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Euclidean Ranking DMUs with Fuzzy Data in DEA

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Abstract

If the inputs and outputs are fuzzy numbers, the DMUs cannot be easily evaluated and ranked using the obtained efficiency scores. In this paper, a new idea based on interactive method for ranking of DMUs with fuzzy data using l_2 norm is introduced. The method is illustrated by solving a numerical example.

Keywords: Data Envelopment Analysis; Efficiency; Ranking; Fuzzy data

1 Introduction

Data Envelopment Analysis (DEA) was suggested by Charnes, Cooper and Rhodes(CCR), [3], and built on the idea of Farrell [4] which is concerned with

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the estimation of technical efficiency and efficient frontiers. In some cases, we have to use imprecise input and output. To deal quantitatively with imprecision in decision progress, Bellman and Zadeh [2] introduce the notion of fuzziness. Some researchers have proposed several fuzzy models to evaluate DMUs with fuzzy data, and introduce a ranking approach with efficiency measure of the model (see[8, 9, 12]). In this paper, we first introduce one approach with Euclidean Norm for ranking of DMUs with crisp data. Second, this model for ranking of DMUs with fuzzy data is used.
The paper is organized as follows: The background on l_2 norm is brought in section 2. An approach for ranking DMUs using l_2-norm is introduced in section 3. The background on fuzzy sets is brought in section 4. An approach for ranking DMUs with fuzzy data in DEA is introduced in section 5. A numerical example and conclusions are drawn in section 6 and 7 respectively.

2 Norms

In order to study ranking model based on l_2-norm, we need to recall definition of norms.

A norm on $\mathbb{R}^n$ is a function that assigns to each $x \in \mathbb{R}^n$ a non-negative real number $\|x\|$, called the norm of x, such that following three properties are satisfied for all $x, y \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$:

- $\|0\| = 0$ and $\|x\| > 0$ if $x \neq 0$ (positive definite property)
- $\|\alpha x\| = |\alpha|\|x\|$ (absolute homogeneity)
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Any norm can be used to measure the lengths or magnitudes (in a generalized sense) of vectors in $\mathbb{R}^n$. In other words, we think of $\|x\|$ as the (generalized) length of x. The (generalized) distance between two vectors x and y is $\|x-y\|$.

For any real number $p \geq 1$, we define p -norms by:

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \tag{1}$$

The most important p -norm is the 2-norm, which is just the Euclidean norm where defined by

$$\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}} \tag{2}$$

Another important p -norm is the 1-norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \tag{3}$$

The ∞ -norm is defined by

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i| \tag{4}$$

This is obvious that

$$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p \tag{5}$$

3 Euclidean Model In DEA

In this section, we introduce ranking model based on l_2-Norm in data envelopment analysis. We assume that the DMU_p is extreme efficient. By omitting (X_p, Y_p) from T_c (PPS of CCR model), we define the production possibility set $T_c^{'}$ as follows:

$$T_c^{'} = \{(X, Y) \mid X \ge \sum_{j=1, j \ne p}^{n} \lambda_j X_j, Y \le \sum_{j=1, j \ne p}^{n} \lambda_j Y_j, \lambda_j \ge 0, j = 1, ..., n, j \ne p\} \tag{6}$$

To obtain the ranking score of DMU_p, we consider the following model:

$$\begin{array}{llll} l_2 - Norm: & min & \Gamma_c^p(X, Y) = \sum_{i=1}^{m} (x_i - x_{ip})^2 & + \sum_{r=1}^{s} (y_r - y_{rp})^2 \\ & s.t & \sum_{j=1, j \ne p}^{n} \lambda_j x_{ij} \le x_i & i = 1, ..., m \\ & & \sum_{j=1, j \ne p}^{n} \lambda_j y_{rj} \ge y_r & r = 1, ..., s \\ & & x_i \ge 0 & i = 1, ..., m \\ & & y_r \ge 0 & r = 1, ..., s \\ & & \lambda_j \ge 0 & j = 1, ..., n, j \ne p \end{array} \tag{7}$$

Where $X = (x_1, ..., x_m)$, $Y = (y_1, ..., y_s)$ and $\lambda = (\lambda_1, ..., \lambda_{p-1}, \lambda_{p+1}, ..., \lambda_n)$ are the variables of the model (7) and $\Gamma_c^p(X, Y)$ is the distance (X_p, Y_p) from (X, Y) by using $l_2 - Norm$. It is evident that the model (2) is nonlinear quadratic problem.

Quadratic programming represents a special class of nonlinear programming in which the objective function is quadratic and the constraints are linear. The KKT conditions of a quadratic programming problem reduce to a linear complementary problem. Thus the complementary pivoting algorithm can be used for solving a quadratic programming problem.

4 Fuzzy Background

A fuzzy number is a fuzzy set $\tilde{A}$ on the real line R whose membership function $\mu_A(.)$ is upper semi-continuous (we will suppose that it is continuous) and such that:

$$r = \mu_A(x) = \begin{cases} f_A(x) & if x \in [a_1, a_2] \\ 1 & if x \in [a_2, a_3] \\ g_A(x) & if x \in [a_3, a_4] \\ 0 & otherwise \end{cases} \tag{8}$$

Such that $f_A(.)$ is increasing function on $[a_1, a_2]$ and $g_A(.)$ is decreasing function on $[a_3, a_4]$. For trapezoidal fuzzy number we have, $f_A(x) = \frac{x-a_1}{a_2-a_1}$ and $g_A(x) = \frac{x-a_4}{a_3-a_4}$.
The α-cut of a fuzzy number $\tilde{A}$ is defined as $[\tilde{A}]^\alpha = \{x|\mu_A(x) \geq \alpha\}$. Since $\mu_A(.)$ is upper semi-continuous then the α-cuts are closed and bounded intervals and we represent by $[\tilde{A}]^\alpha = [f_A^{-1}(r), g_A^{-1}(r)]$.
The $supp(\tilde{A})$ is defined by $supp(\tilde{A}) = cl(\{x|\mu_A(x) > 0\})$.
For two arbitrary fuzzy numbers A and B with α- cuts $[f_A^{-1}(r), g_A^{-1}(r)]$ and $[f_B^{-1}(r), g_B^{-1}(r)]$, respectively, the quantity

$$d(A, B) = \sqrt{\int_0^1 (f_A^{-1}(r) - f_B^{-1}(r))^2 d\alpha + \int_0^1 (g_A^{-1}(r) - g_B^{-1}(r))^2 d\alpha} \tag{9}$$

is the l_2 distance between A and B. For more details we refer the reader to [5].

Following Heilpern [6], we define the expected interval and expected value of a fuzzy number $\tilde{A}$ and noted them by $EI(\tilde{A})$ and $EV(\tilde{A})$, respectively.

$$\begin{gathered} EI(\tilde{A}) = [E_1^A, E_2^A] = [\int_0^1 f_A^{-1}(r)dr, \int_0^1 g_A^{-1}(r)dr] \\ EV(\tilde{A}) = \frac{E_1^A + E_2^A}{2} = \frac{\int_0^1 f_A^{-1}(r)dr + \int_0^1 g_A^{-1}(r)dr}{2} \end{gathered} \tag{10}$$

If $\tilde{A} = (a_1, a_2, a_3, a_4)$ is a trapezoidal fuzzy number then:

$$EI(\tilde{A}) = [\frac{a_1 + a_2}{2}, \frac{a_3 + a_4}{2}] \quad , \quad EV(\tilde{A}) = \frac{1}{4}(a_1 + a_2 + a_3 + a_4) \tag{11}$$

Proposition 1. If $\tilde{A}$ and $\tilde{B}$ are two fuzzy numbers then:

$$\begin{aligned} EI(\lambda\tilde{A}+\mu\tilde{B}) &= \lambda EI(\tilde{A})+\mu EI(\tilde{B}) \\ EV(\lambda\tilde{A}+\mu\tilde{B}) &= \lambda EV(\tilde{A})+\mu EV(\tilde{B}) \end{aligned} \tag{12}$$

5 Ranking In Fuzzy DEA

In this section, we suppose that inputs and outputs of DMUs are fuzzy numbers. Therefore,

$$\tilde{T}_c^{'} = \{(X,Y) \mid X \geq \sum_{j=1,j\neq p}^{n} \lambda_j \tilde{X}_j, Y \leq \sum_{j=1,j\neq p}^{n} \lambda_j \tilde{Y}_j, \lambda_j \geq 0, j=1,...,n, j\neq p\} \tag{13}$$

l_2-Norm model with Eqs. (9) can be extended to the following model:

$$\begin{aligned} min \quad & \Gamma_c^p(X,Y) = \sum_{i=1}^{m}(\int_0^1 (f_{x_i}^{-1}(r) - f_{x_{ip}}^{-1}(r))^2 d\alpha + \int_0^1 (g_{x_i}^{-1}(r) - g_{x_{ip}}^{-1}(r))^2 d\alpha) \\ & + \sum_{r=1}^{s}(\int_0^1 (f_{y_r}^{-1}(r) - f_{y_{rp}}^{-1}(r))^2 d\alpha + \int_0^1 (g_{y_r}^{-1}(r) - g_{y_{rp}}^{-1}(r))^2 d\alpha) \\ s.t \quad & (X,Y) \in \tilde{T}_c^{'} \end{aligned} \tag{14}$$

Where $X = (x_1, ..., x_m)$, $Y = (y_1, ..., y_s)$ and $\lambda = (\lambda_1, ..., \lambda_{p-1}, \lambda_{p+1}, ..., \lambda_n)$ are the variables of the model (14).
For solving the model (14) we consider:

Definition 1. Jimenez [10], For any pair of fuzzy numbers $\tilde{A}$ and $\tilde{B}$ the degree in $\tilde{A}$ bigger than $\tilde{B}$ is the following:

$$\mu_M(\tilde{A},\tilde{B}) = \begin{cases} 0 & if \quad E_2^A - E_1^B < 0 \\ \frac{E_2^A - E_1^B}{E_2^A - E_1^A + E_2^B - E_1^B} & if \quad 0 \in [E_1^A - E_2^B, E_2^A - E_1^B] \\ 1 & if \quad E_1^A - E_2^B > 0 \end{cases} \tag{15}$$

Were $[E_1^A, E_2^A]$ and $[E_1^B, E_2^B]$ are the expected intervals of $\tilde{A}$ and $\tilde{B}$. When $\mu_M(\tilde{A},\tilde{B}) = \frac{1}{2}$, we will say that $\tilde{A}$ and $\tilde{B}$ are indifferent. When $\mu_M(\tilde{A},\tilde{B}) \geq \alpha$ we will say that $\tilde{A}$ is bigger than, or equal to $\tilde{B}$ at least in degree α and we will represent it by $\tilde{A} \geq_\alpha \tilde{B}$.

Definition 2. Given a production possibility $(X, Y) \in \tilde{T}_c^{'}$, we will say that it is product in degree α in $\tilde{T}_c^{'}$ if:

$$min \left\{ \begin{array}{cc} \mu_M(x_i, \sum\limits_{j=1, j\neq p}^{n} \lambda_j \tilde{x}_{ij}), & \mu_M(\sum\limits_{j=1, j\neq p}^{n} \lambda_j \tilde{y}_{rj}, y_r) \\ \mu_M(x_i, \tilde{x}_{ip}) & \mu_M(\tilde{y}_{rp}, y_r) \\ i = 1, ..., m & r = 1, ..., s \end{array} \right\} = \alpha \tag{16}$$

That is to say

$$\begin{array}{l} x_i \geq_\alpha \sum\limits_{j=1, j\neq p}^{n} \lambda_j \tilde{x}_{ij} \quad , \quad i = 1, ..., m \\ y_r \leq_\alpha \sum\limits_{j=1, j\neq p}^{n} \lambda_j \tilde{y}_{rj} \quad , \quad r = 1, ..., s \end{array} \tag{17}$$

With proposition 1:

$$\begin{array}{l} x_i \geq \sum\limits_{j=1, j\neq p}^{n} \lambda_j (\alpha E_2^{x_{ij}} + (1-\alpha) E_1^{x_{ij}}) \quad i = 1, ..., m \\ y_r \leq \sum\limits_{j=1, j\neq p}^{n} \lambda_j (\alpha E_1^{y_{rj}} + (1-\alpha) E_2^{y_{rj}} \quad r = 1, ..., s \end{array} \tag{18}$$

Definition 3. A production possibility $(X_o, Y_o)^\alpha \in \tilde{T}_c^{'}$ is an α-acceptable optimal(nearest) solution of model (14) if it is an optimal solution of the following model:

$$\begin{array}{ll} min & \Gamma_c^p(X, Y)_\alpha = \sum\limits_{i=1}^{m} (\int_0^1 (f_{x_i}^{-1}(r) - f_{x_{ip}}^{-1}(r))^2 d\alpha + \int_0^1 (g_{x_i}^{-1}(r) - g_{x_{ip}}^{-1}(r))^2 d\alpha) \\ & \quad + \sum\limits_{r=1}^{s} (\int_0^1 (f_{y_r}^{-1}(r) - f_{y_{rp}}^{-1}(r))^2 d\alpha + \int_0^1 (g_{y_r}^{-1}(r) - g_{y_{rp}}^{-1}(r))^2 d\alpha) \\ s.t & (X, Y) \in \tilde{T}_c^{'\alpha} \end{array} \tag{19}$$

Where

$$\tilde{T}_c^{'\alpha} = \{(X, Y) \mid X \geq_\alpha \sum_{j=1, j\neq p}^{n} \lambda_j \tilde{X}_j, Y \leq_\alpha \sum_{j=1, j\neq p}^{n} \lambda_j \tilde{Y}_j, \lambda_j \geq 0, j = 1, ..., n\} \tag{20}$$

Proposition 3. If $\alpha_1 < \alpha_2$ then $\tilde{T}_c^{'\alpha_2} \subseteq \tilde{T}_c^{'\alpha_1}$.

We write model (19) as follow:

$$
\begin{array}{lll}
min & \Gamma_c^p(X,Y)_\alpha = \sum\limits_{i=1}^{m}(\int_0^1(f_{x_i}^{-1}(r)-f_{x_{ip}}^{-1}(r))^2 d\alpha + \int_0^1(g_{x_i}^{-1}(r)-g_{x_{ip}}^{-1}(r))^2 d\alpha) & \\
& \quad +\sum\limits_{r=1}^{s}(\int_0^1(f_{y_r}^{-1}(r)-f_{y_{rp}}^{-1}(r))^2 d\alpha + \int_0^1(g_{y_r}^{-1}(r)-g_{y_{rp}}^{-1}(r))^2 d\alpha) & \\
s.t & x_i \geq \sum\limits_{j=1,j\neq p}^{n}\lambda_j(\alpha E_2^{x_{ij}}+(1-\alpha)E_1^{x_{ij}}) & i=1,...,m \\
& y_r \leq \sum\limits_{j=1,j\neq p}^{n}\lambda_j(\alpha E_1^{y_{rj}}+(1-\alpha)E_2^{y_{rj}}) & r=1,...,s \\
& y_r \geq 0 & r=1,...,s \\
& \lambda_j \geq 0 & j=1,...,n
\end{array}
\tag{21}
$$

Model (21) is a crisp α-parametric model. Therefore, we can solve it by the interactive method. Now we are going to explain the interactive method.

5.1 Interactive Method

Regarding to Proposition 3, to obtain the nearest (X,Y) of $\tilde{T}_c'$ implies a lesser degree of production possibility. Then the decision-maker runs into two conflicting objectives: to find the nearest (X,Y) and to improve the degree of production possibility. Following Kaufmann and Gil Aluja [11], we consider 11 scales, which allow for different choice of decision-maker idea in (21) model.
1: $\alpha = 0$ Unacceptable solution
2: $\alpha = 0.1$ Practically unacceptable solution
3: $\alpha = 0.2$ Almost unacceptable solution
4: $\alpha = 0.3$ Very unacceptable solution
5: $\alpha = 0.4$ Quite unacceptable solution
6: $\alpha = 0.5$ Neither acceptable nor unacceptable solution
7: $\alpha = 0.6$ Quite acceptable solution
8: $\alpha = 0.7$ Very acceptable solution
9: $\alpha = 0.8$ Almost acceptable solution
10: $\alpha = 0.9$ Practically acceptable solution
11: $\alpha = 1$ Completely acceptable solution
We choice the α_0 is the minimum acceptable degree with decision-maker idea. Then, we solving the (21) α-parametric model for each $\alpha_k; k = 0,1,...,(10-10\alpha_0)$. We obtain the α_k-acceptable optimal fuzzy value of objective function of original model (14) with α_k-acceptable solution of model (21) in model (14).

6 Numerical example

A simple numerical example with fuzzy single-input and single-output was introduced by C. Kao and S.T. Liu [12]. We will consider this example with its data listed in table 1.

Table 1: Fuzzy data of DMUs in fuzzy data

$DMUs$	$input$	$\alpha - cut$	$output$	$\alpha - cut$
A	$(11, 12, 14)$	$[11 + \alpha, 14 - 2\alpha]$	$(10, 10, 10)$	$[10, 10]$
B	$(30, 30, 30)$	$[30, 30]$	$(12, 13, 14, 16)$	$[12 + \alpha, 16 - 2\alpha]$
C	$(40, 40, 40)$	$[40, 40]$	$(11, 11, 11)$	$[11, 11]$
D	$(45, 47, 52, 55)$	$[45 + 2\alpha, 55 - 3\alpha]$	$(12, 15, 19, 22)$	$[12 + 3\alpha, 22 - 3\alpha]$

These $DMUs(A, B, C$ and $D)$ are evaluated by proposed models in (14) with different α_k. The α-parametric model for θ_A is shown as follows:

$$\begin{array}{ll} min & \int_0^1 (x - 11 - \alpha)^2 d\alpha + 2\int_0^1 (y - 10)^2 d\alpha + \int_0^1 (x - 14 + 2\alpha)^2 d\alpha \\ s.t & x \geq 30\lambda_B + 40\lambda_C + \lambda_D(53.5 - 7.5\alpha), \\ & y \leq \lambda_B(15 - 2.5\alpha) + 11\lambda_C + \lambda_D(20.5 - 7\alpha), \\ & x \geq 0, \\ & y \geq 0, \\ & \lambda_B, \lambda_C, \lambda_D \geq 0 \end{array} \tag{22}$$

The α-parametric model for $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ can be showed similarly. The results is shown in table 2 for α=0.0,.1, .2,...,1. In

Table 2: The optimal values of α-parametric model.

α	Γ_c^A	Γ_c^B	Γ_c^C	Γ_c^D	$(\text{Ranking})_\alpha$
0	25.5667	3.5417	0	55.2083	$(D, A, B, C)_0$
0.1	27.0176	3.5417	0	55.2083	$(D, A, B, C)_{0.1}$
0.2	28.5187	3.5417	0	55.2083	$(D, A, B, C)_{0.2}$
0.3	30.0705	3.5417	0	55.2083	$(D, A, B, C)_{0.3}$
0.4	31.6735	3.5417	0	55.2083	$(D, A, B, C)_{0.4}$
0.5	33.3281	3.5417	0	55.2083	$(D, A, B, C)_{0.5}$
0.6	35.0347	3.5417	0	55.2083	$(D, A, B, C)_{0.6}$
0.7	36.7936	3.5417	0	55.2083	$(D, A, B, C)_{0.7}$
0.8	38.6054	3.5417	0	55.2083	$(D, A, B, C)_{0.8}$
0.9	40.4703	3.5417	0	55.2083	$(D, A, B, C)_{0.9}$
1.0	42.3886	3.5417	0	55.2083	$(D, A, B, C)_1$

(23)

7 Conclusions

In this paper a new approach based on l_2-norm for ranking of DMUs with fuzzy data in DEA is introduced. The method is based on the interactive method. α-acceptable optimal solution of proposed model for $\alpha \geq \frac{1}{2}$ is an acceptable solution. For any DMU, the score of ranking is obtained by solving $\alpha-$parametric model (21).

References

[1] M. S. Bazara, H. D. Sherali, C. M. Shetty, Nonlinear programming theory and algorithms, Second Edition, John Wiley and Sons, 1993

[2] R. E. Bellman, L. A. Zadeh, Decision-making in a fuzzy environment, Management Sci. 17(1970), 141-164.

[3] A. Charnes, W. W. Cooper, E. Rhodes, Measuring the efficiency of decision making units, Eur. J. Operat. Res. 2(1978), 429-444.

[4] M. J. Farrell, The measurement of productive efficiency, Journal of the Royal Statistical Society A 120(1957), 253-281.

[5] P. Grzegorzewski, Metrics and orders in space of fuzzy numbers, Fuzzy sets and systems 97 (1998) 83 - 94.

[6] S. Heilpern, The expected valued of a fuzzy number, Fuzzy Sets and Systems 47(1996), 81-86.

[7] G. R. Jahanshahloo,F. Hosseinzade,N. Shoja,G. Tohidi, SH. Razavian, Ranking by sing l_2-Norm in Data Envelopment Analysis, Applied mathematics and computation 153(2004), 215-224.

[8] G.R. Jahanshahloo, M. Soleimani-damaneh, E. Nasrabadi, Measure of efficiency in DEA with fuzzy input-output levels: a methodology for assessing, ranking and imposing of weights restrictions, Applied Mathematics and Computation 156 (2004), 175-187.

[9] G.R. Jahanshahloo, F. Hosseinzade Lotfi, M. Adabitabar Firozja, E. Zahedy Alamdary, T. Allahviranloo, Ranking DMUs with Fuzzy Data In DEA, Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 5, 203 - 211.

[10] M. Jimenez, Ranking fuzzy numbers through the comparison of its expected intervals, International Journal of Uncertainty, Fuzziness and Knowledge -Based Systems 4(2004), 379-388.

[11] A. Kaufmann,J. Gil Aluja, Tecnicas de Gestion de Empresa. Piramide, Madrid, 1992.

[12] C. Kao, S. T. Liu, A mathematical programming approach to fuzzy efficiency ranking, Internat. J. Production Econom 86 (2003) 45-154.

[13] D. S. Watkins, Fundamentals of Matrix Computations, Second Edition, John Wiley and Sons, 2002.

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