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On the Study of a Fading-Memory System

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Abstract

As a natural continuation of [7], we use in this paper the parabolic regularization approach [12], [19], to establish the existence, uniqueness, regularity and asymptotic behavior of the solution of an integro-differential equation. To motivate the considered system, we give two fading-memory models stem from fluid mechanics theory.

Keywords: Existence- uniqueness- regularity- asymptotic behavior- fading-memory systems

1 Introduction

In this paper, we are concerned with the following fading-memory system

$$\mathcal{P}_\infty \begin{cases} (1.1) \quad \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=1}^{N} A_i \frac{\partial u}{\partial x_i} + Bu - \int_{-\infty}^{t} G(t-s)\Delta u(x,s)ds = f - \nabla p \text{ in } \Omega, t. \\ (1.2) \quad u(x,t) = h(x,t) \quad t \le 0, \quad x \in \Omega. \\ (1.3) \quad div\ u = 0 \qquad \text{Incompressibility condition.} \\ (1.4) \quad u(x,t) = 0 \qquad \text{on} \quad \Gamma \times]0,T[\end{cases}$$

Throughout this paper, $N \in \mathbb{N}^*$, Ω is an open of $\mathbb{R}^n$, Γ denotes the boundary of Ω. B and A_i $(1 \le i \le N)$, matrices of order N independent of the time. G, h, f are given. u, p denote respectively the velocity and the pressure.

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The considered system $\mathcal{P}_\infty$, is inspired from the work of D.D Joseph [14] in which it was used for the modelling of raising problems of Visco-elastic physics. The purpose of this work is to establish the existence, the uniqueness, the regularity and the asymptotic behavior of the solution of $\mathcal{P}_\infty$.
The approach that we use consists in introducing the intermediate system $\mathcal{P}$ by

$$\mathcal{P}\begin{cases} \dfrac{\partial u}{\partial t}+\displaystyle\sum_{i=1}^{3}A_i\frac{\partial u}{\partial x_i}+Bu-\int_0^t G(t-s)\Delta u(s)ds=f-\nabla p \quad \text{in} \quad \Omega,t. \\ u(0)=u_0 \\ div\ u=0 \\ u(x,t)=0 \qquad \text{on} \quad \Gamma\times]0,T[\end{cases}$$

For the existence of the solution of the problem $\mathcal{P}$, we use the techniques of the parabolic regularization, i.e., we demonstrate that the solution u of $\mathcal{P}$ is such that

$$u=\lim_{\alpha\to 0}u_\alpha \qquad \text{(for adequate topology)},$$

where u_α is solution of the system

$$\mathcal{P}_\alpha\begin{cases} \dfrac{\partial u}{\partial t}+\displaystyle\sum_{i=1}^{3}A_i\frac{\partial u}{\partial x_i}+Bu-\alpha\Delta u-\int_0^t G(t-s)\Delta u(s)ds=f-\nabla p \text{ in } \Omega,t. \\ u(0)=u_0 \\ div\ u=0 \\ u(x,t)=0 \qquad \text{on} \quad \Gamma\times]0,T[\end{cases}$$

Note that the existence and the uniqueness of the solution $\mathcal{P}_\alpha$ have been establish by Laklalech and al [7].
The uniqueness of solution u is demonstrated by using techniques which are similar to those used in [7].
In theorems 2, 3 and 4 of the fourth section we give, under some hypothesis some results on the regularity of the solution u.

The section 5 of this paper deals with the asymptotic behavior of u. We establish, using a lemma of Halanay[6], that

$$\lim_{t \to +\infty} \|u(t)\| = 0,$$

where $\|.\|$ is a suitable norm.
In section 6 and as application of previous results, we prove the existence, the uniqueness and the regularity of the solution of the fading-memory problem $\mathcal{P}_\infty$. We also study the asymptotic behavior of the solution of $\mathcal{P}_\infty$.
Finally and to motivate the type of systems considered in this paper, we propose two mathematical models, stem from fluid mechanics theory, one due to Slemord [16] and the other considered by D.D.Joseph [14].

2 Some Technical results

Let Ω is an open of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$). Q_T the cylinder $\Omega \times]0, T[$ with lateral boundary $\Gamma \times]0, T[$.

Let $\mathcal{D}(\Omega)$ (resp. $\mathcal{D}(Q_T)$) denote the space of indefinitely differentiable functions with compact support in Ω (resp. in $\Omega \times]0, T[$). Here we list the basic results that will be used in this work.

Lemma 1 *[19] Let $f = (f_1, ..., f_N)$ be in $(\mathcal{D}(Q^T))^N$, the necessary and sufficient condition so that $f = grad\, p$, $p \in \mathcal{D}'(Q_T)$ is*

$$(f, \varphi) = 0\ ,\ \ \forall \varphi \in V,$$

where $grad\, p = (\partial_1 p, ..., \partial_N p)$ and $V = \{\varphi = (\varphi_1, \varphi_2, ..., \varphi_N) \in (\mathcal{D}(\Omega))^N /\ div\varphi = 0\}$.

If we denote by H the adherence of V in $(L^2(\Omega))^N$ (resp. V the adherence of V in $(H^1(\Omega))^N$) where $H^1(\Omega)$ is the sobolev space, then we have (see [19])

$$H = \{\varphi \in (L^2(\Omega))^N /\ div\varphi = 0 \ and\ \varphi.\overrightarrow{n} = 0\ on\ \Gamma\},$$

where $\overrightarrow{n}$ denotes the external normal to Γ. Consequently, the orthogonal of H in $(L^2(\Omega))^N$ is characterized by.

$$H^\perp\ =\ \{\varphi \in (L^2(\Omega))^N\ /\ \varphi = gradp,\ p \in H^1(\Omega)\}.$$

Lemma 2 *[17]*

1. *If $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V')$ then $u \in \mathcal{C}(0,T,H)$.*
2. *If $f \in L^p(0,T;X)$ and $f' \in L^p(0,T;X)$, where X is Banach space then f is continuous almost everywhere on $[0,T]$.*

Lemma 3 ***(The Inequality of Young)***
If p and q are two reals such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then for all a and b two positif reals, we have

$$ab \ < \ \epsilon a^p \ + \ \frac{p-1}{p^q}\,\frac{b^q}{\epsilon^{\frac{1}{p-1}}} \quad (\epsilon > 0)\ .$$

Lemma 4 *For $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R},H)$ then*

$$f*g \in L^r(\mathbb{R},H) \qquad and \quad \|f*g\|_L^r(\mathbb{R},H) \leq \|f\|_{L^p(\mathbb{R},H)}\|g\|_{L^q(\mathbb{R},H)}.$$

Where $\dfrac{1}{r}=\dfrac{1}{p}+\dfrac{1}{q}-1$.

Lemma 5 ***(Lemma 4.1**[7])*
If $ReF(\tilde{G}(t))>0$ for all $t>0$, where F is the Fourier transform on $\mathbb{R}$, $Re(F)$ denotes the real part of F and

$$\tilde{G}(t) = \begin{cases} G(t) & on \quad [0,T] \\ 0 & elsewhere. \end{cases}$$

Then the function

$$\begin{array}{lll} H_G: & L^2(0,T;H^2\cap V) & \longrightarrow \mathbb{R} \\ & u & \longrightarrow \int_0^t \left[\int_0^s G(s-\sigma) \sum_{|\gamma|=0}^{1}\sum_{i=1}^{N}\left(D^{|\gamma|}\dfrac{\partial u}{\partial x_i}(\sigma); D^{|\gamma|}\dfrac{\partial u}{\partial x_i}(s)d\sigma\right)\right] ds \end{array}$$

is positive.

Lemma 6 *[6]. Let X be a Hilbert space with the inner product (,), G is a function defined on $[0,+\infty[$ that verifies*

$$(-)^k G^k(s) > 0, \quad k=0,1,2.$$

There exist constants $\alpha>0$ and $\beta>0$ such that for all $t>0$, and for all $V \in L^2(0,T;X)$,

$$\int_0^t (V(s);G*v(s))ds \geq \alpha \int_0^t \left(v(s);\int_0^s exp(-\beta(s-t))v(t)dt\right)ds,$$

where $G*v(s)=\int_0^s G(s-t)V(\tau)d\tau$.

3 Existence and uniqueness theorems

Theorem 1 *Suppose that the following hold*

1. $f \in L^2(0,T;V)$
2. $f' \in L^2(0,T;V')$
3. $u_0 \in V$
4. G *and* G' *are bounded and* $ReF(\tilde{G}(\tau)) \geq 0$
5. $((\sum_{i=1}^{n} A_i\partial_i u, u)) \geq 0$
6. $((Bu,u)) \geq 0$

then the following problem has a unique solution (u,p)

$$\begin{cases} u' + \sum_{i=1}^{n} A_i\partial_i u + Bu - \int_0^t G(t-s)\Delta u(s)ds = f - \nabla p \quad on \quad \Omega\times]0,T[\\ div\ u = 0 \\ u(0) = u_0 \\ u(x,t) = 0 \quad on \quad \Gamma\times]0,T[\end{cases}$$

such that $u \in L^\infty(0,T;V)$ *and* $p \in L^\infty(0,T;L^2(\Omega))$. *where* $L^\infty(0,T;X)$, *denote spaces of functions integrable white* $essSup\|f(t)\|_X < +\infty$ *endowed with norms*

$$|f|_{L^\infty(0,T,\infty)} = essSup\|f(t)\|_X.$$

Proof
we consider the parabolic regularization introduced in [7], and defined by the sequence of problem P_α

$$P_\alpha \begin{cases} (3.1) \quad u' + \sum_{i=1}^{n} A_i\partial_i u + Bu - \alpha\Delta u - \int_0^t G(t-s)\Delta u(s)ds = f - \nabla p \\ (3.2) \quad div\ u = 0 \\ (3.3) \quad u(0) = u_0 \\ (3.4) \quad u(x,t) = 0 \qquad \text{for all } (x,t) \in \Gamma\times]0,T[\end{cases}$$

It follows from [7] that the problem (3.1), (3.2), (3.3), (3.4) has a unique solution (u_α, p_α) such that

$$u_\alpha \in L^2(0,T;H^3 \cap V), \quad and \quad p_\alpha \in L^2(0,T;H^2).$$

To establish that the sequence $(u_\alpha)_{\alpha>0}$ is bounded in the space $L^\infty(0,T;V)$ and $(\sqrt{\alpha}u_\alpha)$ is bounded in $L^2(0,T;H^2\cap V)$, we apply $D^{|\gamma|}$ $(0\le|\gamma|\le 1)$ to (3.1) where $D^{|\gamma|}$ is given by[3] and we use the inner product whith $D^{|\gamma|}u$, then we sum to $|\gamma|$ $(0\le|\gamma|\le1)$ and using (3.2) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha(t)\|^2+((\sum_{i=1}^{N}A_i\partial_i u_\alpha;u_\alpha(t)))+((Bu_\alpha;u_\alpha))+\alpha\|u_\alpha(t)\|_2^2+$$
$$\int_0^t G(t-s)\sum_{|\gamma|=0}^{1}\sum_{i=1}^{N}\left(D^{|\gamma|}\frac{\partial u_\alpha}{\partial x_i}(s);D^{|\gamma|}\frac{\partial u_\alpha}{\partial x_i}(t)\right)ds=\sum_{|\gamma|=0}^{1}(D^{|\gamma|}f;D^{|\gamma|}u_\alpha).$$

Under the hypothesis 1, 4, 5 , we deduce after integrating and by the Young inequality, that

$$\|u_\alpha(t)\|^2+2\int_0^t\left[\int_0^s G(s-\sigma)\sum_{|\gamma|=1}^{1}\sum_{i=1}^{N}\left(D^{|\gamma|}\frac{\partial u_\alpha}{\partial x_i}(\sigma);D^{|\gamma|}\frac{\partial u_\alpha}{\partial x_i}(s)d\sigma\right)ds\right] \tag{3.5}$$
$$+2\alpha\int_0^t\|u_\alpha(s)\|_2^2ds\le\|u_0\|^2+\frac{c}{\gamma}\int_0^t\|f(s)\|^2ds+\gamma\int_0^t\|u_\alpha(s)\|^2ds,$$

As in lions [12] which implies that u_α and $\sqrt{\alpha}u_\alpha$ are bounded. We deduce the convergence of the sequence $(u_{\alpha,p_\alpha})_{\alpha>0}$ to $(a,p)\in L^\infty(0,T;V)\times D'(Q_T)$ and by passage to limits on (3.1) (when $\alpha\to0$), u and p are solution of the system

$$u'+\sum_{i=1}^{N}A_i\partial_i u+Bu-\int_0^t G(t-s)\Delta u(s)ds=f-grad\ p \tag{3.6}$$
$$div\ u=0 \tag{3.7}$$
$$u(0)=u_0 \tag{3.8}$$
$$u(x,t)=0\quad\text{on }\Gamma\times]0,T[.$$

The Lemma 5 and the hypothesis of 5, 6 establish the uniqueness of the solution [by same proof of Theorem 4.2, [7]]. ■

4 Theorems of regularity

Now we give some theorems of regularity, according to the nature of matrices B and A_i $(1\le i\le N)$.

Theorem 2 *Let $m\ge2$, we suppose that the following hold*

[3] $D^{|\gamma|}=\dfrac{\partial^{\gamma_1+\gamma_2+\ldots+\gamma_n}u}{\partial x_1^{\gamma_1}\ldots\partial x_n^{\gamma_n}}$, $\qquad|\gamma|=\gamma_1+\gamma_2+\ldots+\gamma_n$.

1. $f \in L^2(0,T;H^m)$
2. $\dfrac{\partial^l f}{\partial t^l} \in L^2(0,T;V)$ *if* $m = 2l+1$
3. $\dfrac{\partial^l f}{\partial t^l} \in L^2(0,T;H)$ *if* $m = 2l$
4. $((\sum_{i=1}^n A, \partial_i u; u))_m \geq 0$
5. $((Bu,u))_m \geq 0$
6. $\dfrac{\partial^k G}{\partial t^k}$ *bounded* $k \leq l-2$ *and* $ReF(\tilde{G}(t)) \geq 0$
7. $u_0 \in H^m \cap V$

$$\begin{array}{l} \dfrac{\partial^l u}{\partial t^l}(0) \in V \quad 0 \leq j \leq l-1 \\ \dfrac{\partial^l u}{\partial t^l}(0) \in V \quad \textit{if} \quad m = 2l+1 \\ \dfrac{\partial^l u}{\partial t^l}(0) \in H \quad \textit{if} \quad m = 2l. \end{array}$$

Then the solution of the problem P verifies

$$\begin{array}{l} u \in L^\infty(0,T;H^m \cap V) \\ p \in L^\infty(0,T;H^{m-1}). \end{array}$$

Moreover, the expression of $\dfrac{\partial^j u}{\partial t^j}(0)$ *according to the initial data is given by*

$$\begin{aligned} \frac{\partial^j u}{\partial t^j}(0) &= (-A)^j u_0 + \left\{ \sum_{i=0}^{j-1} (\frac{d}{dt})^j (-A)^{j-1-i} f + \right. \\ &\left. + \sum_{r=1}^{j-1} (-1)^{j-1-r} \sum_{i=0}^{j-1-r} \frac{\partial^{r-1}}{\partial t^{r-1}} [(A)^{j-1-i-r} G(t) \Delta(u(s)) ds \right\}, \end{aligned}$$

where $Au = P\left(\sum_{i=1}^N A_i \partial_i u + Bu - \int_0^t G(t-s)\Delta u(s) ds\right)$, *and P the operator projection.*

Proof

Using the parabolic regularization considered in [7]. Let's take a sequence $u_{0\alpha} \in H^{m+2} \cap V$, which tends strongly to u_0 on $H^m \cap V$. We have $(A_\alpha)^j u_{0\alpha} \to A^j(u_0)$ on V for $j = 1, 2, ..., l$, where

$$A_\alpha u = P\left(\sum_{i=1}^N A_i \partial_i u + Bu - \alpha \Delta u - \int_0^t G(t-s)\Delta u(s) ds\right).$$

Thus from [7] and by a reasoning similar to the one done for Theorem 1, there exists a unique solution u such that

$$u \in L^\infty(0,T; H^m \cap V), \quad p \in L^\infty(0,T; H^{m-1}). \qquad \blacksquare$$

Theorem 3 *Under the hypothesis of Theorem 1, if moreover we suppose that $u_0 \in H^3 \cap V$, $f' \in L^2(0,T;V)$ and $f'' \in L^2(0,T;V')$ then the found solution verifies*

$$u' \in L^\infty(0,T;V)$$

Proof
We derive the equation (3.6) with respect to t, we have

$$(4.1) \qquad u'' + \sum_{i=1}^{N} A_i \partial_i u' + Bu' - \int_0^t G(t-s)\Delta u'(s)ds = f' - \nabla p' + G(t)\Delta u_0$$

$$(4.2) \qquad u'(0) = f(0) - \sum_{i=1}^{N} A_i \partial_i u_0 - Bu_0 \in V$$

And we have $\quad f' + G(t)\Delta u_0 \in L^2(0,T;V),$
$\qquad f'' + G'(t)\Delta u_0 \in L^2(0,T;V').$

Moreover u' is the solution of (4.1) and (4.2), then it follows from theorem 1

$$u' \in L^\infty(0,T;V).$$

Corollary 1 *Under the assumption of Theorem 3, the problem P has a solution u such that*

$$u \in C(0,T;V) \quad \textit{and} \quad u' \in C(0,T;H).$$

Proof
It is a consequence of the Theorems 1 and 3, and the Lemma [17]. $\blacksquare$

Theorem 4 *Under the hypothesis of theorem 3, if moreover*

$$u_0 \in H^4 \cap V, f'' \in L^2(0,T;V), f''' \in L^2(0,T;V').$$

Then $u \in L^\infty(0,T;H^2 \cap V)$ and $u' \in C(0,T;V)$.

Proof
We differentiate (4.1) with respect to t, we obtain a system similar to P. Then

it follows that $u'' \in C(0,T;V)$.
Now we prove that $u \in L^\infty(0,T,H^2\cap V)$. For this, let us write (4.1) like

$$-G(0)\Delta u(t) + \int_0^t G'(t-s)\Delta u(s)ds = f' - \nabla p' - u'' - \sum_{i=1}^N A_i\partial_i u' - Bu'.$$

which is a Volterra equation of second order (Yoshida [21]), which gives

$$u \in L^\infty(0,T,H^2\cap V). \qquad \blacksquare$$

5 On the Asymptotic Behavior of Solution

Theorem 5 *If We suppose that*

1. $u_0 \in H^2 \cap V$
2. $f \in L^2(0,\infty;H^2\cap V)\cap L^\infty(0,\infty;H)$
3. $f' \in L^2(0,\infty;V)\cap L^1(0,\infty;H)$
4. $f'' \in L^2(0,\infty;V')$
5. $(-1)^k G^{(k)}(s) \geq 0, \quad k=0,1,2$ *where* $G^{(k)} = \dfrac{\partial^k G(s)}{\partial s^k}$
6. $((\sum_{i=1}^N A_i\partial_i u; u))_2 > 0$
7. $((Bu,u))_2 > 0$

then

$$\lim_{t\to+\infty} |u(t)| = 0.$$

Proof
The idea consists of using the following lemma of Halanay [6] (see also S.O. Macamy and J.S. Wong [13]).

Let us consider the inner product by u, in (3.6) and integrate on $[0,t]$, we deduce that

$$(5.1)\quad \frac{1}{2}|u(t)|^2 + \int_0^t \left(\int_0^s G(s-\sigma)\sum_{i=1}^N \frac{\partial_i u}{\partial x_i}(\sigma); \frac{\partial_i u}{\partial x_i}(s)d\sigma\right) ds \leq \frac{1}{2}|u_0| + \int_0^t (f(s);u(s))ds.$$

According to the Lemma 6 and assuming $X=V$. Then there exists $\alpha > 0$

and $\beta > 0$ such that

$$\begin{aligned} (5.2) \qquad & \alpha \int_0^t ((u(s); \int_0^s exp(-\beta(s-\tau)u(\tau)d\tau)))ds \\ & \leq \int_0^t (\int_0^s G(s-\sigma) \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}(\sigma); \frac{\partial u}{\partial x_i}(s) d\sigma \right) ds \end{aligned}$$

then from (5.1) and (5.2), it follows that

$$\begin{aligned} (5.3) \qquad \frac{1}{2}|u(t)|^2 & + \alpha \int_0^t ((u(s); \int_0^s exp(-\beta(s-\tau)u(\tau)d\tau)))ds \\ & \leq \frac{1}{2}|u_0|^2 + \int_0^t (f(s); u(s))ds. \end{aligned}$$

Let $w(s) = \int_0^s exp(-\beta(s-\tau))u(\tau)d\tau$ be the solution of the equation

$$(5.4) \qquad w'(t) + \beta w(t) = u(t)$$

The substitution of (5.4) in (5.3) allows to obtain

$$(5.5) \quad \frac{1}{2}|u(t)|^2 + \alpha \int_0^t (w'(s) + \beta w(s); w(s))\, ds \leq \int_0^t (f(s); w'(s) + \beta w(s))ds.$$

Then, we have

$$\begin{aligned} (5.6) \qquad \frac{1}{2}|u(t)|^2 & + \frac{\alpha}{2} \int_0^t \|w(s)\|^2 ds + \alpha\beta \int_0^t \|w'(s)\|^2 ds \\ & \leq \frac{1}{2}|u_0|^2 + |f(t)||w(t)| + \int_0^t (|f'(s)| + \beta|f(s)|; u(s))\, ds. \end{aligned}$$

From an integration by parts the right terms of (5.6), it results that

$$\begin{aligned} (5.7) \quad \frac{1}{2}|u(t)|^2 & + \frac{\alpha}{2}\|w(t)\|^2 + \alpha\beta \int_0^t \|w(s)\|^2 ds \\ & \leq \frac{1}{2}|u_0|^2 + |f(t)||w(t)| + \int_0^t (|f'(s)| + \beta|f(s)|)\, |w(s)| ds. \end{aligned}$$

By using the Young inequality

$$(5.8) \qquad |f(t)||w(t)| \leq \frac{4}{\alpha}|f(t)|^2 + \frac{\alpha}{4}|w(t)|^2$$

$$(5.9) \qquad (|f'(s)| + \beta|f(s)|)\, |w(s)| \leq \frac{2}{\beta\alpha}(|f'(s)| + \beta|f(s)|)^2 + \frac{\beta\alpha}{2}|w(s)|^2$$

we deduce from (5.7), (5.8), (5.9), that

$$\frac{1}{2}|u(t)|^2 + \frac{\alpha}{4}\|w(t)\|^2 + \frac{\alpha\beta}{2}\int_0^t \|w(s)\|^2 ds$$
$$\leq \frac{1}{2}|u_0|^2 + \frac{4}{\alpha}|f(t)|^2 + \frac{2}{\alpha\beta}\int_0^t \left(|f'(s)| + \beta|f(s)|\right)^2 ds$$

where $f \in L^2(0,\infty; H^2 \cap V) \cap L^\infty(0,\infty; H)$, $f' \in L^2(0,\infty; H)$. We obtain

$$(5.10) \qquad \begin{array}{l} |u(t)| \leq c \\ \dfrac{\alpha}{4}\|w(t)\|^2 + \dfrac{\alpha\beta}{2}\displaystyle\int_0^t \|w(s)\|^2 ds \leq c. \end{array}$$

where c is an positive constant independent of t .

We know that $u \in L^\infty(0,T; H^2 \cap V)$ and $u' \in L^\infty(0,T; H)$, then $u \in C(0,T,V)$, and w is continuous. From (5.4), (5.5), (5.6), we deduce that w' is bounded and w is uniformly continuous. From (5.10), we get

$$(5.11) \qquad \lim_{t\to+\infty} |w(t)| = 0,$$

and

$$(5.12) \qquad \lim_{t\to+\infty} (u(t); w(t)) = 0.$$

We admit the momently the following Lemma,

Lemma 7 *The derivative u' is such that*

$$(5.13) \qquad |u'| \leq c \quad p.p.t$$

c is a constant independent of t. ■

Since $u \in L^\infty(0,T,H^2 \cap V)$, $u' \in L^\infty(0,T,H)$, G and f are continuous, then it result that u' is continuous.
Therefore, as $|u'(t)|$ is bounded and the function u is uniformly continuous, the function

$$(5.14) \qquad \frac{d}{dt}(u(t); w(t)) = (u'(t); w(t)) + (u(t); w'(t))),$$

is uniformly continuous. From (5.12), we have

$$(5.15) \qquad \lim_{t\to+\infty} \frac{d}{dt}(u(t); w(t)) = 0,$$

From (5.11), (5.14) and (5.15), we obtain

$$\lim_{t\to+\infty}(u(t);w'(t))=0. \tag{5.16}$$

Finally, if we consider (5.4), we take the inner product by $u(t)$, we deduce from (5.12) and (5.16)

$$\lim_{t\to+\infty}|u(t)|^2=0. \quad \blacksquare$$

Proof of Lemma 7

The procedure to demonstrate this, is to compute in (3.6), $u'(t+h)-u'(t)$ we get

$$\begin{aligned}(5.17)\qquad u'(t+h) &-u'(t)+\sum_{i=1}^{N}A_i\partial_i(u'(t+h)-u'(t))+B(u(t+h)-u(t))\\ &-\textstyle\int_0^t G(t+h-s)\Delta u(s)ds+\int_0^t G(t-s)\Delta u(s)ds\\ &=f(t+h)-f(t)-\nabla p(t+h)+\nabla p(t).\end{aligned}$$

We take the inner product where $u(t+h)-u(t)$, which gives

$$\begin{aligned}(5.18)\qquad &\frac{1}{2}\frac{d}{dt}|u(t+h)-u(t)|^2-\int_0^{t+h}G(t+h-s)(\Delta u(s);u(t+h)-u(t))ds\\ &+\textstyle\int_0^t G(t-s)\left(\Delta u(s);u(t+h)-u(t)\right)ds\\ &\le(f(t+h)-f(t);u(t+h)-u(t)).\end{aligned}$$

Taking $s'=s-h$, we obtain

$$\begin{aligned}(5.19)\qquad &\frac{1}{2}\frac{d}{dt}|u(t+h)-u(t)|^2-\int_{-h}^{t}G(t-s)(\Delta u(s+h);u(t+h)-u(t))ds\\ &+\textstyle\int_0^t G(t-s)(\Delta u(s);u(t+h)-u(t))ds\\ &=\frac{1}{2}\frac{d}{dt}|u(t+h)-u(t)|^2-\int_{-h}^{0}G(t-s)(\Delta u(s+h);u(t+h)-u(t))ds\\ &-\textstyle\int_0^t G(t-s)(\Delta u(s+h)-u(s);u(t+h)-u(t))ds\\ &\le(f(t+h)-f(t);u(t+h)-u(t)).\end{aligned}$$

If we denode $s'=s+h$, we obtain

$$\begin{aligned}(5.20)\qquad &=\frac{1}{2}\frac{d}{dt}|u(t+h)-u(t)|^2+\int_0^h G(t-s+h)C(u(s);u(t+h)-u(t))ds\\ &+\textstyle\int_0^t G(t-s)C(u(s+h)-u(s);u(t+h)-u(t))ds\\ &\le(f(t+h)-f(t);u(t+h)-u(t)),\end{aligned}$$

where $C(u,v)=\sum_{i,j=1}^{N}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx.$

By integration in $[0,t]$ and division by h^2, for $h\to 0$, it follows that

$$(5.21) \qquad \frac{1}{2}|u'(t)|^2 - \frac{1}{2}|u'(0)|^2 - \int_0^t (u'(s); G(s)\Delta u_0) ds \leq \int_0^t (f'(s); u'(s)) ds.$$

We have $u \in L^\infty(0,\infty;H)$, $u_0 \in H^2 \cap V$ and G is bounded. Then, we obtain

$$|u'(t)|^2 \leq C + \int_0^t |f'(s)||u'(s)| ds.$$

where C is an independent constant of t. From [4],

$$(5.22) \qquad |u'(t)|^2 \leq C + \textstyle\int_0^t |f'(s)| ds, \quad \text{but } f' \in L^1(0,\infty;H), \text{ then}$$

$$|u'(t)| \leq C, \quad \blacksquare$$

As is Theorem 5, we use the Lemma of Halanay [6] for $X = H^2 \cap V$ endowed with the inner product

$$((u;v))_2 = \sum_{\alpha=0}^{1} \sum_{i=1}^{N} \left(D^\alpha \frac{\partial u}{\partial x_i}, D^\alpha \frac{\partial v}{\partial x_i} \right),$$

and the norm $\|.\|$, to establish have the following result.

Proposition 1 *Under the following hypothesis*

1. $u_0 \in H^4 \cap V$
2. $f \in L^\infty(0,\infty;V) \cap L^\infty(0,\infty;H)$
3. $f' \in L^1(0,\infty;V)$
4. $f'' \in L^2(0,\infty;V)$
5. $f''' \in L^2(0,\infty;V')$
6. $(-)^k G^k(s) \geq 0$ *for* $k = 0,1,2$ *where* $G(s) = \dfrac{\partial^k G(s)}{\partial s^k}$.
7. $((\sum_{i=1}^{N} A_i \partial_i u; u)) \geq 0$
8. $((Bu;u)) \geq 0$.

$$\textit{Then} \qquad \lim_{t\to+\infty} \|u(t)\| = 0.$$

6 Application to Fading-memory systems

In this section, we apply the previous result to study existence, uniqueness, regularity and asymptotic behavior of solution corresponding to the fading-memory system described by

$$P_\infty \begin{cases} 6.1 \quad u'(t) + \sum_{i=1}^{N} A_i \partial_i u + Bu - \int_{-\infty}^{t} G(t-s) \triangle u(s) ds \; = \; f - \nabla p \\ 6.2 \quad div\ u = 0 \\ 6.3 \quad u(x,t) = h(x,t) \qquad if \quad t \leq 0 \\ 6.4 \quad u(x,t) = 0 \qquad if \quad x \in \Gamma \times]0,T[\end{cases}$$

Using the equality

$$\int_{-\infty}^{t} G(t-s)\Delta u(s)ds = \int_{-\infty}^{0} G(t-s)\Delta u(s)ds + \int_{0}^{t} G(t-s)\Delta u(s)ds.$$

The system P_∞ takes the form studied previously and so we have the following result.

6.1 Existence and uniqueness theorems

Lemma 8 *We suppose that* $h \in L^1(]-\infty,0]; H^3)$ *then*

$$q(t) = \int_{-\infty}^{0} G(t-s) \triangle h(s) ds \in L^2(0,T,V).$$

Proof

$$\begin{array}{ll} \int_0^T \|q(t)\|^2 dt & \leq \int_0^T \| \int_{-\infty}^0 G(s-t)\Delta(s)ds\|^2 dt \\ & \leq \int_0^T (\int_{-\infty}^0 G(s-t)\|h(s)\|_3 ds)^2 dt \\ & \leq C(T). \qquad \blacksquare \end{array}$$

Theorem 6 *Under the hypothesis of the Theorem 1, and if* $h \in L^1(-\infty,0; H^3 \cap V)$ *and* $h(0) \in V$*. Then, the systems* P_∞ *have a unique solution* (u,p) *such that*

$$u \;\in\; L^\infty(0,T,V), \quad p \in L^2(0,T,L^2(\Omega)).$$

Proof

The proof of theorem 6 is a consequence of Theorem 1 and Lemma 8. ■

6.2 Asymptotic behavior of the solution

Proposition 2 *Under the hypothesis of the Theorem 5 and if*

1. $G \in L^1(0, \infty)$ *et* $\lim_{t \mapsto +\infty} G(t) = 0$,
2. $h \in L^2(-\infty, 0; H^4 \cap V) \cap L^\infty(-\infty, 0; H^2 \cap V)$,
3. $h' \in L^2(-\infty, 0; H^3 \cap V) \cap L^1(-\infty, 0; H^2 \cap V)$,

then

$$\lim_{t \mapsto +\infty} |u(t)| = 0.$$

Proof

Using the lemma 4, for $p = 1, q = 2, r = 2$ and $p = 1, q = \infty, r = \infty$, we establish the result

(6.5) $f(t) - \int_{-\infty}^{0} G(t-s) \triangle h(s) ds \in L^2(0, \infty; H^2 \cap V) \cap L^\infty(0, \infty, H)$.

On other hand, since $G'(t-s) = -\frac{\partial}{\partial s} G(t-s)$, we integrate by parts to obtain

$$\int_{-\infty}^{0} G'(t-s) \Delta h(s) ds = -G(t) \Delta u_0 + \int_{-\infty}^{t} G(t-s) \Delta h'(s) ds.$$

Then by Lemma 3, we have

(6.6) $f'(t) - \int_{-\infty}^{0} G'(t-s) \triangle h(s) ds \in L^2(0, \infty; H^2 \cap V) \cap L^\infty(0, \infty, H^1 \cap V)$.

So, we use theorem 5, to conclude ■

As for the proof of proposition 2, we easily establish the following result

Proposition 3 *Under the hypothesis of Theorem 5, so in more*

1. $G \in L^1(0, \infty)$ *et* $\lim_{t \mapsto +\infty} G(t) = 0$ *and* $\lim_{t \mapsto +\infty} G'(t) = 0$
2. $h \in L^2(-\infty, 0; H^3 \cap V) \cap L^\infty(-\infty, 0; H^2 \cap V)$,
3. $h' \in L^2(-\infty, 0; H^2 \cap V) \cap L^\infty(-\infty, 0; H^2 \cap V)$,
4. $h'' \in L^2(-\infty, 0; V)$,

then

$$\lim_{t \mapsto +\infty} |u(t)| = 0.$$

To achieve this subsection, we use proposition 1, to deduce

Proposition 4 *Under the hypothesis of the theorem 3, if we have*

1. $\lim_{t \mapsto +\infty} G''(t) = 0$,
2. $h'' \in L^2(-\infty, 0; H^2 \cap V)$,
3. $h''' \in L^2(-\infty, 0; V)$,

then

$$\lim_{t \mapsto +\infty} \|u(t)\| = 0.$$

7 Applications

To motivate the Integero differential equations considered in this paper, we give the two following applications.

Example 1: Let Ω be an open space bounded in $\mathbb{R}^3$ of regular border Γ, A linear model of a simple incompressible fluid to Fading-memory is given by the system [16]

$$\begin{array}{lll} (7.1) & u'(x,t) - \int_{-\infty}^{t} G(t-s)\triangle u(s)ds = -\nabla p & x \in \Omega, \quad t \geq 0, \\ (7.2) & div\ u = 0 & x \in \Omega, \quad t \geq 0, \\ (7.3) & u(x,t) = 0 & x \in \Gamma, \quad t \geq 0, \\ (7.4) & u(x,t) = h(x,t) & x \in \Omega \quad t \leq 0. \end{array}$$

where $u(x,t) = (u_1(x,t); u_2(x,t); u_3(x,t)$ is the velocity of one particle to the point x of Ω at time t.

The results developed in previous give

Corollary 2 *Let's suppose that*

1. $u_0 \in V$,
2. $(-1)^k G^k(s) \geq 0 \qquad k = 0, 1, 2,$
3. *G and its derivative are bounded,*
4. $h \in L^1(-\infty, 0; H^3 \cap V)$,

then there exists

$$u \in L^{\infty}(0,T;V), \qquad p \in L^{\infty}(0,T;L^2(\Omega)),$$

a unique solution of the system (7.1), (7.2), (7.3), and (7.4).

Corollary 3 *Under the hypothesis of corollary 2, if we have*

1. $G \in L^1(0,\infty)$ *and* $\lim\limits_{t\mapsto+\infty} G(t) = 0$,
2. $h \in L^2(-\infty,0;H^3\cap V)\cap L^{\infty}(-\infty,0;H^2\cap V)$,
3. $h' \in L^2(-\infty,0;H^3\cap V)\cap L^{1}(-\infty,0;H^2\cap V)$,

then

$$\lim_{t\mapsto+\infty} |u(t)| = 0. \qquad \blacksquare$$

Example 2: Let's consider Joseph's system [14], it is a system as those described by equations (6.1), (6.2), (6.3) and (6.4) which data

$$A_1 = \begin{pmatrix} -\Omega x_2 & 0 & 0 \\ 0 & -\Omega x_2 & 0 \\ 0 & 0 & -\Omega x_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\Omega x_1 & 0 & 0 \\ 0 & -\Omega x_1 & 0 \\ 0 & 0 & -\Omega x_1 \end{pmatrix},$$

$$A_3 = 0, \qquad B = \begin{pmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and $A = \sum\limits_{i=1}^{N} A_i n_i$ where $n = (n_1, n_2, \cdots, n_N)$ vector normal outside to Γ.

Proposition 5 *Let's suppose that*

1. $u_0 \in V$,
2. $(-1)^k G^k(s) \geq 0$
3. G *and its derivative are bounded,*
4. $h \in L^1(-\infty,0;H^3\cap V)$,
5. $f \in L^2(0,T;V)$,
6. $f' \in L^2(0,T;V)$,
7. $(Au,u) \geq 0$ *for all* $u \in H^{\frac{1}{2}}(\Gamma)$ *(where* $(Au,u) = \int_{\Gamma} Au.ud\sigma$ *).*

then there exists

$$u \in L^{\infty}(0, T; V), \qquad p \in L^{\infty}(0, T; L^2(\Omega)),$$

a unique solution of the problem.

Proposition 6 *Under the hypothesis of corollary 5, if we have*

1. $G \in L^1(0, \infty)$, $\lim_{t \mapsto +\infty} G(t) = 0$ *and* $\lim_{t \mapsto +\infty} G'(t) = 0$,
2. $h \in L^2(0, T; H^3 \cap V) \cap L^{\infty}(-\infty, 0; H^2 \cap V)$,
3. $h' \in L^2(-\infty, 0; H^3 \cap V) \cap L^{\infty}(-\infty, 0; H^2 \cap V)$,
4. $h'' \in L^2(-\infty, 0; V)$.

then

$$\lim_{t \mapsto +\infty} |u(t)| = 0.$$

Proposition 7 *Under the hypothesis of the corollary 5, if we have*

1. $\lim_{t \mapsto +\infty} G''(t) = 0$,
2. $h \in L^2(0, T; H^3 \cap V) \cap L^{\infty}(-\infty, 0; H^2 \cap V)$,
3. $h'' \in L^{\infty}(-\infty, 0; H^3 \cap V)$,
4. $h''' \in L^2(-\infty, 0; V)$.

then

$$\lim_{t \mapsto +\infty} \|u(t)\| = 0.$$

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