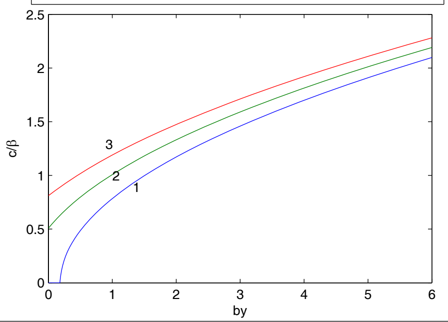


Fig 1: Velocity of propagation of shear wave at different depth from free surface when the wave propagates at 60 degree with horizontal direction

| P/2Q | a/b | b/k | N/Q | by |
|---|---|---|---|---|
| 0.4 | 2 | 0.01 | 2 | 2 |

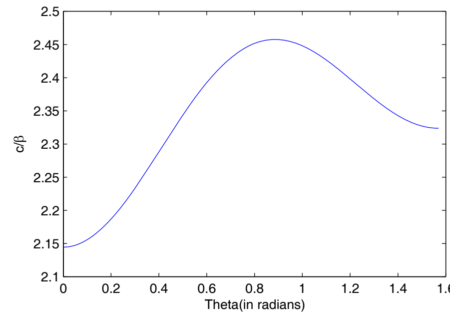


Fig 2: Variation of shear wave with respect to direction of propagation in a non−homogeneous anisotropic initially stressed medium

| Curve No. | P/2Q | a/b | b/k | N/Q | Theta (in degree) |
|---|---|---|---|---|---|
| 1 | 0.4 | 0 | 0.01 | 2 | 60 |
| 2 | 0.4 | 0.5 | 0.01 | 2 | 60 |
| 3 | 0.4 | 1 | 0.01 | 2 | 60 |
| 4 | 0.4 | 2 | 0.01 | 2 | 60 |
| 5 | 0.4 | 3 | 0.01 | 2 | 60 |

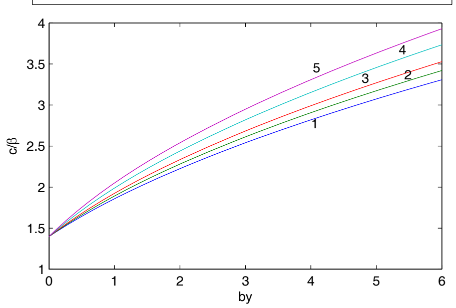


Fig 3: Velocity of shear wave at different depths and due to different variation ratio a/b, the anisotropic factor N/Q

| Curve No. | P/2Q | a/b | b/k | N/Q | Theta (in degree) |
|---|---|---|---|---|---|
| 1 | 0.4 | 3 | 0.01 | 0.5 | 60 |
| 2 | 0.4 | 3 | 0.01 | 1 | 60 |
| 3 | 0.4 | 3 | 0.01 | 2 | 60 |

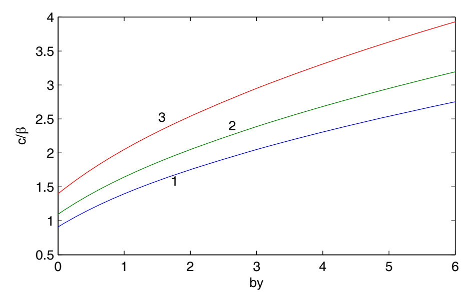


Fig 4: Effect of anisotropy on the velocity of propagation of shear wave

| P/2Q | a/b | b/k | N/Q | Theta (in degree) |
|---|---|---|---|---|
| 0.4 | 1 | 0 | 2 | 60 |

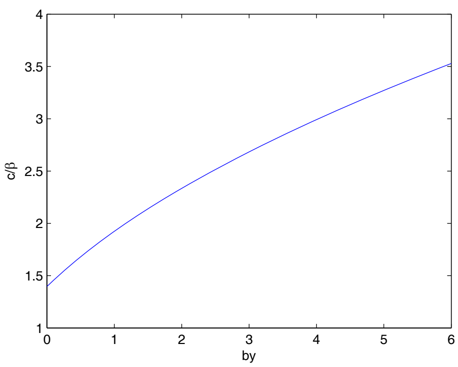


Fig 5: Velocity of shear wave under initial stress for constant N

| Curve No. | P/2Q | a/b | b/k | N/Q | by |
|---|---|---|---|---|---|
| 1 | 0.4 | 3 | 0.01 | 2 | 2 |
| 2 | 0.8 | 3 | 0.01 | 2 | 2 |

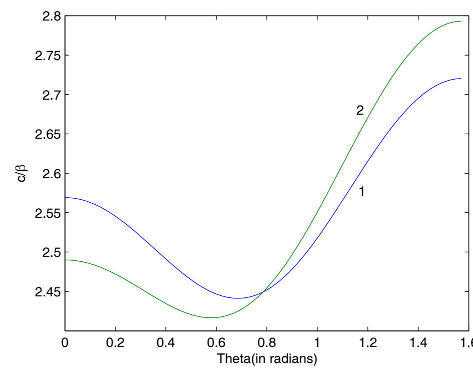


Fig 6: Effect of initial stresses in the propagation of shear waves at different direction of propagation