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Convergence of the Euler-Maruyama Method for Stochastic Differential Equations with Respect to Semimartingales

Yongtian Wang [1] **and Chenggui Yuan**

Department of Mathematics
University of Wales Swansea
Swansea SA2 8PP, UK

Abstract

In this paper, we study the stochastic differential equations with respect to semimartingales and the property of convergence of the Euler-Maruyama scheme approximations to the exact solutions.

Keywords: Itô's formula; Euler-Maruyama method; Lipschitz condition; Convergence in probability

1 Introduction

In this paper we study the numerical solution of stochastic differential equation:

$$dY(t) = f(Y(t))dA(t) + g(Y(t))dM(t) \quad 0 \le t \le T,$$
$$Y(0) = y_0 \in \mathbb{R}^n. \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are sufficiently smooth for the existence and uniqueness of the solution. $M(t) = (M_1(t), \dots, M_m(t))$ is an m-dimensional continuous local martingale with $M(0) = 0$ and $A(t)$ is a continuous adapted increasing process with $A(0) = 0$. Our main objective is to study strong convergence questions for numerical approximations of Eq. (1.1). In fact, when $A(t) = t$ and $M(t)$ is a Brownian motion, there exists an extensive literature in this area, we here only mention Higham, Mao and Stuart [10], Kloeden and Platen [12], Mao [23], Schurz [26] and the references therein.

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t>0}$ satisfying the usual

[1] Corresponding author and e-mail: maytw@swansea.ac.uk

conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$ as well as the matrix trace norm. If A is a vector or matrix, its transpose is denoted by A^T. We will assume there exist $\mathcal{F}_t$-adapted processes $K_{ij}(\cdot)$, $i, j = 1, \dots, m$, such that

$$\langle M_i, M_j\rangle(t) = \int_0^t K_{ij}(s)dA(s), \quad t \geq 0. \tag{1.2}$$

We shall write $K := (K_{ij})_{m\times m}$. Let γ and β be positive numbers such that $A(T) \leq \gamma$ a.s. and $\|K\| \leq \beta$. Let Q be open subset subset of $\mathbb{R}^n$. Denote $C^{2,1}(Q \times \mathbb{R}_+; \mathbb{R}_+)$ the family of all functions $V(x,t) : Q \times \mathbb{R}_+ \to \mathbb{R}_+$ which continuous second partial derivatives in x and first partial derivative in t. Define an operator L acting on $C^{2,1}(Q \times \mathbb{R}_+)$ functions by

$$\begin{aligned}\mathcal{L}V(x, A(t)) =& \frac{\partial V(x, A(t))}{\partial t} + \sum_{i=1}^n f_i(x)\frac{\partial V(x, A(t))}{\partial x_i} \\ &+ \frac{1}{2}\sum_{i,j=1}^n \sum_{k,l=1}^m g_{ik}(x)K_{kl}(t)g_{jl}(x)\frac{\partial^2 V(x, A(t))}{\partial x_i \partial x_j}\end{aligned}$$

In order to prove our results, we need the stochastic integral inequality of the Gronwall-Bellman type (cf. Mao [21]).

Lemma 1.1 *Let ρ be a finite stopping time and γ be a positive constant. Let $(A(t))_{0\leq t\leq\rho}$ be a non-decreasing continuous adapted process such that $A(0) = 0$ and $A(\rho) \leq \gamma$ a.s. and let $(X(t))_{0\leq t\leq\rho}$ be a non-decreasing progressive process. If*

$$EX(\tau) \leq x_0 + E\int_0^\tau X(s)dA(s) \tag{1.3}$$

holds for any stopping time τ with $0 \leq \tau \leq \rho$, where x_0 is a constant, then we have

$$EX(\rho) \leq x_0 e^\gamma. \tag{1.4}$$

Given a stepsize $\Delta > 0$, we can now define the Euler-Maruyama (EM) approximate solution to the Eq. (1.1). Given a stepsize $\Delta > 0$, let $t_k = k\Delta$ for $k \geq 0$. Compute the discrete approximations $X_k \approx Y(t_k)$ by setting $X_0 = y_0$ and forming

$$X_{k+1} = X_k + f(X_k)\Delta A_k + g(X_k)\Delta M_k, \tag{1.5}$$

where $\Delta A_k = A(t_{k+1}) - A(t_k)$ and $\Delta M_k = M(t_{k+1}) - M(t_k)$. Let

$$\bar{X}(t) = X_k, \quad \text{for } t \in [t_k, t_{k+1}) \tag{1.6}$$

and define the continuous EM approximate solution

$$X(t) = X_0 + \int_0^t f(\bar{X}(s))dA(s) + \int_0^t g(\bar{X}(s))dM(s). \tag{1.7}$$

Note that $X(t_k) = \bar{X}(t_k) = X_k$, that is $X(t)$ and $\bar{X}(t)$ coincide with the discrete solution at the gridpoints. Let us now present a lemma for future use.

Lemma 1.2 *Assume that f and g satisfy the linear growth condition:*

(LG) *There is a constant $h > 0$ such that*

$$|f(x)| \vee |g(x)| \le h(1 + |x|) \qquad \textit{for } \forall x \in \mathbb{R}^n.$$

Then for any $p \ge 2$ there is a constant K, which is dependent only on p, T, h, y_0 but independent of Δ, such that the exact solution and the EM approximate solution to the Eq. (1.1) have the property that

$$E\left[\sup_{0\le t\le T} |Y(t)|^p\right] \vee E\left[\sup_{0\le t\le T} |X(t)|^p\right] \le K. \tag{1.8}$$

Proof. It follows from (1.7) and Hölder inequality (cf. Hardy, Littlewood and Polya [9]) that

$$\begin{aligned} |X(t)|^p \le & 3^{p-1}\left[|y_0|^p + |\int_0^t f(\bar{X}(s))dA(s)|^p + |\int_0^t g(\bar{X}(s))dM(s)|^p\right] \\ \le & 3^{p-1}\left[|y_0|^p + T^{p-1}\int_0^t |f(\bar{X}(s))|^p dA(s) + |\int_0^t g(\bar{X}(s))dM(s)|^p\right]. \end{aligned}$$

This implies that for any $0 \le t_1 \le T$,

$$\begin{aligned} E\left[\sup_{0\le t\le t_1} |X(t)|^p\right] \le 3^{p-1}\Big[&|y_0|^p + T^{p-1}\int_0^{t_1} E|f(\bar{X}(s))|^p dA(s) \\ &+ E\big[\sup_{0\le t\le t_1} |\int_0^t g(\bar{X}(s))dM(s)|^p\big]\Big]. \end{aligned} \tag{1.9}$$

By the Burkholder-Davis-Gundy inequality (cf. Daniel and Marc [3]) and the Hölder inequality we compute that

$$E\big[\sup_{0\le t\le t_1} |\int_0^t g(\bar{X}(s))dM(s)|^p\big] \le C_p E\Big[\int_0^{t_1} |g(\bar{X}(s))|^2 d\langle M, M\rangle(s)\Big]^{p/2} \tag{1.10}$$

$$\le C_p T^{p/2-1} E\int_0^{t_1} \|K(s)\||g(\bar{X}(s))|^p dA(s) \le C_p T^{p/2-1}\beta E\int_0^{t_1} |g(\bar{X}(s))|^p dA(s), \tag{1.11}$$

where C_p is a constant. Substituting this into (1.9) and then using the linear growth condition (LG) we obtain

$$E\left[\sup_{0\le t\le t_1}|X(t)|^p\right] \le 3^{p-1}\left[|y_0|^p + 2^{p-1}h^p(T^{p-1}+\beta T^{p/2-1})E\int_0^{t_1}(1+|\bar{X}(s)|^p)ds\right.$$
$$\le K_1 + K_1\int_0^{t_1} E\left[\sup_{0\le t\le s}|X(r)|^p\right]dA(s), \tag{1.12}$$

where $K_1 = K_1(p, T, h, y_0)$ is a constant independent of Δ. Applying Lemma 1.1 to (1.12) yields

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right] \le K_1 e^{\gamma K_1} := K.$$

Similarly, we can show that

$$E\left[\sup_{0\le t\le T}|Y(t)|^p\right] \le K.$$

So the required assertion follows. □

2 Convergence with the Global Lipschitz Condition

In this section we shall show the strong convergence of the EM approximate solution to the exact solution under the following global Lipschitz condition:

(GL) There is a constant $L > 0$ such that

$$|f(x)-f(y)| \vee |g(x)-g(y)| \le L|x-y|$$

for all $x, y \in \mathbb{R}^n$.

Note from this global Lipschitz condition we have

$$|f(x)| \vee |g(x)| \le h(1+|x|) \tag{2.1}$$

with $h = L \vee |f(0)| \vee |g(0)|$. In other words, the global Lipschitz condition (GL) implies the linear growth condition (LG). Lemma 1.2 then shows that under condition (GL) any pth moments, especially the 2nd moments, of the exact solution and the EM approximate solution to Eq. (1.1) are finite.

Theorem 2.1 *Under the global Lipschitz condition (GL) and let*

$$\kappa(\Delta A) = \max\left\{E(\Delta A), E(\Delta A)^2, (E(\Delta A)^4)^{1/2}\right\}.$$

Then

$$E\left[\sup_{0\le t\le T} |X(t) - Y(t)|^2\right] \le C\kappa(\Delta A) + o(\kappa(\Delta A)), \tag{2.2}$$

where C is a positive constant independent of Δ.

Proof. By the Hölder inequality and the Doob martingale inequality, it is not difficult to show that for $0 \le t \le T$,

$$\begin{aligned} E\Big(\sup_{0\le s\le t} |X(s) - Y(s)|^2\Big) \le & 2TE\int_0^t |f(\bar{X}(s)) - f(Y(s))|^2 dA(s) \\ & + 8E\int_0^t |g(\bar{X}(s)) - g(Y(s))|^2 \|K(s)\| dA(s). \end{aligned} \tag{2.3}$$

Note from (GL) that

$$\begin{aligned} & E\int_0^t |f(\bar{X}(s)) - f(Y(s))|^2 dA(s) \le L^2 \int_0^t E|\bar{X}(s) - Y(s)|^2 dA(s) \\ & \le 2L^2 \left[\int_0^t E|X(s) - Y(s)|^2 dA(s) + \int_0^t E|\bar{X}(s) - X(s)|^2 dA(s)\right]. \end{aligned} \tag{2.4}$$

By the Burkholder-Davis-Gundy inequality and (1.2) we obtain

$$E|M(t_2) - M(t_1)|^4 \le C\kappa(\Delta A). \tag{2.5}$$

For $s \in [0, t]$, let $k_s = [s/\Delta]$, the integer part of s/Δ. It then follows from (1.7), (2.1) and (2.5) as well as Lemma 1.2 that

$$\begin{aligned} E|\bar{X}(s) - X(s)|^2 & \le CE\Big[(1 + |X_{k_s}|^2)(|A(s) - A(k_s)|^2 + |M(s) - M(t_{k_s})|^2)\Big] \\ & \le CE\Big[|A(s) - A(k_s)|^2 + |M(s) - M(t_{k_s})|^2\Big] \\ & \quad + C(E|X_{k_s}|^4)^{\frac{1}{2}}\Big[(E|A(s) - A(k_s)|^4)^{\frac{1}{2}} + (E|M(s) - M(k_s)|^4)^{\frac{1}{2}}\Big] \\ & \le C\kappa(\Delta A). \end{aligned} \tag{2.6}$$

Putting (2.6) into (2.4) we see that

$$\begin{aligned} E\Big(\sup_{0\le s\le t} |X(s) - Y(s)|^2\Big) & \le C\int_0^t E|X(s) - Y(s)|^2 ds + C\kappa(\Delta A) \\ & \le C\int_0^t E\Big(\sup_{0\le r\le s} |X(r) - Y(r)|^2\Big) dA(s) + C\kappa(\Delta A) \end{aligned}$$

and the required result (2.2) follows from Lemma 1.1. □

3 Convergence with the Local Lipschitz and Linear Growth Condition

In the previous section we show the strong convergence of the EM method of Eq. (1.1) under the global Lipschitz condition. But in many situations, the coefficients f and g are only locally Lipschitz continuous. It is therefore useful to establish the strong convergence of the EM method under the local Lipschitz condition. By the local Lipschitz condition we mean:

(LL) For each $R = 1, 2, \cdots$, there is a constant $L_R > 0$ such that

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq L_R |x - y|$$

for all those $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq R$.

Theorem 3.1 *Under the local Lipschitz condition (LL) and the linear growth condition (LG), if*

$$\lim_{\Delta \to 0} E(\Delta A)^4 = 0, \tag{3.1}$$

then the EM approximate solution converges to the exact solution of the Eq. (1.1) in the sense that

$$\lim_{\Delta \to 0} E\left[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2\right] = 0. \tag{3.2}$$

Proof Fix a $p > 2$. By Lemma 1.2, there is a positive constant K independent of Δ such that

$$E\left[\sup_{0 \leq t \leq T} |X(t)|^p\right] \vee E\left[\sup_{0 \leq t \leq T} |Y(t)|^p\right] \leq K. \tag{3.3}$$

For sufficiently large integer R, define the stopping times

$$\tau_R = \inf\{t \in [0, T] : |X(t)| \geq R\}, \quad \rho_R = \inf\{t \in [0, T] : |Y(t)| \geq R\}, \quad \theta_R = \tau_R \wedge \rho_R,$$

where throughout this paper we set $\inf \emptyset = T$. Let

$$e(t) = X(t) - Y(t).$$

Recall the Young inequality: for $r^{-1} + q^{-1} = 1$ and $\forall a, b, \delta$

$$ab \leq \frac{\delta}{r} a^r + \frac{1}{q\delta^{q/r}} b^q.$$

Thus, for any $\delta > 0$,

$$\begin{aligned} E\left[\sup_{0\le t\le T}|e(t)|^2\right] =& E\left[\sup_{0\le t\le T}|e(t)|^2 I_{\{\tau_R>T,\rho_R>T\}}\right] + E\left[\sup_{0\le t\le T}|e(t)|^2 I_{\{\tau_R\le T \text{ or } \rho_R\le T\}}\right] \\ \le& E\left[\sup_{0\le t\le T}|e(t\wedge\theta_R)|^2 I_{\{\theta_R>T\}}\right] + \frac{2\delta}{p}E\left[\sup_{0\le t\le T}|e(t)|^p\right] \\ &+ \frac{1-2/p}{\delta^{2/(p-2)}}P(\tau_R\le T \text{ or } \rho_R\le T). \end{aligned} \tag{3.4}$$

Now, by (3.3),

$$P(\tau_R\le T) = E\left[I_{\{\tau_R\le T\}}\frac{|X(\tau_R)|^p}{R^p}\right] \le \frac{1}{R^p}E\left[\sup_{0\le t\le T}|X(t)|^p\right] \le \frac{K}{R^p}.$$

A similar result can be derived for ρ_R, so that

$$P(\tau_R\le T \text{ or } \rho_R\le T) \le \frac{2K}{R^p}.$$

Note also from (3.3) that

$$E\left[\sup_{0\le t\le T}|e(t)|^p\right] \le 2^{p-1}\left(E\left[\sup_{0\le t\le T}|X(t)|^p\right] + E\left[\sup_{0\le t\le T}|Y(t)|^p\right]\right) \le 2^pK.$$

Using these bounds gives

$$\begin{aligned} E\left[\sup_{0\le t\le T}|e(t)|^2\right] \le& E\left[\sup_{0\le t\le T}|X(t\wedge\theta_R)-Y(t\wedge\theta_R)|^2\right] \\ &+ \frac{2^{p+1}\delta K}{p} + \frac{2(p-2)K}{p\delta^{2/(p-2)}R^p}. \end{aligned} \tag{3.5}$$

In the similar way as Theorem 2.1 was proved, we can show that

$$E\left[\sup_{0\le t\le T}|X(t\wedge\theta_R)-y(t\wedge\theta_R)|^2\right] \le C_R\kappa(\Delta A) + o(\kappa(\Delta A)), \tag{3.6}$$

where C_R is a constant independent of Δ. Substituting this into (3.5) gives

$$E\left[\sup_{0\le t\le T}|e(t)|^2\right] \le C_R\kappa(\Delta A) + \frac{2^{p+1}\delta K}{p} + \frac{2(p-2)K}{p\delta^{2/(p-2)}R^p}. \tag{3.7}$$

Now, given any $\varepsilon > 0$, we can choose δ so that

$$\frac{2^{p+1}\delta K}{p} < \frac{\epsilon}{3},$$

then choose R sufficiently large for

$$\frac{2(p-2)K}{p\delta^{2/(p-2)}R^p} < \frac{\epsilon}{3},$$

and finally choose Δ sufficiently small for

$$C_R\kappa(\Delta A)a < \frac{\epsilon}{3},$$

so that, in (3.7),

$$E\left[\sup_{0\le t\le T}|e(t)|^2\right] < \epsilon$$

as required. □

We observe that the proof of Theorem 3.1 uses only the local Lipschitz condition (LL), (3.1) and the bounded pth moment property (3.3), namely

(BM) For some $p > 2$, there is a positive constant K independent of Δ such that

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right] \vee E\left[\sup_{0\le t\le T}|Y(t)|^p\right] \le K.$$

So the following general statement holds.

Theorem 3.2 *Under the local Lipschitz condition (LL), (3.1) and the bounded pth moment condition (BM), the EM approximate solution converges to the exact solution of the Eq. (1.1) in the sense that*

$$\lim_{\Delta\to 0} E\left[\sup_{0\le t\le T}|X(t)-Y(t)|^2\right] = 0.$$

4 Convergence in Probability

Instead let us now concentrate on the the Eq. (1.1) with only the local Lipschitz condition (LL) but without the linear growth condition (LG) or the bounded pth moment property (BM). The following theorem describes the convergence in probability, instead of L^2, of the EM solutions to the exact solution under some additional conditions in terms of Lyapunov-type functions. Let

$$V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \dots, \frac{\partial V(x)}{\partial x_n}\right), \quad V_{xx}(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Theorem 4.1 *Let the local Lipschitz condition (LL) and (3.1) hold. Assume that there exists a C^2 function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the following three conditions:*

(i) $\lim_{|x|\to\infty} V(x) = \infty$;

(ii) *for some* $h > 0$,

$$\mathcal{L}V(x) \vee V_x g(x) \le h(1 + V(x)) \quad \forall (x, i) \in \mathbb{R}^n \times S,$$

where

$$\mathcal{L}V(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial V(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k,l=1}^{m} g_{ik}(x) K_{kl} g_{jl} \frac{\partial^2 V(x)}{\partial x_i \partial x_j};$$

(iii) *for each* $R > 0$ *there exists a positive constant* K_R *such that for all* $x, y \in \mathbb{R}^n$ *with* $|x| \vee |y| \le R$,

$$|V(x) - V(y)| \vee |V_x(x) - V_x(y)| \vee |V_{xx}(x) - V_{xx}(y)| \le K_R |x - y|.$$

Then

$$\lim_{\Delta \to 0} \left(\sup_{0 \le t \le T} |X(t) - Y(t)|^2 \right) = 0 \quad \textit{in probability.} \tag{4.1}$$

Proof. We divide the whole proof into three steps.

Step 1. For sufficiently large R, define the stopping time

$$\theta = \inf\{t \in [0, T] : |Y(t)| \ge R\}.$$

Applying the generalised Itô formula (cf. Mao [21]) and using condition (ii) to $V(Y(t))$ yields

$$\begin{aligned} V(Y(t \wedge \theta)) &= V(y_0) + \int_0^{t\wedge\theta} \mathcal{L}V(Y(s)) dA(s) + \int_0^{t\wedge\theta} V_x(Y(s) g(Y(s)) dM(s) \\ &\le V(y_0) + h \int_0^{t\wedge\theta} (1 + V(Y(s))) dA(s) + \int_0^{t\wedge\theta} V_x(Y(s) g(Y(s)) dM(s) \end{aligned}$$

By Burkholder-Davis-Gundy inequality and Hölder inequality, we obtain

$$\begin{aligned} E\left[\sup_{0 \le t \le t_1} V^2(Y(t \wedge \theta)) \right] &\le [V(y_0) + hT]^2 + h^2 E \left(\int_0^{t_1 \wedge \theta} V(Y(s))) dA(s) \right)^2 \\ &+ E \left(\sup_{0 \le t \le t_1} | \int_0^{t\wedge\theta} V_x(Y(s) g(Y(s)) dM(s)| \right)^2 \\ &\le [V(y_0) + hT]^2 + h^2 T E \int_0^{t_1 \wedge \theta} V^2(Y(s))) dA(s) \\ &+ CE \int_0^{t_1 \wedge \theta} |V_x(Y(s) g(Y(s))|^2 d\langle MM \rangle_s \\ &\le [V(y_0) + 2hT]^2 + h^2(T + 2\beta C) \int_0^{t_1} E \sup_{0 \le r \le s} V^2(Y(r \wedge \theta))) dA(s) \end{aligned}$$

Using Lemma 1.1, we obtain

$$E[V^2(Y(T\wedge\theta))] \le [V(y_0)+2hT]^2\, e^{h^2(T+2\beta C)\gamma}. \tag{4.2}$$

Let

$$v_R = \inf\{V(x) : |x| \ge R\}.$$

By condition (i), $v_R \to \infty$ as $R \to \infty$. Noting that $|Y(\theta)| = R$ whenever $\theta < T$, we derive from (4.2) that

$$\begin{aligned} [V(y_0)+2hT]^2\, e^{h^2(T+2\beta C)\gamma} &\ge E[V^2(Y(\theta))I_{\{\theta<T\}}] \\ &\ge v_R^2 P(\theta < T). \end{aligned}$$

That is

$$P(\theta < T) \le \frac{e^{h^2(T+2\beta C)\gamma}}{v_R^2}\,[V(y_0)+2hT]^2\,. \tag{4.3}$$

Step 2. For sufficiently large R define the stopping time

$$\rho = \inf\{t \in [0,T] : |X(t)| \ge R\}.$$

Using (1.7) and applying the generalized Itô's formula to $V(X(t))$ yields

$$\begin{aligned} dV(X(\rho\wedge t)) &= \sum_{i=1}^n g_i(\bar X(s))\frac{\partial V(X(s))}{\partial x_i}dM(s) \\ &+ \left[\sum_{i=1}^n f_i(\bar X(s))\frac{\partial V(X(s))}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n\sum_{k,l=1}^m g_{ik}(\bar X(s))K_{kl}(s)g_{jl}(\bar X(s))\frac{\partial^2 V(X(s))}{\partial x_i\partial x_j}\right]dA(s) \\ &= LV(\bar X(s))dA(s) + \sum_{i=1}^n g_i(\bar X(s))\left(\frac{\partial V(X(s))}{\partial x_i} - \frac{\partial V(\bar X(s))}{\partial x_i}\right)dM(s) \\ &+ \sum_{i=1}^n g_i(\bar X(s))\frac{\partial V(\bar X(s))}{\partial x_i}dM(s) + \sum_{i=1}^n f_i(\bar X(s))\left(\frac{\partial V(X(s))}{\partial x_i} - \frac{\partial V(\bar X(s))}{\partial x_i}\right)dA(s) \\ &+ \frac{1}{2}\sum_{i,j=1}^n\sum_{k,l=1}^m g_{ik}(\bar X(s))K_{kl}(s)g_{jl}(\bar X(s))\left(\frac{\partial^2 V(X(s))}{\partial x_i\partial x_j} - \frac{\partial^2 V(\bar X(s))}{\partial x_i\partial x_j}\right)dA(s). \end{aligned}$$

Whence on applying condition (iii) we obtain

$$\begin{aligned}
&dV(X(\rho\wedge t))\\
&\le h(1+V(\bar X(s)))dA(s)+\sum_{i=1}^{n} g_i(\bar X(s))\frac{\partial V(\bar X(s))}{\partial x_i}dM(s)\\
&+\sum_{i=1}^{n} g_i(\bar X(s))\left(\frac{\partial V(X(s))}{\partial x_i}-\frac{\partial V(\bar X(s))}{\partial x_i}\right)dM(s)\\
&+\sum_{i=1}^{n} f_i(\bar X(s))\left(\frac{\partial V(X(s))}{\partial x_i}-\frac{\partial V(\bar X(s))}{\partial x_i}\right)dA(s)\\
&+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{m} g_{ik}(\bar X(s))K_{kl}(s)g_{jl}(\bar X(s))\left(\frac{\partial^2 V(X(s))}{\partial x_i\partial x_j}-\frac{\partial^2 V(\bar X(s))}{\partial x_i\partial x_j}\right)dA(s)
\end{aligned}$$

$$\begin{aligned}
&=h(1+V(X(s)))dA(s)+(V(\bar X(s))-V(X(s)))dA(s)\\
&+\sum_{i=1}^{n} g_i(\bar X(s))\frac{\partial V(\bar X(s))}{\partial x_i}dM(s)+\sum_{i=1}^{n} g_i(\bar X(s))\left(\frac{\partial V(X(s))}{\partial x_i}-\frac{\partial V(\bar X(s))}{\partial x_i}\right)dM(s)\\
&+\sum_{i=1}^{n} f_i(\bar X(s))\left(\frac{\partial V(X(s))}{\partial x_i}-\frac{\partial V(\bar X(s))}{\partial x_i}\right)dA(s)\\
&+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{m} g_{ik}(\bar X(s))K_{kl}(s)g_{jl}(\bar X(s))\left(\frac{\partial^2 V(X(s))}{\partial x_i\partial x_j}-\frac{\partial^2 V(\bar X(s))}{\partial x_i\partial x_j}\right)dA(s).
\end{aligned}$$

Integrating from 0 to $\rho\wedge t$ and taking expections gives

$$\begin{aligned}
&\frac{1}{6}E[\sup_{0\le t\le t_1} V^2(X(\rho\wedge t))]\le [V(y_0)+hT]^2+h^2E\left(\int_0^{\rho\wedge t_1} V(X(s))dA(s)\right)^2\\
&+h^2E\left(\int_0^{\rho\wedge t}|V(\bar X(s))-V(X(s))|dA(s)\right)^2\\
&+E\left(\int_0^{\rho\wedge t}|V_x(X(s))-V_x(\bar X(s))||f(\bar X(s))|dA(s)\right)^2\\
&+\frac{1}{4}E\left(\int_0^{\rho\wedge t}|V_{xx}(X(s))-V_{xx}(\bar X(s))||g(\bar X(s))|^2\|K(s)\|dA(s)\right)^2\\
&+E\left(\sup_{0\le t\le t_1}|\int_0^{\rho\wedge t}[V_x(X(s))-V_x(\bar X(s))]g(\bar X(s))dM(s)|^2\right)\\
&+E\left(\sup_{0\le t\le t_1}|\int_0^{\rho\wedge t}V_x(\bar X(s))g(\bar X(s))dM(s)|^2\right).
\end{aligned}\tag{4.4}$$

By Burkholder-Davis-Gundy inequality and Hölder inequality, we obtain

$$\begin{aligned}
&\frac{1}{6}E[\sup_{0\le t\le t_1} V^2(X(\rho\wedge t))]\\
&\le [V(y_0)+hT]^2 + h^2TE\left(\int_0^{\rho\wedge t_1} V^2(X(s))dA(s)\right)\\
&+\beta^2TE\left(\int_0^{\rho\wedge t_1} V^2(\bar X(s))dA(s)\right) + h^2TE\left(\int_0^{\rho\wedge t} |V(\bar X(s))-V(X(s))|^2dA(s)\right)\\
&+\frac{1}{4}TE\left(\int_0^{\rho\wedge t} |V_{xx}(X(s))-V_{xx}(\bar X(s))|^2|g(\bar X(s))|^4\|K(s)\|dA(s)\right)\\
&+E\left(|\int_0^{\rho\wedge t_1}[V_x(X(s))-V_x(\bar X(s))]^2g^2(\bar X(s))K(s)dA(s)|\right)
\end{aligned}$$

By condition (iii) we have

$$E\int_0^{\rho\wedge t}|V(\bar X(s))-V(X(s))|^2dA(s)\le E\int_0^{\rho\wedge t}K_R^2|\bar X(s)-X(s)|^2dA(s)$$

We can similarly estimate the other terms on the right-hand side of (4.4) to get that

$$\begin{aligned}
E[\sup_{0\le t\le t_1} V^2(X(\rho\wedge t))] &\le 6[V(y_0)+hT]^2+6h^2TE\int_0^{\rho\wedge t_1} E\sup_{0\le r\le s}V^2(X(\rho\wedge r))dA(s)\\
&+C_1(R)\int_0^T\left(E|\bar X(\rho\wedge s)-X(\rho\wedge s)|^2\right)dA(s), \qquad (4.5)
\end{aligned}$$

where $C_1(R)$ and the following $C_2(R)$, $C_3(R),\cdots$ are all constants dependent of R but independent of Δ. But, in the same way as (2.6) was proved, we can show that

$$E|\bar X(\rho\wedge s)-X(\rho\wedge s)|^2\le C_2(R)\kappa(\Delta A)\quad \forall s\in[0,T].$$

Substituting this into (4.5) yields that

$$\begin{aligned}
E[\sup_{0\le t\le t_1} V(X(\rho\wedge t), r(\rho\wedge t))] &\le 6[V(y_0)+hT]^2+C_3(R)\kappa(\Delta A)\\
&+6h^2T\int_0^t E\sup_{0\le r\le s}V^2(X(\rho\wedge r))dA(s).
\end{aligned}$$

By the Gronwall inequality,

$$E[V(X(\rho\wedge T), r(\rho\wedge T))]\le e^{6h^2T\gamma}\Big[6[V(y_0)+hT]^2+C_3(R)\kappa(\Delta A)\Big]. \qquad (4.6)$$

In the way as (4.3) was obtained, we can then show that

$$P(\rho < T) \leq \frac{e^{6h^2T\gamma}}{v_R^2}\Big[6[V(y_0)+hT]^2 + C_3(R)\kappa(\Delta A)\Big]. \tag{4.7}$$

Step 3. Let $\tau = \rho \wedge \theta$. In the same way as Theorem 2.1 was prove we can show that

$$E\left[\sup_{0\leq t\leq \tau\wedge T} |X(t)-Y(t)|^2\right] \leq C_4(R)\kappa(\Delta A). \tag{4.8}$$

Now, let $\varepsilon, \delta \in (0,1)$ be arbitrarily small. Set

$$\bar{\Omega} = \{\omega : \sup_{0\leq t\leq T} |X(t)-Y(t)|^2 \geq \delta\}$$

Using (4.8), we compute

$$\begin{aligned}
\delta P(\bar{\Omega}\cap\{\tau\geq T\}) &= \delta E\left[I_{\{\tau\geq T\}} I_{\bar{\Omega}}\right] \\
&\leq E\left[I_{\{\tau\geq T\}} \sup_{0\leq t\leq \tau\wedge T} |X(t)-Y(t)|^2\right] \\
&\leq E\left[\sup_{0\leq t\leq \tau\wedge T} |X(t)-Y(t)|^2\right] \\
&\leq C_4(R)\kappa(\Delta A).
\end{aligned}$$

This, together with (4.3) and (4.7), yields that

$$\begin{aligned}
P(\bar{\Omega}) &\leq P(\bar{\Omega}\cap\{\tau\geq T\}) + P(\tau < T) \\
&\leq P(\bar{\Omega}\cap\{\tau\geq T\}) + P(\theta < T) + P(\rho < T) \\
&\leq \frac{C_4(R)}{\delta}\kappa(\Delta A) + \frac{e^{h^2(T+2\beta C)\gamma}}{v_R^2}\left[V(y_0)+hT\right]^2 \\
&+ \frac{e^{6h^2T\gamma}}{v_R^2}\Big[6[V(y_0)+hT]^2 + C_3(R)\kappa(\Delta A)\Big].
\end{aligned}$$

Recalling that $v_R \to \infty$ as $R\to\infty$, we can choose R sufficiently large for

$$\frac{e^{h^2T\gamma[V(y_0)+hT]^2}}{v_R^2}\left[e^{2h^2\beta C\gamma} + 6e^{5h^2T}\right] < \frac{\varepsilon}{2},$$

and then choose Δ sufficiently small for

$$\frac{C_4(R)}{\delta}\kappa(\Delta A) + \frac{e^{6h^2T\gamma}}{v_R^2} C_3(R)\kappa(\Delta A) < \frac{\varepsilon}{2}$$

to obtain

$$P\left(\bar{\Omega}\right) = P\left(\sup_{0\leq t\leq T} |X(t)-Y(t)|^2 \geq \delta\right) < \varepsilon$$

This proves the assertion (4.1). □

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