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An Alternative Exact Solution Method for the Reduced Wave Equation with a Variable Coefficient

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Abstract. In this paper we present the exact solution of reduced wave equation with a variable coefficient $\Delta u(x) + k^2 n(x)u(x) = 0$ for $n(x) = n(r)$, $r = |x|$ by the solution of a classic Riccati differential equation. By constructing an "iteration" technique for a differential equation of the form z"=λz we present not only partial solution of a classical Riccati differential equation but olso the exact solution of the reduced wave equation with a variable coefficient.In addition we present a simple criterion for the existance of polynomial solution of a differential equation of the form z"=λz .Where λ is a function in C^∞.

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1. Introduction

The wave equation is an important partial differential equation that described a variety of waves, such as sound waves,light waves and water waves . It arises in fields such as acoustic, electromagnetics and fluid dynamics.

The propogation of waves in a homogeneous , isotropic medium is mathematically described by the wave equation

$$\Delta V(x,t) - \frac{1}{c^2} V_{tt} = 0 \tag{1.1}$$

where Δ is the Laplace operator and c denotes the speed of propogation. If the problems involve time-harmonic waves,i.e.waves field of the form

$$V(x,t) = u(x)e^{-iwt} \tag{1.2}$$

where $i = \sqrt{-1}$ and w denotes the frequences of waves, then the wave equation can be reduced to the homogeneous scalar Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, k = \frac{w}{c} \tag{1.3}$$

If we consider an inhomogeneous medium in $\mathbb{R}^3$ and if we assume that the inhomogenity is compactly supported, then the propogation of time-harmonic acoustic waves in the medium is governed by the equation

$$\Delta u(x) + k^2 n(x)u(x) = 0 \tag{1.4}$$

where u describes the pressure field, $k > 0$ is the wave number and $n(x)$is the refractive index of the medium. k and $n(x)$ are related to the frequency w of the wave and to the speed of sound of the medium via $k = \frac{w}{c_0}$ and $n(x) = \left(\frac{c_0}{c(x)}\right)^2$.

Elementary solution of the reduced wave equation for variable index of refraction n , which generalize the well-known solution (point source) $u = \frac{\exp(ikR)}{R}$ where $R = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$,(line source) $u = i\pi H_0^{(1)}(kQ)$ where $\mathrm{Q}^2 = (x-x_0)^2 + (y-y_0)^2$ and $H_0^{(1)}$ is the Hankel function of the first kind of order zero , for a homogeneous medium ($n = 1$) ,are known to be very few in number. In fact , for layered media $[n = n(y)]$, only two such solution have so far been found.These are (i) Pekeris' solution [6] for a point source in a medium specified by $n = y^{-1}$,in which case

$$u = \frac{2(yy_0)^{\frac{1}{2}}}{RR'} \exp\left[2i\left(k^2 - \frac{1}{4}\right)^{\frac{1}{2}} \tanh^{-1}\left(\frac{R}{R'}\right)\right], \tag{1.5}$$

with $R'^2 = (x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2$;and (ii) Kormilitsin's solution [5] for a line source extending parallel to the z axis in a medium specified by $n = y^{\frac{1}{2}}$,in which case

$$u = \int_0^\infty \exp\left[ik\left(\frac{Q^2}{2\zeta} + (y+y_0)\frac{\zeta}{4} - \frac{\zeta^3}{96}\right)\right]\frac{d\zeta}{\zeta}. \tag{1.6}$$

For $n^2 = 1$ Fock [3] and Weinstein [7] have obtained asymtotic solution of reduced wave equation for many important problems. In [4] R.L.Holford has discussed the elemantery solution of reduced wave equation in two dimension for which the refraction index is the form $n = (A + Bx + +Cy + Dx^2 + Exy +$

$Fy^2)^{\frac{1}{2}}$. In [1] Daniel J.Arrigo and Fred Hickling have considered the reduced wave equation with a variable wave speed and its parabolic approxsimation.

2. Formation of exact solution

One important tool for constructing exact solution to a differential (partial differential) equation is to link its solution to the known solutions of another differential (partial differential) equation . This tecnique , now commonly referred to as Darboux transformations. In this paper Darboux transformation is constructed from the Fundamental solution of scalar Helmholtz equation.

In the rest of this work we assume that $n(x) = n(|x|) = n(r)$ and there exist a continious function $P(r)$ such that $k^2 n(r) = k^2 - P(r)$.

Lemma 2.1. *Let $f : \mathbb{R}^3 \to \mathbb{R} \setminus \{0\}$ be a continuous function that has first and second derivatives and satisfies the equation*

$$P(r) = -\frac{f''(r)}{f(r)} + 2\frac{(f'(r))^2}{f^2(r)} - (2k\cot kr)\frac{f'(r)}{f(r)} \tag{2.1}$$

then

$$u(x) = \frac{\sin kr}{rf(r)}, r = |x| \tag{2.2}$$

satisfies the equation (1.4)

Note that in the rest of this paper,for the sake of simplicity we use only f and P instead of $f(r)$ and $P(r)$.

Proof. Using the chain rule we get for i=1,2,3. □

$\frac{\partial u}{\partial x_i} = \left[\frac{k\cos kr}{rf} - \frac{\sin kr}{r^3 f} - \frac{f'\sin kr}{r^2 f^2}\right] x_i$

Moreover,

$\frac{\partial^2 u}{\partial x_i^2} = \frac{\cos kr}{rf}\left(\frac{k}{r}\right) - \frac{\sin kr}{rf}\left(\frac{1}{r^2} + \frac{f'}{rf}\right) +$
$\frac{x_i^2}{r}\left\{\frac{\sin kr}{rf}\left(-\frac{k^2}{r} + \frac{3}{r^3} + \frac{3f'}{r^2 f} - \frac{f''}{rf} + \frac{2f'^2}{rf^2}\right) + \frac{\cos kr}{rf}\left(-\frac{3k}{r^2} - \frac{2kf'}{rf}\right)\right\}$

Adding, we find

$$\triangle u = \frac{\sin kr}{rf}\left(-k^2 - \frac{f''}{f} + \frac{2f'^2}{f^2}\right) - \frac{\cos kr}{rf}\left(\frac{2kf'}{f}\right) \tag{2.3}$$

If we rearrange the second term of the right side of the equation (2.3), we get

$$\triangle u = \frac{\sin kr}{rf}\left(-k^2 - \frac{f''}{f} + \frac{2f'^2}{f^2} - (2k\cot kr)\frac{f'}{f}\right) \tag{2.4}$$

This completes the proof of the lemma by (2.1) and (2.2).

To get a solution of the equation (1.4) in the form(2.2) we must solve the equation (2.1) which is a second order nonlinear ordinary differential equation with a variable coefficient.In addition the coefficient function $\cot(kr)$ may have some singular points.Thus it seems that to get a solution f from the equation (2.1) may not be easy.

As an initial simplification,we will choose a constant k such that (2.1) will have no critical point.

Lemma 2.2. *Let ϕ be a continuous differentiable function,such that ϕ satisfies the Riccati differential equation*

$$\phi' + \phi^2 = P - k^2 \tag{2.5}$$

then

$$u = \frac{1}{r} \exp\left(\int^{r} \phi(\tau) d\tau \right) \tag{2.6}$$

is the solution of equation (1.4)

Proof. If we multiply by f^2 the equation (2.1) we get ☐

$$ff'' - 2f'^2 + (2k \cot kr) ff' + Pf^2 = 0 \tag{2.7}$$

Let ψ is a continuous differentiable function and

$$f' = -f\psi \tag{2.8}$$

then we have

$$f'^2 = f^2\psi^2, ff' = -f^2\psi, ff'' = f^2\left(\psi^2 - \psi'\right) \tag{2.9}$$

If we use equalities (2.9) in the equation (2.7) we get

$$\psi' + \psi^2 = -(2k \cot kr)\psi + P \tag{2.10}$$

Again if we consider the change of variable

$$\psi = \phi - k \cot kr \tag{2.11}$$

where ϕ is a continuous differentiable function , then the equation (2.10) becames a classical Riccati equation in the form $\phi' + \phi^2 = P - k^2$, which completes the proof of lemma.

Remark 2.3. It is well known that if $g(r) \neq 0$ then a rational solution of the equation $\phi' + \phi^2 = g$ is equivalent to an exponential solution $\exp\left(\int \phi(v) dv\right)$ of the linear differential equation

$$z'' = g(r)z \tag{2.12}$$

3. The asymtotic itereation method

Let $\lambda_0 \in C^{\infty}(a,b)$and consider the equation

$$z''(t) = \lambda_0(t)z(t) \tag{3.1}$$

For some λ_0 function we shall give a new method to obtain the general solution of (3.1) .This method depends on finding same symmetric structure by using asymptotic behavior of equation (3.1). Thus for this purpose if we differentiate (3.1) with respect to t, we find that

$$z''' = \lambda_0 z' + \lambda_1 z \tag{3.2}$$

where $\lambda_1 = \lambda_0'$.

If we write second derivative of the equation (3.1), we get

$$z^{(4)} = \lambda_2 z' + \lambda_3 z \tag{3.3}$$

where $\lambda_2 = \lambda_0' + \lambda_1$ and $\lambda_3 = \lambda_0^2 + \lambda_1'$.

Thus if we continue in this way ,we get for $n \geq 3$

$$z^{(n)} = \lambda_{2n-6} z' + \lambda_{2n-5} z \tag{3.4}$$

where ,for $k = 3, 4, ..., n$

$$\lambda_{2k-1} = \lambda_0 \lambda_{2k-4} + \lambda_{2k-3}', and \lambda_{2k} = \lambda_{2k-2}' + \lambda_{2k-1} \tag{3.5}$$

Similarly,for the $(n+1)\,th$ and $(n+2)\,th$ derivatives, we get

$$z^{(n+1)} = \lambda_{2n-4} z' + \lambda_{2n-3} z \tag{3.6}$$

and

$$z^{(n+2)} = \lambda_{2n-2} z' + \lambda_{2n-1} z \tag{3.7}$$

for $n = 3, 4, ...$

From the ratio of the $(n+2)\,th$ and $(n+1)\,th$ derivatives , we get

$$\frac{d}{dt}\left(\ln z^{(n+1)}\right) = \frac{z^{(n+2)}}{z^{(n+1)}} = \frac{\lambda_{2n-2}\left(z' + \frac{\lambda_{2n-1}}{\lambda_{2n-2}} z\right)}{\lambda_{2n-4}\left(z' + \frac{\lambda_{2n-3}}{\lambda_{2n-4}} z\right)} \tag{3.8}$$

We now introduce the "asymptotic" aspect of the method . If we have for sufficiently large $n \geq 3$

$$\frac{\lambda_{2n-1}}{\lambda_{2n-2}} = \frac{\lambda_{2n-3}}{\lambda_{2n-4}} := \beta \tag{3.9}$$

then (3.8) reduces to

$$\frac{d}{dt}(\ln z^{(n+1)}) = \frac{\lambda_{2n-2}}{\lambda_{2n-4}} \tag{3.10}$$

which yields

$$z^{(n+1)} = c_1 \exp\left(\int^t \frac{\lambda_{2n-2}(\tau)}{\lambda_{2n-4}(\tau)} d\tau\right) \tag{3.11}$$

But in equation (3.11) the integrant function is

$$\frac{\lambda_{2n-2}}{\lambda_{2n-4}} = \frac{\lambda'_{2n-4}}{\lambda_{2n-4}} + \frac{\lambda_{2n-3}}{\lambda_{2n-4}}. \tag{3.12}$$

Then (3.11) becomes

$$z^{(n+1)} = c_1 \lambda_{2n-4} \exp\left(\int^t \beta(\tau)\, d\tau\right) \tag{3.13}$$

Substituting (3.13) into (3.6) we obtain the first order differential equation

$$z' + \beta z = c_1 \exp\left(\int^t \beta(\tau)\, d\tau\right). \tag{3.14}$$

Which, in turn, yields the general solution of (3.1) as

$$z(t) = \exp\left(-\int^t \beta(\tau)\, d\tau\right)\left[c_1 \int^t \exp(\int^\tau 2\beta(\zeta)\, d\zeta) d\tau + c_2\right] \tag{3.15}$$

Lemma 3.1. *The function β, which is defined by (3.9) must differ from zero.*

Proof. If for sufficiently large $n \geq 3$ there exist a β such that the equality (3.9) hold , then $\lambda_{2n-2}\lambda_{2n-4} \neq 0. If\ \beta = 0$ then $\lambda_{2n-1}$and λ_{2n-3}could be zero and if $\lambda_{2n-1} = 0$ then $\lambda'_{2n-1} = 0$, similarly $\lambda'_{2n-3} = 0$. If we use this fact in (3.5) we get $\lambda_0\lambda_{2n-4} = 0$ .This is a contradiction because $\lambda_0 \neq 0$and $\lambda_{2n-4} \neq 0$ .Thus $\beta \neq 0$. □

If we use above fact (3.1)-(3.15), we have proved the following theorem.

Theorem 3.2. *Given $\lambda_0 \in C^\infty(a, b)$, then the differential equation (3.1) has a general solution (3.15) if for same $n \geq 3$*

$$\frac{\lambda_{2n-1}}{\lambda_{2n-2}} = \frac{\lambda_{2n-3}}{\lambda_{2n-4}} = \beta \tag{3.16}$$

equality holds.Where

$$\lambda_{2k-1} = \lambda_0 \lambda_{2k-4} + \lambda'_{2k-3}, and \lambda_{2k} = \lambda'_{2k-2} + \lambda_{2k-1} for k = 3, 4, ..., n. \tag{3.17}$$

Example 3.3. Consider a simple Euler's differential equation

$$z''(t) = \frac{2}{t^2} z(t) \tag{3.18}$$

Then $\lambda_0 = \frac{2}{t^2}, \lambda_1 = \frac{-4}{t^3}, \lambda_2 = \frac{-8}{t^3}, \lambda_3 = \frac{16}{t^4}, \lambda_4 = \frac{40}{t^4}$, ...and for $n = 3, 4, 5, ...$

$$\beta = \frac{\lambda_{2n-1}}{\lambda_{2n-2}} = \frac{\lambda_{2n-3}}{\lambda_{2n-4}} = -\frac{2}{t} \tag{3.19}$$

Thus from Theorem1 the general solution of (3.18) is

$$z(t) = \exp\left(-\int^t \left(-\frac{2}{\tau}\right) d\tau\right) \left[c_1 \int^t \exp\left(\int^\tau 2\left(-\frac{2}{\zeta}\right) d\zeta\right) d\tau + c_2\right] \tag{3.20}$$
$$= c_1(\frac{-1}{3t}) + c_2 t^2$$

If we choose $c_1 = 0$,$c_2 = 1$ in the equation (3.20) we get polynomial solution $z(t) = t^2$ of the equation (3.18).

Now we want to construct a characterization of polynomial solution ,at least mathematically, of the equation (3.1) .For this purpose multiply (3.6) by λ_{2n-2} and (3.7) by -(λ_{2n-4}) , and add.We obtain

$$\lambda_{2n-2} z^{(n+1)} - \lambda_{2n-4} z^{(n+2)} = T_n z \tag{3.21}$$

where

$$T_n = \lambda_{2n-2}\lambda_{2n-3} - \lambda_{2n-1}\lambda_{2n-4} \tag{3.22}$$

Thus, if equation (3.1) has a polynomial solution $z(t)$ whose degree at most n we have $z^{(n+1)} = z^{(n+2)} = 0$. Consequently we conclude that from (3.21) that $T_n = 0$. Conversely if $T_n = 0$ and $\lambda_{2n-2}\lambda_{2n-4} \neq 0$, then from (3.22) we have $\frac{\lambda_{2n-1}}{\lambda_{2n-2}} = \frac{\lambda_{2n-3}}{\lambda_{2n-4}} = \beta$, and , from Theorem 1,we conclude that a solution of (3.1) is given by

$$z(t) = \exp\left(-\int^t \beta(\tau)\, d\tau\right) \tag{3.23}$$

If we differentiate (3.23) with respect to t, we find that

$$z'(t) = -\beta(t)\exp\left(-\int^{t}\beta(\tau)\,d\tau\right) = -\frac{\lambda_{2n-3}}{\lambda_{2n-4}}z(t) \tag{3.24}$$

thus we have

$$\lambda_{2n-4}z' + \lambda_{2n-3}z = 0 \tag{3.25}$$

If we insert (3.25) in the equation (3.6) , we get $z^{(n+1)} = 0$, or equivalently, that $z(t)$ is a polynomial of degree at most n.

Consequently, we have proved the following theorem.

Theorem 3.4. *a) Given $\lambda_0 \in C^\infty(a,b)$, if the differential equation (3.1) has a polynomial solution whose degree at most n then $T_n = 0$.*

b)If $T_n = 0$ and $\lambda_{2n-2}\lambda_{2n-4} \neq 0$, then the differential equation (3.1) has a polynomial solution whosedegree at most n.

Lemma 3.5. *Let $\lambda_0 \in C^\infty(0,r)$. If there exist a β such that the equality (3.16) holds and $P(r) = k^2 - k^2n(r)$, then*

$$u = \frac{1}{r}\exp\left(-\int^{r}\beta(\tau)d\tau\right) \tag{3.26}$$

is the solution of (1.4) .

Proof. If there exist a β , which satisfies (3.16) , then a solution of (3.1) is □

$$z = \exp\left(-\int^{r}\beta(\tau)d\tau\right) \tag{3.27}$$

Because of the function (3.27) satisfies the equation (3.1) for $\lambda_0 = -k^2n(r) = P - k^2$, we get

$$\beta' - \beta^2 = P - k^2. \tag{3.28}$$

If we take $\beta = -\phi$ in equation (3.28) then the equation (3.28) becomes (2.5).This completes the proof of lemma by using the Lemma 2 and Theorem 1 respectively.

Remark 3.6. Consider the differential equation

$$u''(t) + \left(\frac{2}{t}\right)u'(t) + f(t)\,u(t) = 0 \tag{3.29}$$

The substitution $u(t) = t^{-1}z(t)$ gives

$$z''(t) + f(t)\,z(t) = 0 \tag{3.30}$$

which is the equation (3.1) for $\lambda_0 = -f(t)$ .Thus if there exist a β such that (3.9) holds for $\lambda_0 = -f(t)$, then general solution of the equation (3.29) can be given by (3.15).

Remark 3.7. In [2] (2004) Guoting Chen and Yujie Ma proved that if the Riccati differential equation $\phi' + \phi^2 = g(r)$ with $g(r) \neq 0$ has a general rational solution , then $g(r)$ has the form

$g(r) = \sum_{i=1}^{m} \left(\frac{\delta_i}{(r-r_i)^2} + \frac{\eta_i}{r-r_i} \right)$,

in which $4\delta_i = \theta_i^2 - 1$ where $r_1, r_2, ..., r_m$ be the poles of r and θ_i is an integer ≥ 2.

4. Conclusion

In this paper first we considered a kind of Darboux transformation to solve the reduced wave equation with a variable coefficient (in particular , refractive index of medium is a function of $r = |x|$).For this, a solution of Riccati differential equation gives us an exact solution of the reduced wave equation with a variable coefficient .Second , we present an asymtotic iteration method for the differential equation $z'' = \lambda_0 z$ where $\lambda_0 \in C^\infty(a, b)$ which works under the existance of $\frac{\lambda_{2n-1}}{\lambda_{2n-2}} = \frac{\lambda_{2n-3}}{\lambda_{2n-4}} = \beta$ for $n \geq 3$. So that we show that if there exist a β for sufficiently large $n \geq 3$ then -β is a particular solution of the Riccati differential equation and an exact solution of the reduced wave equation with a variable coefficient also obtained with the aid of β.

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