Applied Mathematical Sciences, Vol. 1, 2007, no. 60, 2955 - 2960

Weak Convergence Theorems for Asymptotically Nonexpansive Mappings

Yongfu Su

Department of Mathematics
Tianjin Polytechnic University
Tianjin,300160, P.R. China
suyongfu@tjpu.edu.cn

Liping Chen

Department of Computer
Hebei Vocational Institute of Political and Law
Shijiazhuang 050061, P.R. China

Abstract. Let K be a nonempty closed convex subset of a uniformly convex Banach space E and $T : K \to K$ be an asymptotically nonexpansive mapping with nonempty fixed points set $F(T)$. The $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0,1]$. The purpose of this article is to study the modified Ishikawa iteration process $\{x_n\}$ of T, for any initial guess $x_1 \in K$, defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n.$$

Not only weak and strong convergence are obtained but also the restriction $0 < a \leq \alpha_n \leq b < 1$ on $\{\alpha_n\}$ are relaxed. The results of this article extend and improve the results of many authors.

Keywords: Asymptotically nonexpansive; Iterative scheme; Fixed point; Weak convergence; Strong convergence; Opial's condition

1. Introduction and Preliminaries

Let K be a nonempty subset of a Banach space E, a mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $n \geq 1$ and for all $x, y \in E$. This class of mappings, as a natural extension to that of nonexpansive mappings, was introduced by Geobel and Kirk[1] in 1972.

Recall that T is said to be uniformly L-Lipschitzian mappings, if $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $n \geq 1$ and for all $x, y \in E$, where $L > 0$ is a constant.

It is obvious that, every asymptotically nonexpansive mapping is also uniformly L-Lipschitzian mapping.

Recall that a Banach space E is said to satisfy Opial's condition if, whenever $\{x_n\}$ is a sequence in E which converges weakly to x, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall\, y \in E \quad y \neq x.$$

Since Schu's results [2,3], the modified Mann and Ishikawa iteration schemes have been studied extensively by various authors to approximate fixed points of asymptotically nonexpansive mappings(see [2-8] and references therein).

Tan and Xu[4] extended Schu's result[2,3] from Hilbert spaces to the case of uniformly convex Banach spaces, and from the modified Mann iteration process to the modified Ishikawa iteration process defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n. \end{aligned} \tag{1.1}$$

In the present convergence theorems [1−8], the condition $0 < a \leq \alpha_n \leq b < 1$ is must needed. The purpose of this article is still to study the modified Ishikawa iteration process (1.1), not only weak and strong convergence theorems are obtained but also the restriction $0 < a \leq \alpha_n \leq b < 1$ on $\{\alpha_n\}$ are relaxed. The results of this article extend and improve the results of many authors.

In order to prove our theorems, the following lemmas will be useful.

Lemma 1.1.[2] *Let K be a nonempty convex subset of a linear normed space and $T : K \to K$ be a uniformly L-Lipschitzian mappings. For any given $x_0 \in E$ and real sequences $\{\alpha_n\}$,$\{\beta_n\}$ in $[0,1]$, the sequence $\{x_n\}$ is the modified Ishikawa iteration sequence defined by (1.1). Then*

$$\|Tx_n - x_n\| \leq c_n + c_{n-1}L(1 + 3L + 2L^2) \quad \forall n \geq 2$$

where $c_n = \|T^n x_n - x_n\|$.

Lemma 1.2.[3] *Let E be a uniformly convex Banach space, $\{t_n\}$ a real sequence such that $0 < a \leq t_n \leq b < 1, \forall\ n \geq 1$ for some $a, b \in (0,1)$, and $\{x_n\}, \{y_n\}$ are sequences in E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \quad \limsup_{n\to\infty} \|y_n\| \leq d$$

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \geq 0$ is a constant.

Lemma 1.3.[5] *Let$\{a_n\}$, $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + b_n)a_n, n \geq 1,$$

if $\sum_{n=1}^{\infty} b_n < +\infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

2. Main results

Theorem 2.1 *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E and $T : K \to K$ an asymptotically nonexpansive mapping with nonempty fixed points set $F(T)$ and with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$. $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$ and there exists a subsequence $\{n_k\}_{k=0}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that, $0 < a \le \alpha_{n_k} \le b < 1, 0 \le \beta_{n_k} \le b < 1$, for any given constants $a, b \in (0, 1)$. Then for any given $x_1 \in K$, we have*

$$\lim_{n\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0,$$

where $\{x_n\}$ is modified Ishikawa iteration sequence defined by (1.1).

Proof For any given $p \in F(T)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(T^n y_n - p)\| \\ &\le (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^n y_n - p\| \\ &\le (1 - \alpha_n)\|x_n - p\| + \alpha_n k_n\|y_n - p\|. \end{aligned} \tag{2.1}$$

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)(x_n - p) + \beta_n(T^n x_n - p)\| \\ &\le (1 - \beta_n)\|x_n - p\| + \beta_n\|T^n x_n - p\| \\ &\le (1 - \beta_n)\|x_n - p\| + \beta_n k_n\|x_n - p\|. \end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), we get

$$\|x_{n+1} - p\| \le= [1 + \alpha_n(1 + k_n\beta_n)(k_n - 1)]\|x_n - p\|$$

by the convergence of $\{k_n\}$ and $\alpha_n, \beta_n \in [0, 1]$, then there exists some $M > 0$ such that

$$\|x_{n+1} - p\| \le [1 + M(k_n - 1)]\|x_n - p\|.$$

Therefore, by condition $\sum_{n+1}^{\infty}(k_n - 1) < +\infty$ and Lemma 1.3,we know that, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. Setting $\lim\limits_{n\to\infty} \|x_n - p\| = d$, then we have

$$\lim_{k\to\infty} \|x_{n_k} - p\| = d. \tag{2.3}$$

Next, it follows from (2.2) that

$$\begin{aligned} &\|T^{n_k} y_{n_k} - p\| \le k_{n_k}\|y_{n_k} - p\| \\ &\le k_{n_k}(1 - \beta_{n_k})\|x_{n_k} - p\| + k_{n_k}^2\beta_{n_k}\|x_{n_k} - p\| \\ &\le k_{n_k}^2\|x_{n_k} - p\|. \end{aligned}$$

Thus

$$\limsup_{k\to\infty} \|T^{n_k} y_{n_k} - p\| \le \limsup_{k\to\infty} k_{n_k}^2\|x_{n_k} - p\| = d. \tag{2.4}$$

Since

$$\lim_{k\to\infty} \|\alpha_{n_k}(T^{n_k} y_{n_k} - p) + (1 - \alpha_{n_k})(x_{n_k} - p)\| = \lim_{k\to\infty} \|x_{n_{k+1}} - p\| = d. \tag{2.5}$$

By condition of theorem2.1, lemma 1.2 and (2.3)(2.4)(2.5) we obtain that

$$\lim_{k\to\infty} \|T^{n_k} y_{n_k} - x_{n_k}\| = 0. \tag{2.6}$$

It follows from the condition $0 \le \beta_{n_k} \le b < 1$ that

$$\begin{aligned}\|T^{n_k} x_{n_k} - x_{n_k}\| &\le \|T^{n_k} x_{n_k} - T^{n_k} y_{n_k}\| + \|T^{n_k} y_{n_k} - x_{n_k}\| \\ &\le k_{n_k}\|x_{n_k} - y_{n_k}\| + \|T^{n_k} y_{n_k} - x_{n_k}\| \\ &\le k_{n_k}\beta_{n_k}\|T^{n_k} x_{n_k} - x_{n_k}\| + \|T^{n_k} y_{n_k} - x_{n_k}\| \\ &\le k_{n_k} b\|T^{n_k} x_{n_k} - x_{n_k}\| + \|T^{n_k} y_{n_k} - x_{n_k}\|,\end{aligned}$$

which implies that

$$(1 - k_{n_k} b)\|T^{n_k} x_{n_k} - x_{n_k}\| \le \|T^{n_k} y_{n_k} - x_{n_k}\|.$$

Therefore, it is easy to see that

$$\lim_{n\to\infty} \|T^{n_k} x_{n_k} - x_{n_k}\| = 0,$$

using lemma 1.1, we obtain that

$$\lim_{n\to\infty} \|x_{n_k} - T x_{n_k}\| = 0.$$

This completes the proof.

Theorem 2.2 *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E and $T : K \to K$ a completely continuously asymptotically nonexpansive mapping with nonempty fixed points set and with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$. Let $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$ and there exists a subsequence $\{n_k\}_{k=0}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that, $0 < a \le \alpha_{n_k} \le b < 1, 0 \le \beta_{n_k} \le b < 1$, for some $a, b \in (0, 1)$. Then for any given $x_1 \in K$, the modified Ishikawa iteration sequence (1.1) converges strongly to a fixed point of T.*

Proof From the proof of theorem 2.1 we know the $\{x_n\}$ is bounded, since T is completely continuous, then there exists subsequence $\{T x_{n_{k_i}}\}$ of $\{T x_{n_k}\}$ such that $\lim_{i\to\infty} T x_{n_{k_i}} = p_0$. Thus it follows from Theorem2.1 that $\lim_{i\to\infty} x_{n_{k_i}} = p_0$. then it is easy to see that $Tp_0 = p_0$, that is $p_0 \in F(T)$, where $F(T)$ denote the fixed points set of T. Because $\lim_{n\to\infty} \|x_n - p_0\|$ exists, consequently, we have $\lim_{n\to\infty} x_n = p_0$. This completes the proof.

Theorem 2.3 *Let E be a uniformly convex Banach space which satisfies the Opial's condition, K a nonempty closed convex subset of E and $T : K \to K$ be an asymptotically nonexpansive mapping with nonempty fixed points set $F(T)$, if $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that $0 < a \le \alpha_{n_k} \le b < 1, \beta_{n_k} \le b < 1$ for any given constants $a, b \in (0, 1)$, then subsequence $\{x_{n_k}\}$ defined by (1.1) converges weakly to a fixed point of T.*

Proof It follows from the proof of Theorem 2.1 that $\{x_n\}$ is bounded, since E is uniformly convex, every bounded subset of E is weakly compact, so that there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_i}}\}$ converges weakly to a point $q \in K$. Therefore, it follows from theorem2.1 that

$$\lim_{n\to\infty} \|x_{n_{k_i}} - Tx_{n_{k_i}}\| = 0.$$

By lemma1.4, we know $I - T$ is demi-closed, so that $q \in F(T)$.

Finally, we prove that the sequence $\{x_{n_k}\}$ converges weakly to q. In fact, suppose this is not true, then there must exists a subsequence $\{x_{n_{k_j}}\} \subset \{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to another $q_1 \in K$, $q_1 \neq q$. Then, by the same method given above, we can also prove that $q_1 \in F(T)$.

Because, we have proved that, for any $p \in F(T)$, the limit $\lim_{n\to+\infty} \|x_n - p\|$ exists. Then we can let

$$\lim_{n\to\infty} \|x_n - q\| = d_1, \quad \lim_{n\to\infty} \|x_n - q_1\| = d_2,$$

by Opial's condition of E, we have

$$d_1 = \limsup_{i\to\infty} \|x_{n_{k_i}} - q\| < \limsup_{i\to\infty} \|x_{n_{k_i}} - q_1\| = d_2,$$

$$= \limsup_{j\to\infty} \|x_{n_{k_j}} - q_1\| < \limsup_{j\to\infty} \|x_{n_{k_j}} - q\| = d_1.$$

This is a contradiction, hence $q = q_1$. This implies that $\{x_{n_k}\}$ converges weakly to a fixed point of T, this completes the proof.

Theorem 2.4 *Let E be a uniformly convex Banach space which satisfies the Opial's condition, K a nonempty closed convex subset of E and $T : K \to K$ an asymptotically nonexpansive mapping with nonempty fixed points set $F(T)$ and with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$. $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$ which satisfy the following conditions*
(1) *There exists a constant $d \in (0, 1)$ such that $0 \le \alpha_n \le d < 1$;*
(2) *There exists a subsequence $\{n_k\}$ of $\{n\}$ such that $n_{k+1} - n_k \le N$ for some given constant $N > 0$ and $0 < a \le \alpha_{n_k} \le b < 1$ for some given constants $a, b \in (0, 1)$;*
(3) *$0 \le \beta_{n_k} \le c < 1$ for some given constant $c \in (0, 1)$.*
Then the Ishikawa iteration sequence $\{x_n\}$ defined by (1.2) converges weakly to a fixed point of T.

Proof. By using the theorem 2.1 we have

$$\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{2.7}$$

On the other hand, if $n \neq n_k$, then $\alpha_n \to 0$ as $n \to \infty$, so that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \alpha_n \|x_n - T^n y_n\| = 0. \tag{2.8}$$

Therefore, we obtain that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n_k}\| + \|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tx_n\| \\ &\leq (1+k_1)\|x_n - x_{n_k}\| + \|x_{n_k} - Tx_{n_k}\| \\ &\leq (1+k_1)\sum_{i=n}^{n_k-1}\|x_i - x_{i+1}\| + \|x_{n_k} - Tx_{n_k}\| \end{aligned} \tag{2.9}$$

where $0 < n_k - n \leq N$. It follows from (2.7)-(2.9) that

$$\lim_{n\to\infty, n\neq n_k} \|x_n - Tx_n\| = 0.$$

Hence this together with (2.7) we have, for all $n \geq 1$, that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

By using the standard method as in the theorem 2.3, we know the Ishikawa iteration sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. This completes the proof.

Remark. For example: In the theorem2.4, the parameters $\{\alpha\}, \{\beta_n\}$ can be chosen as follows

$$\alpha_n = \begin{cases} \frac{1}{n+1} & if \quad n \neq 10m, \\ \frac{1}{2} & if \quad n = 10m. \end{cases} \quad m = 1, 2, 3, \cdots$$

$$\beta_n = \begin{cases} \beta_n \ chosen\ arbitrarily & if \quad n \neq 10m, \\ \frac{1}{2} & if \quad n = 10m. \end{cases} \quad m = 1, 2, 3, \cdots$$

References

1. K. Goebel, W. Kirk, *A Fixed point theorem for asymptotically nonexpansive mappings*, Proc, Amer. Math. Soc. 35(1972), 171-174.
2. J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 158(1991),407-413
3. J. Schu, *Weak and strong convergence to fixed of asymptotically nonexpansive mappings* , Bull Austral Math soc. 43(1991), 153-159
4. K. K. Tan and H. K. Xu, *Fixed point iteration processes for asymptotically nonexpansive mappings* , Proc. Amer. Math. Soc. 122(1994), 733-739.
5. H. H. Bauschke, *The approximation of fixed points of composition of nonexpansive Mappings in Hilbert space*,J. Math. Anal. Appl,202(1996),150-159.
6. H. K. Xu, *Existence and convergence for fixed points of mappings of asymtotically nonexpansive type*, Nonlinear Anal. 16(1991), 1139-1146.
7. B. L. Xu and M. A. Noor, *Fixed points iterations for Asymptotically nonexpansive Mappings in Banach spaces,* , J Math.Anal.Appl, 267(2002), 444-453.
8. Yeol Je Cho, Haiyun Zhou and Ginti Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymtotically nonexpansive mappings*, Computers and Mathematics with Applications, 47(2004), 707-717.

Received: May 28, 2007