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Existence and Blow-up for a Nonlocal Degenerate Parabolic Equation

Shuo Yang[2] [1] and Yi Zhang[2,3]

[2] Department of Mathematics, Sichuan University
Chengdu 610064, P. R. China

[3] Yibin Vocational and Technical College
Yibin 644003, P. R. China

Abstract

In this paper, we establish the local existence and uniqueness of the solution for the degenerate parabolic equation with a nonlocal source and homogeneous Dirichlet boundary condition. Moreover, we prove that the solution blows up in finite time and obtain the blow-up set in some special case.

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1 Introduction

In this paper, we consider the following degenerate parabolic equation with a nonlocal source

$$\begin{aligned} &u_t = (u^m)_{xx} + au^p \int_{-l}^{l} u^q dx - ku^r, && x \in (-l,l), t > 0, \\ &u(\pm l,t) = 0, && t > 0, \\ &u(x,0) = u_0(x), && x \in [-l,l], \end{aligned} \tag{1.1.1}$$

where $a, k > 0, p, q \geq 0$, $p + q > r > m > 1$.

Problem (1.1) arises in the study of the flow of a fluid through a porous medium or in study of population dynamics. Over the last few years, many physical phenomena were formulated into non-local mathematical models.

[1]E-mail: yangshuo3721@gmail.com

In [3], Deng et al. studied the following problem

$$\begin{aligned} &u_t = (u^m)_{xx} + a\int_{-l}^{l} u^q dx, && x \in (-l,l), t>0,\\ &u(\pm l,t)=0, && t>0,\\ &u(x,0)=u_0(x), && x\in[-l,l], \end{aligned}$$

where $a>0$, $q>m>1$. Under certain conditions, they obtained that the solution blows up in finite time and got the estimate of blow-up rate.

Souplet [8] and Wang [10] considered the following problem

$$\begin{aligned} &u_t - \triangle u = \int_\Omega u^p(y,t)dy - u^q(x,t), && x\in\Omega, t>0,\\ &u(x,t)=0, && x\in\partial\Omega, t>0,\\ &u(x,0)=u_0(x), && x\in\Omega, \end{aligned}$$

where $p>q\geq 1$. They proved that the solution blows up in finite time for large data $u_0(x)$, and obtained the following blow-up rate

$$\lim_{t\to T^*}(T^*-t)^{\frac{1}{p-1}}u(x,t) = [(p-1)|\Omega|]^{\frac{-1}{p-1}},$$

where T^* is the blow-up time of $u(x,t)$.

Motivated by these results, in this paper, we will study the blow-up and blow-up set of (1.1). Firstly, we give a definition of classical solution for problem (1.1).

Definition 1.1 *If there exists some $T^*\in(0,+\infty]$ such that for any $T\in(0,T^*)$, function $u(x,t)\in C^{2,1}(D_T)\cap C(\overline{D_T})$ and satisfies (1.1), then $u(x,t)$ is a classical solution of (1.1) on $[0,T^*)$, where $D_T=(-l,l)\times[0,T]$, $\overline{D_T}=[-l,l]\times[0,T]$. If $T^*=+\infty$, then u is a global solution.*

Definition 1.2 *A point $x_0\in[-l,l]$ is a blow-up point of $u(x,t)$ if there exists a sequence $\{x_n,t_n\}$ such that $t_n\to T^*$, $x_n\to x_0$, and*

$$\lim_{n\to\infty} u(x_n,t_n)=\infty.$$

We call the set of all blow-up points the blow-up set. If the blow-up set is $(-l,l)$, we say that (1.1) has global blow-up.

Before stating our main results, we make some assumptions for initial data $u_0(x)$ as follows:
(H1) $u_0(x)\in C^{2+\alpha}([-l,l])$ for some $0<\alpha<1$, $u_0(x)>0$ in $(-l,l)$;
(H2) $u_0(\pm l)=0$, $u_{0x}(l)<0$, $u_{0x}(-l)>0$, $(u_0^m)_{xx}+au_0^p\int_{-l}^{l}u_0^q dx - ku_0^r|_{\pm l}=0$;
(H3) $(u_0^m)_{xx}+au_0^p\int_{-l}^{l}u_0^q dx - ku_0^r\geq 0$, $x\in(-l,l)$;

(H4) $(u_0^m)_{xx} \le 0$, $x \in [-l, l]$.

Remark 1.1 we can choose $u_0 = C(\cos \frac{\pi}{2l} x)^{\frac{1}{m}}$ to satisfy the above conditions (H1)-(H4), where C is a sufficiently large positive constant.

Our main results are stated as follows.

Theorem 1.1 *Suppose that $u_0(x) \in C^2(-l, l) \cap C([-l, l])$ and satisfies (H1)-(H3), then (1.1) there exists a unique classical solution.*

Theorem 1.2 *Suppose that $u_0(x)$ satisfies (H1)-(H3). Then*
(i) (1.1) has a global solution if $u_0(x) \le (\frac{k}{2al})^{\frac{1}{p+q-r}}$;
(ii) the solution of (1.1) blows up in finite time if $u_0(x)$ is sufficiently large.

Theorem 1.3 *Suppose that $u_0(x)$ satisfies (H1)-(H4), $p + m < 2$. Then the solution $u(x, t)$ of (1.1) blows up globally.*

This paper is organized as follows. In section 2, we state the local existence and uniqueness of the solution and prove that the solution is a classical one by adding some assumptions on $u_0(x)$. The results of global existence and finite time blow-up are shown in section 3. In section 4, we prove that (1.1) has global blow-up.

2 Local existence and uniqueness

To investigate the local existence and uniqueness of the solution of problem (1.1), let $u^m = v$, $t = \frac{1}{m}\tau$, then (1.1) becomes

$$\begin{aligned} &v_\tau = v^{m_1}(v_{xx} + av^{p_1}\int_{-l}^{l} v^{q_1} dx - kv^{r_1}), && x \in (-l, l), \tau > 0, \\ &v(\pm l, \tau) = 0, && \tau > 0, \\ &v(x, 0) = v_0(x), && x \in [-l, l], \end{aligned} \qquad (2.2.1)$$

where $0 < m_1 = \frac{m-1}{m} < 1$, $p_1 = \frac{p}{m} \ge 0$, $q_1 = \frac{q}{m} \ge 0$, $r_1 = \frac{r}{m}$, $p_1 + q_1 > r_1 > 1$, $v_0(x) = u_0^m(x)$.

Under this transformation, assumptions (H1)-(H4) become
(H1)′ $v_0(x) \in C^{2+\alpha}([-l, l])$ for some $0 < \alpha < 1$, $v_0(x) > 0$ in $(-l, l)$;
(H2)′ $v_0(\pm l) = 0$, $v_{0x}(l) < 0$, $v_{0x}(-l) > 0$, $v_{0xx} + av_0^{p_1}\int_{-l}^{l} v_0^{q_1} dx - kv_0^{r_1}|_{\pm l} = 0$;
(H3)′ $v_{0xx} + av_0^{p_1}\int_{-l}^{l} v_0^{q_1} dx - kv_0^{r_1} \ge 0$, $x \in (-l, l)$;
(H4)′ $v_{0xx} \le 0$, $x \in [-l, l]$.

Because (2.1) is a degenerate equation, the standard parabolic theory can't be used to give the local existence of the solution, we consider the regularized problem

$$\begin{array}{lll} v_{\varepsilon\tau} = (v_\varepsilon + \varepsilon)^{m_1}(v_{\varepsilon xx} + av_\varepsilon^{p_1}\int_{-l}^{l} v_\varepsilon^{q_1}dx - kv_\varepsilon^{r_1}), x \in (-l,l), \tau > 0, & & \\ v_\varepsilon(\pm l,\tau) = 0, & \tau > 0, & (2.2.2) \\ v_\varepsilon(x,0) = v_0(x), & x \in [-l,l]. & \end{array}$$

Lemma 2.1 *Suppose that* $w(x,\tau) \in C^{2,1}(D_T) \cap C(\overline{D_T})$ *and satisfies*

$$\begin{array}{ll} w_\tau - d(x,\tau)w_{xx} \geq c_1(x,\tau)w + c_3(x,\tau)\int_{-l}^{l} c_2(x,\tau)w(x,\tau)dx, (x,\tau) \in D_T, & \\ w(\pm l,\tau) \geq 0, & \tau \in (0,T], \\ w(x,0) \geq 0, & x \in (-l,l), \end{array}$$

where $c_1(x,\tau)$, $c_2(x,\tau)$, $c_3(x,\tau)$ *are bounded functions and* $c_2(x,\tau) \geq 0$, $c_3(x,\tau) \geq 0$, $d(x,\tau) \geq 0$ *in* D_T. *Then* $w(x,\tau) \geq 0$ *on* $\overline{D_T}$.

Proof. The proof is similar to the proofs of Lemma 1 in [7] or Lemma 2.1 in [10], we omit it.

Lemma 2.2 *Assume that* $v_\varepsilon(x,\tau) \in C^{2,1}(D_T) \cap C(\overline{D_T})$ *is a nonnegative solution of (2.2) and a nonnegative function of* $w(x,\tau) \in C^{2,1}(D_T) \cap C(\overline{D_T})$, *and satisfies*

$$\begin{array}{ll} w_\tau \geq (\leq)(w+\varepsilon)^{m_1}(w_{xx} + aw^{p_1}\int_{-l}^{l} w^{q_1}dx - kw^{r_1}), \quad (x,\tau) \in D_T, & \\ w(\pm l,\tau) \geq (\leq) 0, & \tau \in (0,T], (2.2.3) \\ w(x,0) \geq (\leq) v_0(x), & x \in (-l,l), \end{array}$$

Then $w(x,\tau) \geq (\leq) v_\varepsilon(x,\tau)$ *on* $\overline{D_T}$.

Proof. We only consider the case $\geq$. Let $z(x,\tau) = w(x,\tau) - v_\varepsilon(x,\tau)$. Subtracting (2.2) from (2.3), we obtain

$$\begin{aligned} z_\tau &= w_\tau - v_{\varepsilon\tau} \\ &\geq m_1(\xi_2+\varepsilon)^{m_1-1}(w_{xx} + aw^{p_1}\int_{-l}^{l} w^{q_1}dx)z + a(v_\varepsilon+\varepsilon)^{m_1}w^{p_1}q_1\int_{-l}^{l}\xi_3^{q_1-1}zdx \\ &\quad +[a(v_\varepsilon+\varepsilon)^{m_1}p\xi_4^{p_1-1}\int_{-l}^{l} v_\varepsilon^{q_1}dx]z + (v_\varepsilon+\varepsilon)^{m_1}z_{xx} \\ &\quad +(-k)[m_1(\xi_1+\varepsilon)^{m_1-1}\xi_1^{r_1} + r_1(\xi_1+\varepsilon)^{m_1}\xi_1^{r_1-1}]z \end{aligned}$$

with the initial-boundary conditions

$$z(x,0) \geq 0 \qquad \textit{and} \qquad z(\pm l,\tau) \geq 0,$$

where $\xi_1, \xi_2, \xi_3, \xi_4 > 0$. By Lemma 2.1, it follows that $w(x,\tau) \geq v_\varepsilon(x,\tau)$.
By Lemma 2.2, we have the following result of monotonicity.

Lemma 2.3 *Let $0 < \varepsilon_2 < \varepsilon_1 < 1$ and suppose that $v_0(x)$ satisfies (H1)′-(H3)′,and $v_{\varepsilon_1}(x,\tau)$and $v_{\varepsilon_2}(x,\tau)$ are solution of (2.2) on $\overline{D_{T_0}}$. Then $v_{\varepsilon_1} \geq v_{\varepsilon_2}$ on $\overline{D_{T_0}}$.*

Lemma 2.4 *Suppose that $v_0(x)$ satisfies (H1)′-(H3)′, and $v_\varepsilon(x,\tau)$ is the solution of (2.2) on $\overline{D_{T_0}}$. Then $u_{\varepsilon\tau} \geq 0$ on $\overline{D_{T_0}}$.*

Proof. Let $w = v_{\varepsilon\tau}$. Differentiating (2.2) with respect to τ gives

$$\begin{aligned} w_\tau &= m_1(v_\varepsilon+\varepsilon)^{m_1-1}(v_{\varepsilon xx} + av_\varepsilon^{p_1}\int_{-l}^{l} v_\varepsilon^{q_1}dx - kv_\varepsilon^{r_1})w \\ &\quad +(v_\varepsilon+\varepsilon)^{m_1}(w_{xx} + ap_1v_\varepsilon^{p_1-1}w\int_{-l}^{l} v_\varepsilon^{q_1}dx + aqv_\varepsilon^{p_1}\int_{-l}^{l} v_\varepsilon^{q_1-1}wdx - kr_1v_\varepsilon^{r_1-1}w) \end{aligned}$$

By (H1)′ and (H3)′, we have

$$w(x,0) = v_{\varepsilon\tau}(x,0) = (v_0(x)+\varepsilon)^{m_1}(v_{0xx} + av_0^{p_1}\int_{-l}^{l} v_0^{q_1}dx - kv_0^{r_1}) \geq 0$$

In view of $w(\pm l,\tau) = v_{\varepsilon\tau}(\pm l,\tau) = 0$, by Lemma 2.1, it follows that $w(x,\tau) \geq 0$ for any $(x,\tau) \in \overline{D_{T_0}}$.

Lemma 2.5 *Suppose $v_0(x)$ satisfies (H1)′-(H3)′. Then there exist T_0 and a priori bound M such that for all $\varepsilon \in (0,1)$, the solution of (2.2) satisfies*

$$v_0(x) \leq v_\varepsilon(x,\tau) \leq M, \quad (x,\tau) \in \overline{D_{T_0}}.$$

Proof. By Lemma 2.3, we know that $v_\varepsilon(x,\tau)$ is monotone with respect to ε. Suppose the solution of (2.2) is $v_1(x,\tau)$ when $\varepsilon = 1$, and T_1 is the maximal existence time of v_1. For any $T_0 \in (0,T)$, we can conclude that

$$v_\varepsilon(x,\tau) \leq v_1(x,\tau) \leq v_1(x,T_0) \leq \max_{x\in[-l,l]} v_1(x,T_0) = M, \quad (x,\tau) \in \overline{D_{T_0}}.$$

Since $v_{\varepsilon\tau} \geq 0$, $v_\varepsilon(x,0) = v_0(x)$. It follows that

$$v_0(x) \leq v_\varepsilon(x,\tau), \qquad (x,\tau) \in \overline{D_{T_0}}.$$

According to Lemma 2.3-2.5, we know that v_ε is monotone with respect to ε on $\overline{D_{T_0}}$ and is bounded from below to above. Thus we have

$$v(x,\tau) = \lim_{\varepsilon\to 0} v_\varepsilon(x,\tau), \quad (x,\tau) \in \overline{D_{T_0}}. \tag{2.2.4}$$

Proposition 2.1 *Suppose that $v_0 \in C^2((-l,l)) \cap C([-l,l])$ and satisfies (H1)′-(H3)′, then the function $v(x,\tau)$ defined by (2.4) is a classical solution of (2.1) in $\overline{D_{T_0}}$.*

Proof. It is required to prove that v belongs to $C^{2,1}(D_{T_0}) \cap C(\overline{D_{T_0}})$. Choose a point $(x_1, \tau_1) \in (-l, l) \times (0, T_0)$. Then select a domain $Q = (a_1, a_2) \times (0, \tau_2)$ such that

$$-l < a_1 < x_1 < a_2 < l \qquad and \qquad 0 < \tau_1 < \tau_2 < T_0$$

Let $C_0 = \inf_{x \in [a_1, a_2]} u_0(x)$. By Lemma 2.5, we have that $v_\varepsilon \geq C_0 > 0$ in Q, then $(v_\varepsilon)^{m_1} \geq C_0^{m_1}$. By Schauder interior estimate, we have

$$\|v_\varepsilon\|_{C^{2+\alpha}(Q)} \leq M,$$

where M depends only on $C_0^{m_1}, v_0, \alpha, Q$.
Now an appeal to Ascli-Arzelá Theorem show that $v \in C^{2+\alpha'}(Q), 0 < \alpha' < \alpha < 1$, with $\|v\|_{C^{2+\alpha'}}(Q) \leq M$. This shows that v is in $C^{2,1}$ at (x_1, τ_1). Notice that

$$0 \leq \lim_{x \to \pm l} v(x, \tau) \leq \lim_{x \to \pm l} v_\varepsilon(x, \tau) = 0, \qquad (\varepsilon \to 0),$$

we have that v is continuous on $\{\pm l\} \times (0, T_0)$.

Proposition 2.2 *Suppose that* $v_0 \in C^2((-l, l)) \cap C([-l, l])$ *and satisfies (H1)′-(H3)′, then the function* $v(x, \tau)$ *defined by (2.4) is unique.*

Proof. Suppose that $v(x, \tau), u(x, \tau)$ are two classical solution of (2.1). By using the same method used in Lemma 2.2, we can easily prove that $v \geq u$ and $v \leq u$. So $v \equiv u$.
The proof of Theorem 1.1 According to Proposition 2.1 and 2.2, we can easily get Theorem 1.1.

3 Global existence and blow-up

In this section, by constructing sub- and super-solution, we shall prove Theorem 1.2.

Proposition 3.1 *Let* $v(x, \tau)$ *be the solution of problem (2.1). Suppose that* v_0 *satisfies (H1)′-(H3)′. Then (2.1) has a global solution if* $v_0(x) \leq (\frac{k}{2al})^{\frac{1}{p_1 + q_1 - r_1}}$.

Proof. Let $w = (\frac{k}{2al})^{\frac{1}{p_1 + q_1 - r_1}}$, then

$$\begin{aligned}
& w_\tau = (w + \varepsilon)^{m_1}(w_{xx} + a w^{p_1} \int_{-l}^{l} w^{q_1} dx - k w^{r_1}) = 0, && (x, \tau) \in D_T, \\
& w \geq 0, && \tau \in (0, T], \\
& w \geq v_0(x), && x \in (-l, l).
\end{aligned}$$

Thus $w(x)$ is a supersolution of (2.1), which means that (2.1) has a global solution.

Proposition 3.2 *Suppose that $u_0(x)$ satisfies (H1)-(H3). Then the solution of (1.1) blows up in finite time if $u_0(x)$ is sufficiently large.*

Proof. Since problem(1.1) does not a prior make sense for negative values of u, we actually consider the following problem

$$\begin{aligned} &u_t = (u^m)_{xx} + au_+^p \int_{-l}^{l} u_+^q dx - ku^r, && x \in (-l,l), t > 0,\\ &u(\pm l, t) = 0, && t > 0,\\ &u(x,0) = u_0(x), && x \in [-l,l], \end{aligned}$$

We set

$$z(x,t) = \frac{1}{(T-t)^{\gamma}} V^{\frac{1}{m}}[\frac{|x|}{(T-t)^{\sigma}}], \quad V(y) = 1 + \frac{A}{2} - \frac{y^2}{2A}, y \geq 0,$$

where $\gamma, \sigma > 0$, $A > 1$, and $0 < T < 1$ are to be determined. First note that

$$suppz(t) = \overline{B(0, R(T-t)^{\sigma})} \subset \overline{B(0, RT^{\sigma})} \subset (-l,l), \tag{3.3.1}$$

for sufficiently small $T > 0$ with $R = (A(2+A))^{\frac{1}{2}}$.

Calculating directly, we obtain

$$-[z^m(x,t)]_{xx} = \frac{N/A}{(T-t)^{m\gamma+2\sigma}}.$$

For all $(x,t) \in (-l,l) \times (0,T)$, we find

$$|z(x,t)| \leq \frac{1+A+4l^2}{(T-t)^{\gamma+2\sigma}}.$$

The remaining terms are estimated in two different ways according to the size of $y = \frac{|x|}{(T-t)^{\sigma}}$. If $0 \leq y \leq A$, we have $1 \leq V(y) \leq 1 + \frac{A}{2}$ and $V'(y) \leq 0$, then

$$z_t(x,t) = \frac{m\gamma V^{\frac{1}{m}}(y) + \sigma y V'(y) V^{\frac{1-m}{m}}}{m(T-t)^{\gamma+1}} \leq \frac{\gamma(1+\frac{A}{2})^{\frac{1}{m}}}{(T-t)^{\gamma+1}}$$

$$z_+^p \int_{-l}^{l} z_+^q dx \;=\; \frac{V_+^{\frac{p}{m}}(y)}{(T-t)^{\gamma(p+q)}} \int_{B(0,R(T-t)^{\sigma})} V_+^{\frac{q}{m}}[\frac{|x|}{(T-t)^{\sigma}}]dx \geq \frac{M}{(T-t)^{\gamma(p+q)-N\sigma}},$$

where $M = \int_{B(0,R)} V_+^{\frac{q}{m}}(|\xi|)d\xi$.
Hence,

$$\begin{aligned} z_t - (z^m)_{xx} - az_+^p \int_{-l}^{l} z_+^q dx + kz^r \leq & \frac{\gamma(1+\frac{A}{2})^{\frac{1}{m}}}{(T-t)^{\gamma+1}} + \frac{N/A}{(T-t)^{m\gamma+2\sigma}}\\ & - \frac{M}{(T-t)^{\gamma(p+q)-N\sigma}} + \frac{k(1+A+4l^2)^r}{(T-t)^{(\gamma+2\sigma)r}}. \end{aligned} \tag{3.3.2}$$

On the other hand, if $y > A$, we have $V(y) \leq 1$ and $V'(y) \leq -1$, so that

$$z_t(x,t) \leq \frac{\gamma - \sigma A/m}{(T-t)^{\gamma+1}}.$$

Therefore, for all $(x,t) \in (-l,l) \times [0,T)$ such that $y \geq A$, we obtain

$$z_t - (z)_{xx} - az_+^p \int_{-l}^{l} z_+^q dx + kz^r \leq \frac{\gamma - \sigma A/m}{(T-t)^{\gamma+1}} + \frac{N/A}{(T-t)^{m\gamma+2\sigma}} + \frac{k(1+A+4l^2)^r}{(T-t)^{(\gamma+2\sigma)r}}. \quad (3.3.3)$$

Since $p+q > r > 1$, we can choose $\sigma > 0$ and $\gamma > 0$, such that

$$\gamma(p+q) - N\sigma > \gamma + 1 > (\gamma + 2\sigma)r > m\gamma + 2\sigma.$$

Select $A > \max\{1, \frac{m\gamma}{\sigma}\}$, then for $T > 0$ sufficiently small, (3.2) and (3.3) imply that

$$z_t - (z^m)_{xx} - az_+^p \int_{-l}^{l} z_+^q dx + kz^r \leq 0, (x,t) \in (-l,l) \times (0,T).$$

Let $\varphi \in C^1([-l,l])$, $\varphi(x) \geq 0$, $\varphi(x) \not\equiv 0$, and $\varphi(\pm l) = 0$. By translation, we may assume without loss of generality that $\varphi(0) > 0$. Since $\varphi(0) > 0$ and φ is continuous, there exist two positive numbers ρ and $\varepsilon > 0$, such that $\varphi(x) > \varepsilon$, for all $x \in B(0,\rho) \subset (-l,l)$. Taking T small enough to insure $B(0, RT^\sigma) \subset B(0,\rho)$, and hence $z \leq 0$ on $\{\pm l\} \times (0,T)$. From (3.1), it follows that $z(x,0) \leq \lambda\varphi(x)$ for sufficiently large λ. By Lemma 2.2, we have $z \leq u$ provided that $u_0(x) > \lambda\varphi(x)$ and u can exist no later than $t = T$. This shows that u blows up in finite time.

The proof of Theorem 1.2 By Proposition 3.1 and 3.2, we can prove Theorem1.2.

4 Blow-up set

In this part, we assume that $v_0(x)$ is sufficiently large, the solution $v(x,\tau)$ of (2.1) blows up in finite time and the blow-up time is ${T_1}^*$.

Lemma 4.1 *Suppose that $v_0(x)$ satisfies (H1)′-(H4)′, $p_1 + 2m_1 < 1$. Then $v_{xx} < 0$ in any compact subsets of $(-l,l) \times [0,T^*)$.*

Proof. Let $w = v_{\varepsilon xx}$. According to (2.2), we have

$$\begin{aligned} w_\tau = \; & (v_\varepsilon + \varepsilon)^{m_1} w_{xx} + 2m_1(v_\varepsilon+\varepsilon)^{m_1-1} v_{\varepsilon x} w_x \\ & + [m_1(v_\varepsilon+\varepsilon)^{-1} v_{\varepsilon\tau} + (v_\varepsilon+\varepsilon)^{m_1}(ap_1 v_\varepsilon^{p_1-1} \int_{-l}^{l} v_\varepsilon^{q_1} dx - kr_1 v_\varepsilon^{r_1-1})]w \\ & + m_1(m_1-1)(v_\varepsilon+\varepsilon)^{-2}(v_{\varepsilon x})^2 v_{\varepsilon\tau} \\ & + (v_\varepsilon+\varepsilon)^{m_1}[ap_1(p_1-1)v_\varepsilon^{p_1-2}(v_{\varepsilon x})^2 \int_{-l}^{l} v_\varepsilon^{q_1} dx - kr_1(r_1-1)v_\varepsilon^{r_1-2}(v_{\varepsilon x})^2] \\ & + 2m_1(v_\varepsilon+\varepsilon)^{m_1-1}(v_{\varepsilon x})^2[ap_1 v_\varepsilon^{p_1-1} \int_{-l}^{l} v_\varepsilon^{q_1} dx - kr_1 v_\varepsilon^{r_1-1}] \end{aligned}$$

Since $v_\varepsilon, v_{\varepsilon\tau} \geq 0$ and $p_1 + 2m_1 < 1$, we have

$$w_\tau - (v_\varepsilon + \varepsilon)^{m_1} w_{xx} - 2m_1(v_\varepsilon + \varepsilon)^{m_1 - 1} v_{\varepsilon x} w_x$$
$$-[m_1(v_\varepsilon + \varepsilon)^{-1} v_{\varepsilon\tau} + (v_\varepsilon + \varepsilon)^{m_1}(ap_1 v_\varepsilon^{p_1 - 1} \int_{-l}^{l} v_\varepsilon^{q_1} dx - kr_1 v_\varepsilon^{r_1 - 1})]w \leq 0.$$

By $v_{\varepsilon\tau}(\pm l, \tau) = 0$ and (H4)$'$, we conclude that

$$w(x, 0) \leq 0, \quad w(\pm l, \tau) = -av_\varepsilon^{p_1}(\pm l, \tau) \int_{-l}^{l} v_\varepsilon^{q_1} dx + kv_\varepsilon^{r_1}(\pm l, \tau) = 0.$$

It follows that $w \leq 0$. That is to say $v_{xx} \leq 0(\varepsilon \to 0)$.

Proposition 4.1 *Suppose that* $v_0(x)$ *satisfies (H1)$'$-(H4)$'$*, $p_1 + 2m_1 < 1$. *Then the solution* $v(x, \tau)$ *of (2.1) blows up globally.*

Proof. Let $x_0 \in (-l, l)$ be a blow-up point. Namely, there exists a sequence $\{x_n, \tau_n\}$, such that

$$\tau_n \to {T_1}^*, x_n \to x_0, and \lim_{n\to\infty} v(x_n, \tau_n) = +\infty.$$

It is obvious that for any point $y \in (x_0, l)$ there exists μ and $0 < \mu < 1$ such that $y = (1 - \mu)l + \mu x_0$. Defining the sequence $y_n = (1 - \mu)l + \mu x_n$, it follows that

$$\lim_{n\to\infty} y_n = y.$$

By Lemma 4.1 we have

$$v(y_n, \tau_n) = v[(1 - \mu)l + \mu x_n, \tau_n] \geq (1 - \mu)v(l, \tau_n) + \mu v(x_n, \tau_n),$$

which means y is a blow-up point. Similarly we may prove that any $x \in (-l, x_0)$ is a blow-up point too. That is to say, v blows up globally in $(-l, l)$.

The proof of Theorem 1.3 For problem(1.1), letting $m_1 = \frac{m-1}{m}$, $p_1 = \frac{p}{m}$, $q_1 = \frac{q}{m}$, $r_1 = \frac{r}{m}$, $T^* = \frac{1}{m}{T_1}^*$, we can obtain Theorem 1.3.

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