Applied Mathematical Sciences, Vol. 1, 2007, no. 60, 2961 - 2966

Obtuse Angle Principle for Approximation of Fixed Points of Nonlinear Mappings

Yongfu Su, Dongxing Wang and Meijuan Shang

Department of Mathematics
Tianjin Polytechnic University, Tianjin 300160, China
suyongfu@gmail.com (Y. F. Su)
wdxwangdongxing@163.com (D. X. Wang)
meijuanshang@sina.com (M. J. Shang)

Abstract. Let E be a Hilbert space , $T : D(T) \to R(T)$ be nonlinear mappings with nonempty fixed points set $F(T)$, and $\{x_n\}$ be iteration sequences of T. Assume $\{x_n\}$ satisfy the following conditions

(i) $\|x_{n+1} - p\| \le [1 + \sigma_n]\|x_n - p\| + \omega_n,\ n \ge 1,\ \forall\ p \in F(T);$

(ii) $\Sigma_{i=n}^{\infty}\sigma_i = o(\|x_n - x_0\|),\ \Sigma_{i=n}^{\infty}\omega_i = o(\|x_n - x_0\|);$

(iii) $x_n \to x_0 \in F(T).$

Where $\{\sigma_n\}, \{\omega_n\}$ are two real sequences. Then x_0 solves the following geometric variational inequality

$$I(p,\ x_0) = \limsup_{n\to\infty}\langle p - x_0, \frac{x_n - x_0}{\|x_n - x_0\|}\rangle \le 0,\ \forall\ p \in F(T).$$

This geometric result is said to be *Obtuse angle principle.*

Mathematics Subject Classification: 47H05; 47H09; 47H10

Keywords: Uniformly smooth Banach spaces; Hilbert spaces; Approximation of fixed points ; Obtuse angle principle

1. Preliminaries

We assume that E is a Banach space, E^* is the dual space of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\ \forall\ x \in E,$$

where $\langle\cdot,\cdot\rangle$ denote the generalized duality pairing. If E is uniformly smooth Banach space, then J is single-valued.

In 1978, Reich[1] established the following will-known inequality in uniformly Banach spaces, which has been extensively used by various authors.

Theorem R *Let E be a real uniformly smooth Banach space. Then there exists a nondecreasing continuous function $b : [0,\infty) \to [0,\infty)$ satisfying the following conditons*

(*i*) $b(ct) \leq cb(t), \quad \forall\ c \geq 1$;

(*ii*) $\lim_{t\to 0^+} b(t) = 0$.

(*iii*) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|),\ \forall\ x, y \in E$.

The inequality (iii) is usually called Reich's inequality. If E is a Hilbert space, then the following equality holds

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2,\ \forall\ x, y \in E.$$

The purpose of this paper is to study the general geometric structure for approximation of fixed points of nonlinear mappings by iteration sequences. We shall establish a general geometric result which is said to be the *Obtuse angle principle*.

2. Main results

Theorem1. *Let E be a real uniformly smooth Banach space, F be a nonempty subset of E and $\{x_n\} \subset E$ is a sequence . Assume the following conditions are satisfied*

(*i*) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|^2,\ \forall\ x, y \in E$;

(*ii*) $x_n \to x_0 \in F,\ x_n \neq x_0$;

(*iii*) *For any real number $r > 0$, there exists a integer N, if $n > N$ then*

$$\|x_{n+m} - p\|^2 - \|x_n - p\|^2 \leq r\|x_n - x_0\|,\ \forall\ p \in F, \forall\ m \geq 1.$$

Then x_0 is the solution of following geometric variational inequality

$$I(p,\ x_0) = \limsup_{n\to\infty}\langle p - x_0, J(\frac{x_n - x_0}{\|x_n - x_0\|})\rangle \leq 0,\ \forall\ p \in F.$$

Proof. Since $x_n \to x_0$, then there exists a integer N, if $n > N$ then $\|x_n - x_0\| < 1$, by using theorem1 condition (i) we have that

$$\|x_n - p\|^2 \leq \|x_n - x_0\|^2 + 2\langle x_0 - p, J(x_n - x_0)\rangle + \|x_0 - p\|^2,$$

which leads to

$$2\langle p - x_0, J(x_n - x_0)\rangle \leq \|x_n - x_0\|^2 + \|x_0 - p\|b(\|x_0 - p\|) - \|x_n - p\|^2.$$

Therefore

$$2\langle p - x_0, J(\frac{x_n - x_0}{\|x_n - x_0\|})\rangle \leq \|x_n - x_0\| + \frac{\|x_0 - p\|b(\|x_0 - p\|) - \|x_n - p\|^2}{\|x_n - x_0\|}.$$

$$2\langle p - x_0, J(\frac{x_n - x_0}{\|x_n - x_0\|})\rangle \leq \|x_n - x_0\| + \frac{\|x_0 - p\|^2 - \|x_n - p\|^2}{\|x_n - x_0\|}. \tag{1}$$

On the other hand, for any fix $r > 0$ and $n > N$, letting $m \to \infty$, we get

$$\|x_0 - p\|^2 - \|x_n - p\|^2 \leq r\|x_n - x_0\|, \ \forall\, p \in K. \qquad (2)$$

Combining (1)(2), if $n > N$ then

$$2\langle p - x_0, J(\frac{x_n - x_0}{\|x_n - x_0\|})\rangle \leq \|x_n - x_0\| + r.$$

Which implies that

$$\limsup_{n\to\infty}\langle p - x_0, J(\frac{x_n - x_0}{\|x_n - x_0\|})\rangle \leq \frac{r}{2}\ , \ \forall\, p \in K.$$

Since $r > 0$ is arbitrary, so that

$$\limsup_{n\to\infty}\langle p - x_0, J(\frac{x_n - x_0}{\|x_n - x_0\|})\rangle \leq 0\ , \ \forall\, p \in K.$$

This proof is complete.

Theorem2. *Let E be a real uniformly smooth Banach space, F be a nonempty subset of E and $\{x_n\} \subset E$ is a sequence . Assume the following conditions are satisfied*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|^2, \ \forall\, x, y \in E;$

(ii) $x_n \to x_0 \in F, \ x_n \neq x_0;$

(iii) $\|x_{n+1} - p\| \leq [1 + \sigma_n]\|x_n - p\| + \omega_n, \ n \geq 1, \ \forall\, p \in F;$

(iv) $\Sigma_{i=n}^{\infty}\sigma_i = o(\|x_n - x_0\|), \ \Sigma_{i=n}^{\infty}\omega_i = o(\|x_n - x_0\|);$

Where $\{\sigma_n\}, \{\omega_n\}$ are two real sequences. Then x_0 is the solution of following geometric variational inequality

$$I(p,\ x_0) = \limsup_{n\to\infty}\langle p - x_0, J(\frac{x_n - x_0}{\|x_n - x_0\|})\rangle \leq 0, \ \forall\, p \in F.$$

Proof. By the conditions of theorem2, we have that

$$\begin{aligned}
&\|x_{n+m} - p\| \leq [1 + \sigma_{n+m-1}]\|x_{n+m-1} - p\| + \omega_{n+m-1}\\
&\leq [1 + \sigma_{n+m-1}][1 + \sigma_{n+m-2}]\|x_{n+m-2} - p\| + [1 + \sigma_{n+m-1}]\omega_{n+m-2} + \omega_{n+m-1}\\
&\leq [1 + \sigma_{n+m-1}][1 + \sigma_{n+m-2}][1 + \sigma_{n+m-3}]\|x_{n+m-3} - p\|\\
&+[1 + \sigma_{n+m-1}][1 + \sigma_{n+m-2}]\omega_{n+m-3} + [1 + \sigma_{n+m-1}]\omega_{n+m-2} + \omega_{n+m-1}\\
&\leq \textstyle\prod_{i=n}^{n+m-1}[1 + \sigma_i]\|x_n - p\| + \prod_{i=n}^{n+m-1}[1 + \sigma_i]\sum_{i=n}^{n+m-1}\omega_i, \qquad (3)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_{n+m} - p\| - \|x_n - p\|\\
&\leq \{\prod_{i=n}^{n+m-1}[1 + \sigma_i] - 1\}\|x_n - p\| + \prod_{i=n}^{n+m-1}[1 + \sigma_i]\sum_{i=n}^{n+m-1}\omega_i
\end{aligned}$$

$$\leq \{\exp \sum_{i=n}^{n+m-1} \ln[1+\sigma_i] - 1\}\|x_n - p\| + \prod_{i=n}^{n+m-1} [1+\sigma_i] \sum_{i=n}^{n+m-1} \omega_i$$

$$\leq \{\exp \sum_{i=n}^{n+m-1} \sigma_i - 1\}\|x_n - p\| + \exp \sum_{i=n}^{n+m-1} \sigma_i \sum_{i=n}^{n+m-1} \omega_i$$

$$= \frac{\exp \sum_{i=n}^{n+m-1} \sigma_i - 1}{\sum_{i=n}^{n+m-1} \sigma_i} \sum_{i=n}^{n+m-1} \sigma_i \frac{\|x_n - p\|}{\|x_n - x_0\|}\|x_n - x_0\| + \exp \sum_{i=n}^{n+m-1} \sigma_i \sum_{i=n}^{n+m-1} \omega_i \quad (4)$$

since $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, $\lim_{n\to\infty} \|x_n - p\|$ exists, it follows from (4) that, there exists real number $M > 0$ such that

$$\|x_{n+m} - p\| - \|x_n - p\| \leq M \frac{\sum_{i=n}^{\infty} \sigma_i}{\|x_n - x_0\|}\|x_n - x_0\| + 2\sum_{i=n}^{\infty} \omega_i$$

$$\leq [M \frac{\sum_{i=n}^{\infty} \sigma_i}{\|x_n - x_0\|} + \frac{2\sum_{i=n}^{\infty} \omega_i}{\|x_n - x_0\|}]\|x_n - x_0\| \quad (5)$$

Combining (5) and conditions of theorem2, we know that, for any real number $r > 0$, there must exists integer N, if $n > N$ then

$$\|x_{n+m} - p\| - \|x_n - p\| < r\|x_n - x_0\|, \ \forall p \in F.$$

Because $\lim_{n\to\infty} \|x_n - p\|$ exists, so there exists real number $M_1 > 0$, if $n > N$ then

$$\|x_{n+m} - p\|^2 - \|x_n - p\|^2$$

$$= (\|x_{n+m} - p\| + \|x_n - p\|)(\|x_{n+m} - p\| - \|x_n - p\|)$$

$$< (\|x_{n+m} - p\| + \|x_n - p\|)r\|x_n - x_0\|$$
$$\leq M_1 r\|x_n - x_0\|, \ \forall p \in F.$$

By using theorem1, we get

$$I(p, x_0) = \limsup_{n\to\infty} \langle p - x_0, J(\frac{x_n - x_0}{\|x_n - x_0\|})\rangle \leq 0, \forall p \in F.$$

This proof is complete.

Theorem3. *Let E be a Hilbert space , $T : D(T) \to R(T)$ be a nonlinear mapping with nonempty fixed points set $F(T)$, and $\{x_n\}$ be a iteration sequence of T. Assume $\{x_n\}$ satisfies the following conditions*

(*i*) $\|x_{n+1} - p\| \leq [1 + \sigma_n]\|x_n - p\| + \omega_n, \ n \geq 1, \ \forall \ p \in F(T)$;
(*ii*) $\Sigma_{i=n}^{\infty} \sigma_i = o(\|x_n - x_0\|), \ \Sigma_{i=n}^{\infty} \omega_i = o(\|x_n - x_0\|)$;
(*iii*) $x_n \to x_0 \in F(T)$.

Where $\{\sigma_n\}, \{\omega_n\}$ are two real sequences. Then x_0 is the solution of following geometric variational inequality

$$I(p, \ x_0) = \limsup_{n\to\infty} \langle p - x_0, \frac{x_n - x_0}{\|x_n - x_0\|}\rangle \leq 0, \ \forall \ p \in F(T).$$

This geometric result is said to be *Obtuse angle principle.*

Proof. Since E is Hilbert space, as we well know that

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \ \forall\, x, y \in E,$$

which leads to

$$\|x+y\|^2 \le \|x\|^2 + 2\langle x, y\rangle + \max\{\|x\|, 1\}\|y\|^2, \ \forall\, x, y \in E.$$

By using the theorem2, we get

$$I(p,\ p_0) = \limsup_{n\to\infty}\langle p - x_0, \frac{x_n - x_0}{\|x_n - x_0\|}\rangle \le 0, \ \forall\, p \in F(T).$$

This proof is complete.

Remark. In the Hilbert spaces, for all nonexpansive mappings or some other nonlinear mappings T with nonempty fixed points set $F(T)$, the Mann, Ishikawa and Noor or other[1-16] iteration sequences $\{x_n\}$ satisfy the following condition

$$\|x_{n+1} - p\| \le \|x_n - p\|, \ \forall\, p \in F(T).$$

Therefore, if iteration sequences $\{x_n\}$ converges strongly to a fixed point $x_0 \in F(T)$, the *Obtuse angle principle* must be true.

In addition, for all asymptotically nonexpansive mappings or some other[1-16] nonlinear mappings with nonempty fixed points set $F(T)$, the modified Mann, Ishikawa and Noor or other iteration sequences $\{x_n\}$ satisfy the following condition

$$\|x_{n+1} - p\| \le (1+\sigma_n)\|x_n - p\| + \omega_n, \ \forall\, p \in F(T),$$

where $\sum_{n=1}^{\infty}\sigma_n < \infty, \sum_{n=1}^{\infty}\omega_n < \infty$. Therefore, if iteration sequences $\{x_n\}$ converges strongly to a fixed point $x_0 \in F(T)$, and adjoin the conditions

$$\Sigma_{i=n}^{\infty}\sigma_i = o(\|x_n - p_0\|), \ \Sigma_{i=n}^{\infty}\omega_i = o(\|x_n - p_0\|),$$

the *Obtuse angle principle* must be true.

The *Obtuse angle principle* not only represent the geometric structure of iteration process of approximation for fixed points of nonlinear mappings, but also is necessary condition for iteration sequences converges strongly to $x_0 \in F(T)$. In particular, the limit point x_0 of iteration sequences $\{x_n\}$ must be the solution of geometric variational inequality

$$I(p,\ x_0) = \limsup_{n\to\infty}\langle p - x_0, \frac{x_n - x_0}{\|x_n - x_0\|}\rangle \le 0, \ \forall\, p \in F(T).$$

Let

$$\theta_n(p, x_0) = \arccos\langle \frac{p - x_0}{\|p - x_0\|}, \frac{x_n - x_0}{\|x_n - x_0\|}\rangle,$$

then *Obtuse angle principle* may be written

$$\theta(p, x_0) = \liminf_{n\to\infty}\theta_n(p, x_0) \ge \frac{\pi}{2}.$$

In 2006, Yongfu Su and Haiyun Zhou[17] established the Obtuse angle principle in Hilbert spaces by using different proved method.

Conjecture. In the uniformly smooth Banach spaces, the *Obtuse angle principle* is still true.

References

1. Reich.S, *An iterative procedure for constrncting zero of accretive sets in Banach spaces*, Nonlinear Anal. TMA,2(1978)85-92.
2. Ishikawa.S, *Fixed points by a new iteration method*, Proc.Amer.Math.Soc,144(1974)147-150.
3. Xu. L, Noor.M.A, *Fixed point iterations for asymptotically nonexpansive mappings in Bnanch spaces*, J. Math. Anal. Appl, 267(2002)444-453.
4. Liu L S, *Ishikawa and Mann iterative processes with errors for nonlinear strong accretive mappings in Banach spaces*, J. Math. Anal. Appl, 194(1995)114-125.
5. Reich. S, *Weak convergence theorem for nonexpansive mappings in Banach spaces*, J.Math,Anal. Appl, 67(1979)274-276.
6. Tan. K.K and Xu. H.K, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process* , J,Math.Anal.Appl, 178(1993)301-308.
7. Zeng L.C, *A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J.Math.Anal.Appl, 226(1998)245-250.
8. Zeng L.C, *Ishikawa Iteration Process for Approximating Fixed Points of Nonexpansive mappings*, Journal of Mathematical Research and Exposition, 123(2003)33-39.
9. Schu. J, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc, 143(1991)153-159.
10. Ishikawa.S,*Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer.Math.Soc, 159(1976)65-71.
11. H.K.Xu, *Another control condition in an iterative method for nonexpansive mapping* , Bull. Austral Math.Soc , 156(2002)109-113.
12. Jung Im Kang, Yeol Je Cho and Haiyun Zhou , *Convergence theorems of iteration sequences for nonexpansive mappings*, Common. Korean. Mtah.Soc, 119(2004)321-382.
13. Osilike. M. O, *Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps*, J.Math.Anal.Appl, 294(2004)73-84.
14. Qihou. L , *Iterative sequenecs for asymptotically Quasi-nonexpansive mappings*, J. Math. Anal. Appl, 259(2001)1-7.
15. Nakajo.K and Takahashi.W, *Strong Convergence theorems for nonexpansive mappings and nonexpansive semigroups* , J. Math. Anal. Appl, 64(2006)1140-1152.
16. Yongfu S and Suhong L, *Composite implicit iteration process for common fixed points of a finite family of strctly pseudocontractive maps* , J. Math. Anal. Appl, 320(2006)882-891.
17. Yongfu S and Haiyun Zhou, *A geometric result for approximating fixed points of nonlonear mappings by iteration sequence* , Acta Mathematica Sinica, Chinese Series, 49(2006)1321-1326.

Received: July 14, 2007