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The Complexity of a Minimal Subsystem on Compact Spaces[1]

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Abstract. In this paper , we discuss the complexity of the compact system generated by the continuous map on compact spaces and prove if there exists a topological semi-conjugate which is from the compact space to the symbolic space, then the compact space has a minimal sub-system which displays Wiggins chaos, Martelli chaos and strictly ergodic.

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1. Introduction

In 1975, T.Y.Li and J.Yorke gave the first mathematical definition of chaos. Since then the definition of chaos has became one of the main research subjects that are concerned by the scientists in all fields. Now chaos has became the field which is very active in the nonlinear science and has very wide prospect. However the scholars in different fields go on their research according to their own understanding of chaos and search for their own application. Consequently there isn't a common definition of chaos which can be used for all fields by now. Surely, it is an important question to understanding the relation among the various definitions of chaos and to unify the definition of chaos.

Throughout this paper, (X, f) will denote a compact system, i.e. (X, d) is a compact metric space and $f : X \to X$ is continuous. The meanings of $O(x)(x \in X)$, $P(f)$, $R(f)$, $\omega(x, f)(x \in X)$ and $A(f)$ are as usual.

If the compact subset $X_0 \subset X$ is invariable to f, i.e. $f(X_0) \subset X_0$, then the compact system $(x_0, f|_{X_0})$ or $f|_{X_0}$ which is generated by the restriction map $f|_{X_0} : X_0 \to X_0$ is called the sub-system of (X, f) or f.

The set $Y \subset X$ is a minimal set of f if for any $x \in Y$, $\omega(x, f) = Y$.

The function $f : X \to X$ is said to be topologically transitive if for each pair of non-empty open sets U, V of X, there is $k \in N$ such that $f^k(U) \cap V \neq \emptyset$.

For every $x_0 \in X$, $O(x_0)$ is unstable with respect to X if there is $r(x_0) > 0$ such that for each $\varepsilon > 0$ there are $y_0 \in X$ with $d(y_0, x_0) \leq \varepsilon$ and $n \in N$ such that $d(f^n(y_0), f^n(x_0)) > r(x_0)$.

The function $f : X \to X$ is said to be sensitive dependence on initial conditions if there is $r_0 > 0$ such that for each $x_0 \in X$ and each $\varepsilon > 0$ there are $y_0 \in X$ with $d(y_0, x_0) \leq \varepsilon$ and $n \in N$ such that $d(f^n(y_0), f^n(x_0)) > r_0$.

Let ß denote the $\sigma-$algebra of Borel sets of X. Call a probability measure μ on $(X, ß)$ invariant under f, if $\mu(f^{-1}(B)) = \mu(B)$ for any $B \in ß$. The set of all the invariant measure of f will be denoted by $M(X, f)$. $\mu \in M(X, f)$ is said to be ergodic if for $B \in ß$, $f^{-1}(B) = B$ implies $\mu(B) = 0$ or 1. If μ is the only member of $M(X, f)$, then it must be ergodic [8]. In this case, we call f uniquely ergodic. A minimal and uniquely ergodic map is simply said to be strictly ergodic.

Let (X, f) and (Y, g) be two compact systems and $f(X) = X$, $g(Y) = Y$. If there exists a continuous surjection $h : X \to Y$ such that $h \circ f = g \circ h$, then f and g are topologically semi-conjugate, h is said to be the topologically semi-conjugate which is from f to g.

Let (X, f) and (Y, g) be two compact systems, $h : X \to Y$ is the topologically semi-conjugate. Define $\bar{h} : M(X, f) \to M(Y, g)$ as $\bar{h}(\mu) = \mu(h^{-1}(A))$, for each

$A \in \text{ß}$. Obviously $\bar{h}$ is continuous and $\bar{h}(M(X,f)) = M(Y,g)$. Call $\bar{h}$ a continuous map induced by h.

Let $S = \{0,1\}$, $\Sigma = \{x = x_1x_2\cdots \mid x_i \in S, i = 1,2,\cdots\}$. Define $\rho : \Sigma\times\Sigma \to R$ as follows: for any $x\,, y\, \in \Sigma$, if $x = x_1x_2\cdots$ and $y = y_1y_2\cdots$, then

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{1}{2^k} & \text{if } x \neq y \text{ , and } k = \min\{n \mid x_n \neq y_n\} - 1 \end{cases}$$

It is not difficult to check that ρ is a metric on Σ.(Σ,ρ) is compact and called the one-sided symbolic space(with two symbols).

Define $\sigma : \Sigma \to \Sigma$ by

$$\sigma(x_1x_2\cdots) = x_2x_3\cdots \quad for \;\; any \;\; x = x_1x_2\cdots \in \Sigma,$$

σ is continuous and called the shift on Σ. Then (Σ,σ) is a compact system.

Call A a tuple (over $S = \{0,1\}$), if it is a finite arrangement of elements in S. If $A = a_1a_2\cdots a_m$, where $a_i \in S$, $1 \le i \le m$, then the length of A is said to be m, denoted by $|A| = m$.

For any tuple $B = b_1\cdots b_n$, $[B] = \{x = x_1x_2\cdots \in \Sigma; x_i = b_i, 1 \le i \le n\}$ is called a cylinder generated by B.

For any $n \ge 1$, let

$$\text{ß}_n = \{\,[b_1b_2\cdots b_n] \mid b_i = 0 \; or \; 1, 1 \le i \le n\}$$

then the collection $\bigcup_{n=1}^{\infty} \text{ß}_n$ is a subalgebra which generates the σ-algebra of Borel subsets of Σ.

Let $h : X \to \Sigma$ be a continuous map, we use $I_{[B]}$ to denote $h^{-1}[B]$ for any $[B] \in \text{ß}_n$.

The main results are stated as follows.

The Main Theorem. *Let $f : X \to X$ and $\sigma : \Sigma \to \Sigma$ are continuous. If f and σ are topologically semi-conjugate, then there exists a minimal sub-system $f|_D$ of f such that*

(P1)$f|_D$ is Wiggins chaotic,

(P2)$f|_D$ is Martelli chaotic,

(P3)$f|_D$ is strictly ergodic.

2. Basic Definitions and Preparations

Definition 2.1. *According to Wiggins, a continuous map $f : X \to X$ is chaotic provided that*

(1) f is topologically transitive in X, and

(2) f has in X sensitive dependence on initial conditions.

Definition 2.2. [1] *According to Martelli, a continuous map $f : X \to X$ is chaotic provided that there exists $x_0 \in X$ such that*

(1) $\omega(x_0) = X$, and

(2) $O(x_0)$ is unstable with respect to X.

Lemma 2.1. *There exists a minimal set $\mathcal{T} \subset \Sigma$ such that*
(1)$\sigma|_{\mathcal{T}}$ is Wiggins chaotic,
(2)$\sigma|_{\mathcal{T}}$ is Martelli chaotic.

For a proof see [2].

Lemma 2.2. *Let (X, d) be the compact metric space and $f : X \to X$ is continuous, then the followings are equivalent*
(1)$x \in A(f)$,
(2)For any $n > 0$, $x \in A(f^n)$,
(3)$x \in \omega(x, f)$ and $\omega(x, f)$ is a minimal set.

For a proof see [3] and [4].

Lemma 2.3. *Let X and Y be compact metric spaces, $f : X \to X$ and $g : Y \to Y$ be continuous maps. If there exists a continuous surjection $h : X \to Y$ such that $g \circ h = h \circ f$, then $h(A(f)) = A(g)$.*

For a proof see [5].

Lemma 2.4. *Let $X, ß, M(X, f)$ be defined as section 1. The following are equivalent.*

(1)There exists $\mu \in M(X, f)$ such that for all $x \in X$, $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to \mu$, where $\delta_y(B)$ is 1 if $y \in B$ and 0 otherwise for any $B \in ß$.
(2)f is uniquely ergodic.

For a proof see [6]

Lemma 2.5. *Let (X, f) and (Y, g) be two compact systems, $\bar{h}$ is the continuous map induced by h. If for each $x \in X$ and an increasing sequence of positive integers $\{n_j\}$,*

$$\frac{1}{n_j}\sum_{i=0}^{n_j-1} \delta_{f^i(x)} \to \mu \in M(X, f),$$

then

$$\bar{h}(\frac{1}{n_j}\sum_{i=0}^{n_j-1} \delta_{f^i(x)}) = \frac{1}{n_j}\sum_{i=1}^{n_j-1} \delta_{g^i(h(x))} \to \bar{h}(\mu) \in M(Y, f).$$

For a proof see [7].

3. PROOF OF THE MAIN THEOREM

By the hypothesis, there exists a continuous surjection $h : X \to \Sigma$ such that $\sigma \circ h = h \circ f$. According to Lemma 2.1, there exists a minimal set $\mathcal{T} = \omega(a, \sigma) \subset A(\sigma)$ such that $\sigma|_{\mathcal{T}}$ is Wiggins chaotic and Martelli chaotic. Since

$a \in \omega(a, \sigma)$ and by Lemma 2.3, there exists $x_0 \in A(f)$ such that $h(x_0) = a$. According to Lemma 2.2, $x_0 \in \omega(x_0, f)$ and $\omega(x_0, f)$ is a minimal set. Let $D = \omega(x_0, f) \subset A(f)$.

We will proof that $h(D) = \omega(a, \sigma)$.

For any $x \in D$, there exists an increasing sequence $\{n_i\}$ of positive integer such that

$$x = \lim_{i\to\infty} f^{n_i}(x_0) \Rightarrow h(x) = \lim_{i\to\infty} h(f^{n_i}(x_0)) = \lim_{i\to\infty} \sigma^{n_i}(a),$$

i.e.$h(x) \in \omega(a, \sigma)$. So $h(D) \subset \omega(a, \sigma)$.

Furthermore,

$$\sigma(h(x)) = \sigma \circ h(\lim_{i\to\infty} f^{n_i}(x_0)) = h \circ f(\lim_{i\to\infty} f^{n_i}(x_0)) = h(\lim_{i\to\infty} f^{n_i+1}(x_0)) \in h(D),$$

i.e.$\sigma(h(D)) \subset h(D)$.

By the minimality of $\omega(a, \sigma)$, we know that $h(D) = \omega(a, \sigma)$.

The proof of (P1):

Firstly, we proof that $f|_D$ is topologically transitive.

For any two open sets $U, V \subset D$ which are not empty. Since $\omega(x_0, f)$ is a minimal set and by it's definition, we know that $D = \omega(x_0, f) = \overline{\{f^n(x_0) | n \geq 0\}}$. So there exists $n \in N$ such that $f^n(x_0) \in U$. Since there is no isolated point in D, there exists $m \in N (m > n)$ such that

$$f^m(x_0) \in V - \{x_0, f(x_0), \cdots, f^n(x_0)\},$$

Let $k = m - n$, then

$$f^m(x_0) = f^{k+n}(x_0) = f^k(f^n(x_0)) \in f^k(U) \cap V,$$

So f is topologically transitive on D.

Secondly, we proof that $f|_D$ has sensitive dependence on initial conditions.

By the definition of cylinder, there exists $N > 0$ such that for any $[B] \in ß_N$, $diam[B] < \frac{1}{2}$. Let

$$t_0 = \min\{d(I_{[B]}, I_{[C]}) \mid [B], [C] \in ß_N \textit{ and } [B] \neq [C]\},$$

where $d(I_{[B]}, I_{[C]}) = \inf\{d(p, q) \mid p \in I_{[B]}, q \in I_{[C]}\}$.

By the properties of h, $d(I_{[B]}, I_{[C]}) > 0$ for any district $[B], [C] \in ß_N$ and so $t_0 > 0$.

Next to prove is that for the above-mentioned t_0, any $\varepsilon > 0$ and $x \in D = \omega(x_0, f)$, there exists $x_1 \in D$ and $k \geq 1$ such that $d(f^k(x_1), f^k(x)) \geq t_0$ when $d(x_1, x) < \varepsilon$.

Since $x \in D$, there exists an increasing sequence $\{n_i\}$ of positive integer such that $x = \lim\limits_{i\to\infty} f^{n_i}(x)$. So for any $\varepsilon > 0$, there exists $i > 0$ such that $d(f^{n_i}(x), x) < \varepsilon$. Let $x_1 = f^{n_i}(x)$, $h(x) = y$, $h(x_1) = y_1$. It is easy to see $y_1 = \sigma^{n_i}(y)$. Because there is no periodic point in $\omega(a, \sigma)$, $\sigma^n(y) \neq \sigma^n(y_1)$ for any $n \geq 1$. So there exists $k \geq 1$ such that

$$\rho(\sigma^k(y_1), \sigma^k(y)) > \frac{1}{2}$$

$\Rightarrow \sigma^k(y_1) \in [B], \sigma^k(y) \in [C]$ for some distinct $[B], [C] \in ß_N$ (since $diam[B] < \frac{1}{2}$)

$\Rightarrow f^k(x_1) \in I_{[B]}, f^k(x) \in I_{[C]}$ and $d(I_{[B]}, I_{[C]}) \geq t_0$

$\Rightarrow d(f^k(x_1), f^k(x)) \geq t_0$

Hence $f|_D$ has sensitive dependence on initial conditions.

Thus we have proved that f is Wiggins chaotic on D.

The proof of (P2):

Firstly, $\omega(x_0, f) = D$, $h(x_0) = a$.

Secondly, by the proof of (1), we know that $f|_D$ has sensitive dependence on initial conditions. And by the definitions, for any $x_0 \in D$, $O(x_0)$ is unstable respect to D.

Hence f is Martelli chaotic on D.

The proof of (P3):

By the hypothesis, there is a continuous surjection h such that $\sigma \circ h = h \circ f$. By lemma 2.1, there is a minimal set $\mathcal{T} \subset \Sigma$ such that $\sigma_{\mathcal{T}}$ is strictly ergodic. So there exists the only member μ of $M(X, f)$ such that

$$\frac{1}{n}\sum_{i=1}^{n-1} \delta_{\sigma^i(x)} \to \mu, \ \ for \ \ each \ \ x \in \mathcal{T}.$$

By lemma 2.5,

$$\bar{h}(\frac{1}{n}\sum_{i=1}^{n-1} \delta_{\sigma^i(x)}) \to \bar{\mu} \in M(D, f).$$

Since $\bar{h}(M(\mathcal{T}, \sigma)) = M(D, f)$, $\bar{h}(\mu)$ is the only member of $M(D, f)$.

So $f|_D$ is uniquely ergodic.

Furthermore, D is minimal, then $f|_D$ is strictly ergodic.

4. COROLLARY OF THE MAIN THEOREM

Let $f : X \to X$ is continuous. If f has a regular shift invariant set(for the definition see [8]), then there has a minimal sub-system $f|_D$ of f such that

(1)$f|_D$ is distributionally chaotic,

(2)$f|_D$ is Li-Yorke chaotic,

(3)$f|_D$ is Wiggins chaotic,

(4)$f|_D$ is Martelli chaotic.

Proof. For the proof of (1) see [8].

The proof of (2):

By the definitions, it is easily to see that if the continuous map f is distributionally chaotic, then it is Li-Yorke chaotic.

The proof of (3) and (4) can be obtained from [8] and the definition of the regular shift invariant set.

$\square$

References

[1] Mario Martelli, Defining Chaos, *Mathematics Magazine*, **71**(1998), no.2, 112-122.

[2] L.D.Wang, Z.Z.Chen and G.G.Liao, The Complexity of a Minimal Sub-Shift on Symbolic Spaces, *Journal of Mathematical Analysis and Applications* in printing.

[3] P.Erdös and A.H.Stone, Some remarks on almost periodic transformations, *Bull. Amer. Math. Soc.***51**(1945), 126-130.

[4] W.H.Gottschalk, Orbit-closure decompositions and almost periodic properties, *ibid.***50**(1944), 915-919.

[5] J.C.Xiong, Set of almost periodic points of a continuous self-map of an interval, *Acta. Math. Sci, New series*, **2**(1986), 73-77.

[6] P.Walters, An introduction to ergodic theory, Springer-Verlag, *New York*(1982)

[7] Denker M, Grillenberger C, Sigmund K, Theorem on Compact Spaces, *Springer Lecture Notes in Math*, **527**(1976).

[8] G.F.Liao and L.D.Wang, Almost periodicity and SS chaotic set, *Chinese Annals of Mathematics Series A***6**(2002), 112-122.

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