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The Free-Surface Flow due to a Jet Against an Infinite Vertical Plate in Presence of Surface Tension

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Abstract

In the present work, we are interested by the study of a bidimensional and potential jet with a free surface. The fluid is assumed to be inviscid, incompressible and irrotational against a vertical plate.With the presence of the surface tension T on the free surface condition, we computed numerically the solutions via a series tuncation method.Our results obtained dependant of a physical parameter Weber number α and the solution exist for any $\alpha \geq 0.1$.

Mathematics Subject Classifications: 76B10, 76C05, 76M45

Keywords: Free surface, Potential flow, Weber number

1 Introduction

In this paper, we consider the steady two dimensional potential flow of a jet past a vertical wall with finite depth. To calculate this flow, we adopt a series-truncation technique similar to the one described by B. Bouderah and H. Mekias ([1] , [2]), H. Mekias and J. M. Vanden-broeck [3], A Gasmi and H. Mekias[4] and E. O. Tuck and J. M. Vanden-Broeck[5] The fluid is assumed to be inviscid, incompressible and irrotationnal. The effects of surface tension T is take into account. As we shall see, the problem is characterized by the Weber number parameter α. It is shown that there is a solution for each value of the Weber number $\alpha > \widetilde{\alpha} = 0.1$.

The problem is formulated in Section 2 and the numerical scheme is described in section 3. The form of the free surface and the calculation of degree of contraction are done in section 4. Some concluding remarks for the problem and the results are discussed and presented in Section 5.

2 Formulation of the problem

We consider the fluid flow due to a jet past a vertical wall of an inviscid, incompressible and irrotationnal where the effects of gravity is neglected and surface tension is take into account.When the flow is symmetrical, one can then study the problem only on the higher half-plane. One takes as locates co-ordinates, wall AB on $x'ox$ and wall BC on the axis $y'oy$. When $x \longrightarrow -\infty$ and $y \longrightarrow +\infty$, one supposes that the flow is uniform speed U and of amplitude H. the flow is highly limited by the free thread of current AC, for X negative. When $x \to 0$, speed tends towards zero. See Figure.1.

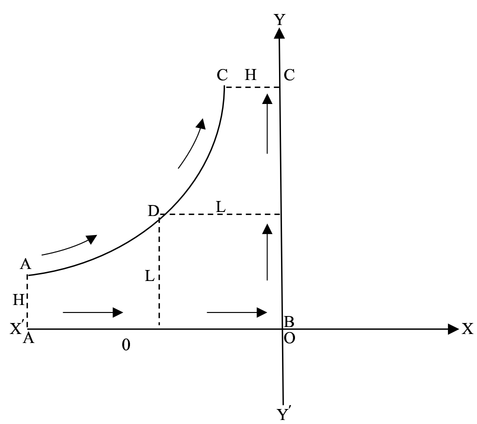

Figure 1: The $z-$plane .

We introduce the complex velocity potential $f = \phi + i\psi$. See Figure 2.We consider the comlplex velocity $\xi(t) = u - iv$ as a function of the variable t in the lower half disk of Figure.3.

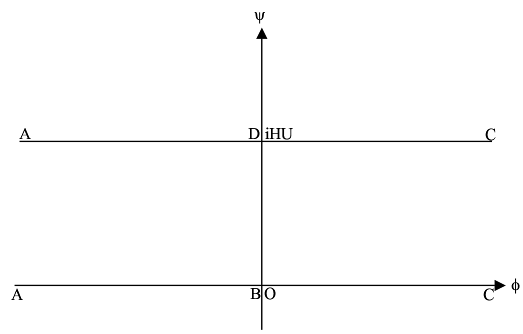

Figure 2: The potential $f - plane$

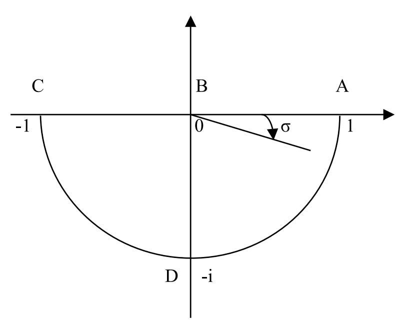

Figure 3: The $t-$plane.

where $u = \phi_x$ and $v = \psi_y$ are the potential function and the streamline function, respectively.The equation of Bernoulli on the free surface is given by

$$\frac{1}{2}q^2 - \frac{1}{\alpha}K = \frac{1}{2}. \tag{1}$$

Where q indicates the module flow speed, α indicates the number of Weber and K indicates the curve. Since $(u - iv)$ is analytical, one defines the function $(\tau - i\theta)$ by the relation:

$$\xi = u - iv = \exp(\tau - i\theta). \tag{2}$$

Where θ is the angle between the flight path vector and the horizontal one. We have: $K = \exp(\tau)\left|\frac{\partial\theta}{\partial\phi}\right|$.

Then, the equation of Bernoulli becomes:

$$\frac{1}{2}q^2 - \frac{1}{\alpha}q\left|\frac{\partial\theta}{\partial\phi}\right| = \frac{1}{2}. \tag{3}$$

In addition, it is known that $\theta(\phi)$ is an increasing function when $-\infty < \phi < +\infty$ on the free surface AC, then the equation of Bernoulli in the f plan is written:

$$\frac{1}{2}\exp(2\tau) - \frac{1}{\alpha}\exp(\tau)\left|\frac{\partial\theta}{\partial\phi}\right| = \frac{1}{2}. \tag{4}$$

With the conditions :

$$\begin{cases} \theta = 0, \psi = 0, \phi < 0. & sur\ AB \\ \theta = \frac{\pi}{2}, \psi = 0, \phi > 0. & sur\ BC \end{cases} \tag{5}$$

The mathematical problem is to determine the function $(\tau - i\theta)$ which is analitycal in the band $0 < \psi < 1$ and checks the conditions (4) and (5).

3 Numerical procedure

To solve this problem numerically, the technique of truncation of the series is applied.

One transforms the field occupied by the fluid in the plan f .See Figure.2, in a half of disc unit in the plan of the variable t. See Figure.3. By the transformation:

$$f = \frac{2}{\pi}\log\left(\frac{1-t}{1+t}\right). \tag{6}$$

The points A, B, D and C in the plan f are transformed respectively at $t = 1$, $t = 0$, $t = -i$ et $t = -1$.The free surface AC is transformed into a circumference of circle See Figure.3. The points of the free surface in the plan t are given by the relation:

:

$$t = |t|\exp(i\sigma) = \exp(i\sigma). \quad \text{such as } -\pi < \sigma < 0 \tag{7}$$

And in the plan f by the relation:

$$f = \phi. \qquad \text{such as} \quad -\infty < \phi < +\infty \tag{8}$$

One as follows writes the equation of Bernoulli in the $t-$plan as follows:

$$\exp(2\tau) + \frac{\pi}{\alpha}\sin(\sigma)\exp(\tau)\frac{\partial\theta}{\partial\sigma} = 1. \tag{9}$$

At the point of stagnation B, we have a flow in an angle equal to $\frac{\pi}{2}$, therefore the flow is characterized by the following potential function:

$$f \sim \frac{a}{2} z^2. \tag{10}$$

The development of the complexe potential function f in the vicinity of the point $t = 0$, is given by:

$$f \sim t + O\left(t^2\right). \tag{11}$$

3.1 Formulation of the series

Let us define the function $\Omega(t)$ as follows:

$$\xi(t) = g(t)\,\Omega(t).$$

Where $g(t)$ contains the singularities and the zeros. The function $\Omega(t)$ is analytical inside the disc unit $|t| < 1$, then it develops in series. And by using the boundary conditions (5) and the development (11), then:

$$u - iv = \sqrt{t}\exp\left(\sum_{k=1}^{+\infty} a_k t^{2k}\right). \tag{12}$$

Where the a_k are real constants to determine. The equation (12) checks all the boundary conditions except the condition of Bernoulli. The a_k are determined so that the equation of Bernoulli is checked.

According to (7) and (12), we have:

$$\exp(\tau - i\theta) = \exp\left(\sum_{k=0}^{+\infty} a_k \cos(2k\sigma) + i\left(\frac{\sigma}{2} + \sum_{k=0}^{+\infty} a_k \sin(2k\sigma)\right)\right).$$

Then

$$\tau(\sigma) = \sum_{k=0}^{+\infty} a_k \cos(2k\sigma). \tag{13}$$

$$\theta(\sigma) = -\left(\frac{\sigma}{2} + \sum_{k=0}^{+\infty} a_k \sin(2k\sigma)\right). \tag{14}$$

To determine the coefficients a_k, one makes a truncation of the series after N terms. Thus one introduces the discretization of the interval $\left[-\frac{\pi}{2}, 0\right]$ in $N+1$ points.

$$\sigma(I) = -\frac{\pi}{2(N+1)}\left(I - \frac{1}{2}\right). \qquad I = 1, 2, ..., N \tag{15}$$

In substituent (13) and (14) in the equation of Bernoulli, one obtains the system of $N+1$ equations in $N+1$ inknowns:

$$\begin{aligned} &\exp\left(2\sum_{k=0}^{N} a_k \cos(2k\sigma(I))\right) - \frac{\pi}{\alpha}\exp\left(\sum_{k=0}^{N} a_k \cos(2k\sigma(I))\right) \\ &\times \sin(\sigma(I))\left(\frac{1}{2} + \sum_{k=0}^{N} 2k a_k \cos(2k\sigma(I))\right) = 1. \end{aligned} \tag{16}$$

The resulting $N+1$ equations are solved by using Newton method.

4 Form of the free surface and value of degree of contraction

4.1 Calculation of degree of contraction

We have:

$$\xi = C\frac{df}{dz}.$$

After calculations, one obtains the system of equations according to:

$$\left\{ \begin{aligned} &\frac{\partial x}{\partial \sigma}(\sigma(I)) = \frac{2C}{\pi \sin(\sigma(I))}\exp\left(-\sum_{k=0}^{N} a_k \cos(2k\sigma(I))\right) \\ &\qquad \times \cos\left(-\frac{\sigma(I)}{2} - \sum_{k=0}^{N} a_k \sin(2k\sigma(I))\right). \\ &\frac{\partial y}{\partial \sigma}(\sigma(I)) = \frac{2C}{\pi \sin(\sigma(I))}\exp\left(-\sum_{k=0}^{N} a_k \cos(2k\sigma(I))\right) \\ &\qquad \times \sin\left(-\frac{\sigma(I)}{2} - \sum_{k=0}^{N} a_k \sin(2k\sigma(I))\right). \end{aligned} \right. \tag{17}$$

By the derivative

$$\frac{\partial y}{\partial \sigma}\left(\sigma\left(I\right)\right) \approx \frac{y\left(\sigma\left(I\right)\right)-y\left(\sigma\left(I-1\right)\right)}{h}. \qquad I=1,N+1$$

Where

$$h=\frac{\pi}{2\left(N+1\right)}.$$

and where $y\left(N+1\right)=1$ $(at\ A)$ and $y\left(1\right)=\frac{L}{H}=\frac{1}{C}$ $(at\ D)$, so:

$$C=1/\left(\begin{array}{c} 1-h\sum_{I=0}^{N}\frac{2}{\pi\sin\left(\sigma\left(I\right)\right)}\exp\left(-\sum_{k=0}^{N}a_k\cos\left(2k\sigma\left(I\right)\right)\right) \\ \times\sin\left(-\frac{\sigma\left(I\right)}{2}-\sum_{k=0}^{N}a_k\sin\left(2k\sigma\left(I\right)\right)\right)\end{array}\right). \tag{18}$$

4.2 Form of the free surface AC

The form of the free surface is found by integrating the system (17) after replacing C given by the relation (18) into (17). The Euler methods was sufficient to solve (17) numerically where we find the values of x and y for each values $\sigma\left(I\right)$, for $I=1,N+1$, with the initial conditions $y\left(0\right)=1$ and $y\left(-\frac{\pi}{2}\right)=\frac{1}{C}$.

5 Discussion and presentation of results

5.1 Solution without surface tension

When the number of Weber tends towards the infinite one, the effect of surface tension is neglected and the equation (9) becomes:

$$\exp\left(2\tau\right)=u^2+v^2=1. \qquad \textit{on the free surface}$$

And the system (16) is reduced in:

$$\exp\left(2\sum_{k=1}^{N}a_k\cos\left(2k\sigma\left(I\right)\right)\right)-1=0. \qquad I=1,N+1 \tag{19}$$

Where $\sigma\left(I\right)$ is given by the relation (15).

In this case, one finds the same form of free surface that have can find analytically. See Figure.4.

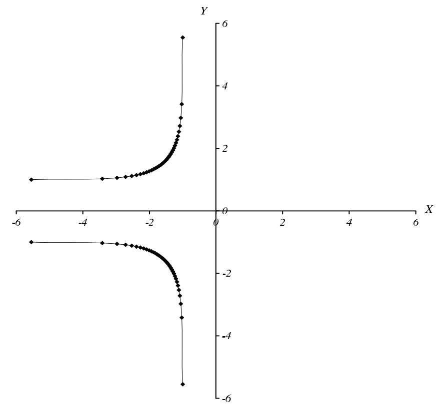

Figure 4: Shape of the free surface flow for the Weber number $\alpha = 10^{+8}$, compared to the exact solution.

5.2 Surface tension effect $(T \neq 0)$.

We use the numerically procedure described above to compute solutions for various values of Weber number α. The a_k coefficients decrease very rapidly as N increases and α decreases. Table.1 shows some of the coefficients of the series (12) for different values of Weber number α.

α	0.1	1	10	50	150	$\rightarrow \infty$
C	0.139	0.44	0.6	0.63	0.638	0.64

Table1. Some Values a_k of coefficients of the serie (12) for differents values of the Weber number α

It is also noted that the series $\sum\limits_{k=1}^{+\infty} a_k t^{2k}$ is absolutely convergent because we have :$\sum\limits_{k=1}^{+\infty} \left|a_k t^{2k}\right| \leq \sum\limits_{k=1}^{+\infty} |a_k| \leq \sum\limits_{k=1}^{+\infty} \left(\frac{4}{5}\right)^k$ .And one presents the comparison of the general term of the series (12) with $\sum\limits_{k=1}^{+\infty} \left(\dfrac{4}{5}\right)^k$ for $\alpha = 50$ and $\alpha = 1000$.See Table.2

	$\alpha = 50$	$\alpha = 1000$	
k	a_k	a_k	$(4/5)^k$
1	$0.6474 10^{-2}$	$0.3325 10^{-3}$	0.80
10	$0.6406 10^{-4}$	$0.3081 10^{-5}$	0.11
20	$0.1414 10^{-4}$	$0.6725 10^{-6}$	$1.15 10^{-2}$
50	$0.1516 10^{-5}$	$0.7219 10^{-7}$	$1.43 10^{-5}$
70	$0.5625 10^{-7}$	$0.2827 10^{-8}$	$0.65 10^{-7}$

Table2. Comparison of the serie (12) versus the serie $\sum(\frac{4}{5})^n$,for the constant $\alpha = 50$ and $\alpha = 1000$

And it is noted that as α decreases the coefficient of contraction is decreases when the surface tension increase. We remark that the Figure.5 confirmed these results. Its observed that there exist a critical value $\widetilde{\alpha} = 0.1$ for all $\alpha > \widetilde{\alpha}$, there exist a solution for the problem. Finally, let us mention that the family of solutions described in this paper is unique solution .we deduce for $\alpha < \widetilde{\alpha}$, the numerical scheme ceases to converge. One expresses the variation of degree of contraction C according to the Weber number α. See Table.3.

α	1	10	50	10^3
a_1	-0.3664	-0.477410^{-1}	-0.497510^{-2}	-0.499210^{-3}
a_{10}	0.460610^{2}	0.364810^{-3}	0.640610^{-4}	0.308110^{-5}
a_{30}	0.572910^{-3}	0.349210^{-4}	0.578310^{-5}	0.273810^{-6}
a_{50}	0.149810^{-3}	0.904410^{-5}	0.151610^{-6}	0.721910^{-7}

Table3. Variation of the contraction coefficients C versus the Weber number α.

It is noticed that the coefficient of contraction C decrease when α decrease See Figure.5. We represent the form of the free surface for various values of the Weber number α See Figure.6. According to Figure.6.

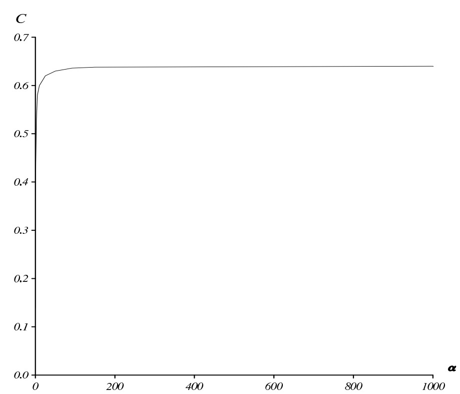

Figure 5: Numerical values of of the contraction coefficients C versus the Weber number α

We concludes that for each value of the Weber number α there is one and only one solution of the problem considered. Most of the results presented here are obtained with $N = 40$.

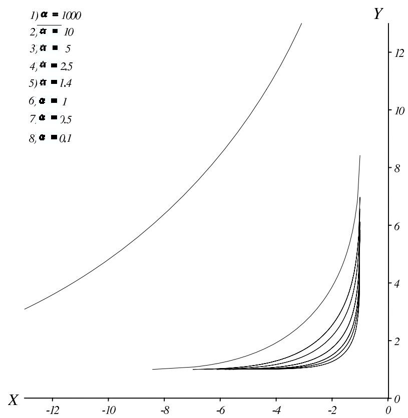

Figure 6: Form of the free surface for different values of the importance measure of the Weber number α.

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