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An Exact Solution for the Deflection of a Clamped Rectangular Plate under Uniform Load

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Abstract

Rectangular plates under uniform load, $x = \pm a$, $y = \pm b$, are considered. An exact solution is presented in which each term of the series is trigonometric and hyperbolic, and identically satisfies the boundary conditions on all four edges. The solution has three terms in which the first term corresponds to the case of a strip and the other two terms denote the effects of the edges. The method used to obtain the solution is simple and straightforward. In order to illustrate the method the numerical values of the deflections are calculated and compared with those of the previous papers. It is found that there is a reasonable agreement between the results of them and those of this paper.

Mathematics Subject Classification: 74K20, 42A32, 35L20

Keywords: Rectangular plate; clamped plate; deflection; uniform load

1 Introduction

Thin plates are common structural elements employed in many engineering applications and are subject to a wide variety of excitations, including acoustic excitations. The problem of a rectangular plate clamped on four sides and carrying a uniformly distributed load is of great importance and many papers have been devoted to this subject. Many authors have calculated the deflections of uniformly loaded rectangular plates with clamped edges using different methods. Some of them are approximate methods. The accuracy of the analytic solutions compiled and developed in the literature varies: those for simply supported plates are exact, others approximate. For fixed rectangular thin plates no accurate results appear to be available. Approximate solutions were also suggested, but these resulted in a notable loss of accuracy [1]. Many authors have calculated the deflections of rectangular plates with various supports using different methods [2-9]. Some of them are approximate methods. Two main methods of approach have been used for obtaining the solution of the maximum deflection for fixed thin rectangular plates under uniform load; these are the double cosine series and the superposition method as a generalization of Hencky's solution [10].

The problem of the uniformly loaded rectangular plate with fixed at all edges has been solved by Hencky and independently by Boobnoff. Boobnoff made exact calculations for several aspect ratios of the plate while Hencky made refined calculations only for the case of a square plat [3]. Hutchinson has used the solution from which was presented in [11] and tabulated deflections for uniformly loaded rectangular plates. Obtaining the numerical values of deflections for a rectangular plate may be difficult. A single cosine series for rectangular fixed plates have been presented in [12, 13].

This paper analyses the deflections of a rectangular fixed thin plates under uniformly distributed loads. In this paper, a comprehensive method is presented for the numerical solution of the fixed rectangular plate problem under uniformly distributed loads and boundary conditions. A solution of the governing equation in terms of trigonometric and hyperbolic function is given. The method is based upon the classical cosine series expansion and found to be easier and more effective. By using this method, a rectangular plate having four edges fixed and subjected to uniformly distributed loading has been modeled. For an isotropic plate with a Poisson coefficient $\nu = 0.3$, tables giving the values of the deflections, each with a different value of the aspect ratio b/a are presented. A numerical method for dealing uniformly normal loaded rectangular plates is compared to the similar numerical techniques used in [5,9] and the results have been compared with those in the literature. The results show reasonable agreement with the other available results, but with a much simpler and a more practical approach.

2 Deflections of laterally loaded rectangular plates

According to the classic theory of plate bending, a small deflection is defined as small compared with the plate thickness. The governing differential equation for isotropic homogenous thin plates is

$$\nabla^4 \left(D \cdot W(x,y) \right) = p(x,y) \tag{2.1}$$

where W is the small deflections of the plate midsurface, $\nabla^4 \equiv \nabla^2 \left(\nabla^2 \right)$ denotes the bi-harmonic operator and ∇^2 is Laplacian operator, p is intensity of lateral pressure on the plate and D is the constant flexural rigidity terms of the material properties, Young's modulus of the material E and Poisson's ratio ν (taken equal to 0.3), and the plate thickness, h,

$$D = \frac{Eh^3}{12(1-\nu)} \tag{2.2}$$

The problem of a rectangular plate fixed at four sides with a uniform load is considered. Taking the origin of coordinates at the center of the plate and x- and y-axes parallel, to the side a and b of the plate, and it is supposed throughout the paper that $a \geq b$. The boundary conditions for (2.1) are obtained by requiring the solution to satisfy two prescribed conditions at each boundary point [14]. The boundary conditions are combinations of

$$W = 0 \quad \text{and} \quad \frac{\partial W}{\partial n} = 0 \tag{2.3}$$

along all edges. When a boundary point is fixed, W and $\partial W / \partial n$ are taken to be zero there. A direct series solution was not obtained for the boundary condition of fixed edges.

The solution to the homogenous biharmonic equation (2.1), the known solution for the simply supported plate with the uniform loading giving the deflection function for the strip case is combined with that for a solution of deflection function which shows the effects due to the edges $W = W_s + W_e$. A method is for efficiently determining a very large number of terms in the series. The series is given by

$$D \cdot W(x,y) = \frac{pb^4}{24} \left[\left(1 - \frac{y^2}{b^2} \right)^2 + \sum_{m=1,3\ldots}^{\infty} \frac{\cos \frac{m\pi y}{2b}}{\cosh \alpha_m} \left(E_m \cosh \frac{m\pi x}{2b} + G_m \frac{x}{a} \sinh \frac{m\pi x}{2b} \right) \right.$$

$$\left. + \sum_{m=1,3\ldots}^{\infty} H_m \frac{\cos \frac{m\pi x}{2a}}{b \cosh \alpha_m} \left(y \sinh \frac{m\pi y}{2a} - b \tanh \alpha_m \cosh \frac{m\pi y}{2a} \right) \right], \tag{2.4}$$

where $\alpha = m\pi a / 2b$ is deflection coefficient. It is seen that $W(x,y)$ given by Eq.(2.4) vanishes in the middle of the plate and satisfies Eq.(1). The first term gives the deflection function for the strip case in $-b \leq y \leq b$. The others show the effects due to the edges. The coefficients E_m, G_m and H_m in Eq.(2.4) must be chosen that the boundary conditions (2.3) are satisfied. These coefficients are to be determined from the condition that the slope at the boundaries is zero. Substuting Eq.(2.4) in Eq.(2.3), the following system of linear equations for determining the coefficients E_m, G_m and H_m are obtained.

$$\sum_{m=1,3,...} \frac{\cos\frac{m\pi y}{2b}}{\cosh\alpha_m}\left(E_m \cosh\alpha_m + G_m \sinh\alpha_m\right) = \left(\frac{y^2}{b^2}-1\right)^2, \qquad (2.5)$$

$$\sum_{m=1,3,...}\left[E_m\alpha_m \tanh\alpha_m + G_m \frac{\alpha_m + \tanh\alpha_m}{a}\right]\cos\frac{m\pi y}{2b} - \sum_{m=1,3...} H_m \frac{(-1)^m m\pi}{2b^2\cosh\beta_m}$$

$$\left(y\sinh\frac{m\pi y}{2a} - b\tanh\beta_m \cosh\frac{m\pi y}{2a}\right) = 0, \qquad (2.6)$$

$$\sum_{m=1,3...}\frac{(-1)^m m\pi}{2a\cosh\alpha_m}\left(E_m \cosh\frac{m\pi x}{2b} + G_m\frac{x}{a}\sinh\frac{m\pi x}{2b}\right)$$

$$-\sum_{m=1,3...} H_m \frac{\cos\frac{m\pi x}{2a}}{b}\left(\tanh\beta_m + \frac{\beta_m}{\cosh^2\beta_m}\right) = 0, \qquad (2.7)$$

where $\alpha = m\pi a/2b$ is deflection coefficient and $\beta = m\pi a/2b$ is moment coefficient. Thus, taking $W = 0$ we obtain

$$E_m + G_m \tanh\alpha_m = (-1)^m\left[\frac{128}{m^3\pi^3} - \frac{1536}{\pi^5 m^5}\right] \qquad (2.8)$$

where G_m is found by expressing the slope at $y = \pm b$ in a cosine series of the form

$$E_m \frac{m^2\pi}{2b}\tanh\alpha_m + G_m \frac{b\tanh\alpha_m}{a} + G_m \frac{m\pi}{2}$$

$$+\sum_{s=1,3,..} H_s \frac{(-1)^{m+s} s^3\pi^3}{2b^2a^2\left(\frac{m^2\pi^2}{4b^2} + \frac{m^2\pi^2}{4a^2}\right)^2} = 0, \qquad (2.9)$$

Similarly, for zero slope at $x = \mp a$ we find H_m is obtained in a series of the form

$$-\sum_{s=1,3...} 2(-1)^{\left(\frac{s-1}{2}\right)+\left(\frac{m-1}{2}\right)} \left\{ E_s \frac{ms}{ab\left(\frac{s^2}{b^2}+\frac{m^2}{a^2}\right)} + G_s \left[\frac{ms \tanh \alpha_s}{ab\left(\frac{s^2}{b^2}+\frac{m^2}{a^2}\right)} - \frac{s^2 m\pi}{a^3 b\left(\frac{s^2}{b^2}+\frac{m^2}{a^2}\right)^2} \right] \right\}$$

$$+H_m \frac{a}{b}\left(\tanh \alpha_m + \frac{\beta_m}{\cosh^2 \beta_m} \right) = 0 \qquad (2.10)$$

After the usual procedure of determining the coefficients of E_m, G_m and H_m a Fourier series, the equations :

$$E_m + G_m \tanh \alpha_m = (-1)^m \left[\frac{128}{m^3\pi^3} - \frac{1536}{\pi^5 m^5} \right], \qquad (2.11)$$

$$E_m\alpha_m \tanh \alpha_m + G_m \left[\frac{b}{a} \tanh \alpha_m + \frac{\alpha_m a}{b} \right]$$

$$+\frac{32\alpha_m^2}{\pi^3} \sum_{s=1,3...} H_s \frac{(-1)^{\left(\frac{m-1}{2}\right)+\left(\frac{s-1}{2}\right)}}{\left(\frac{m^3}{s^2}+\frac{b^2}{ma^2}\right)^2} = 0, \qquad (2.12)$$

$$\sum_{s=1,3,..} E_m \frac{b^2 m}{a^2 s} \frac{2(-1)^{\left(\frac{m-1}{2}\right)+\left(\frac{s-1}{2}\right)}}{1+\frac{m^2b^2}{s^2a^2}} - \sum_{s=1,3...} G_m \frac{b^2}{a^2} 2(-1)^{\left(\frac{m-1}{2}\right)+\left(\frac{s-1}{2}\right)} \left[\frac{m/s}{1+\frac{m^2b^2}{s^2a^2}} \tanh \beta_s \right.$$

$$\left. \left. - \frac{8\alpha_m}{s\pi^2\left(1+\frac{m^2}{s^2}\right)^2} \right] \right\} - H_m \left[\tanh \alpha_m + \frac{\alpha_m}{\cosh^2 \alpha_m} \right]. \qquad (2.13)$$

where $\alpha = m\pi a/2b$ is deflection coefficient and $\beta = m\pi a/2b$ is moment coefficient.

The numerical calculation of the coefficients completes the solution of Eq.(2.1). Values of the deflection at the center point may be determined from the known relations in the theory of plates. These equations are solved numerically by neglecting all terms higher than a given order, which results in a system of simultaneous equations. The formula is valid for most commonly used metal materials that have Poission's ratios around 0.3. In fact, the Poisson's ratio has a very limited effect on the displacement and the above calculation normally gives a very good approximation for most practical cases.

The increasing usage of flat plates in the construction of panels in such steel structure as bridges and decks has called attention to the need for more information on the behavior of rectangular plates with uniformly distributed loads. The center deflection of rectangular plates with fixed at four edges and subject to the action of uniformly distributed loads is an important problem that has received considerable attention because of its technical importance. The value of the deflection at the point $x = 0$, $y = 0$ is very important and it is given in the following form

$$W(0,0) = \frac{pb^4}{24D}\left[1 + \sum_{m=1,3...} \frac{E_m}{\cosh \alpha_m} - \sum_{m=1,3...} \frac{H_m \tanh \beta_m}{\cosh \beta_m}\right] \qquad (2.14)$$

From the boundary condition (2.3), $G_m = 0$ for the center deflection of the rectangular plate. It is clearly seen that there is no contribution of the coefficient G_m on the deflection. The series in this expression converges very rapidly, and sufficient accuracy is obtained by taking only the first term. The expansion of the center deflection function of the rectangular plate is in a series form and the second term of the series is negligible and that by taking only the first term the formula for deflection is obtained. When the aspect ratio goes to infinity, the center deflection tends to $0.00260417qb^4/D$. This result is in good agreement with [5,9].

3 Numerical example and results

The problem to be considered is a uniformly loaded rectangular plate with its edges fixed as in Figure 1. This problem was solved using a variety of numerical methods and compared to the exact solutions.

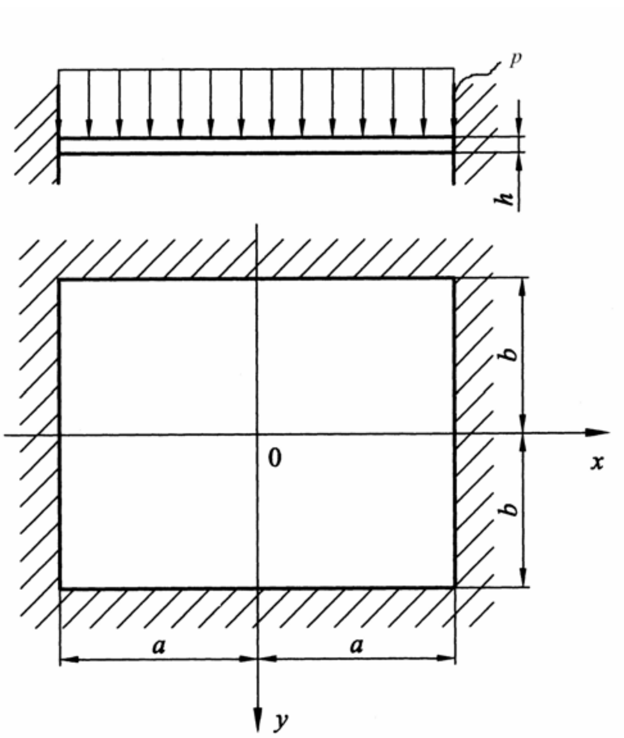

Figure 1. A rectangular plate uniformly loaded and fixed at four edges.

The numerical values of the deflections can be calculated if the coefficients E_m, G_m and H_m are known for all values of b/a and can be determined by Eqs.(2.11), (2.12) and (2.13). The numerical values of coefficients are obtained by a set of linear equations as given in [3]. Accuracy of the numerical values of them depends on the number of linear equations. Computations of the coefficients E_m, G_m and H_m were also made for b/a equal to 1.0, 1.2, 1.4, 1.6, 1.8, 2.0 and ∞. Using the values of the coefficients E_m, G_m and H_m, we proceed to find the maximum deflection in the plate under considerations.

The deflection at the center is then found from Eq.(2.14) and may be reduced to the form

$$W_{\substack{x=0\\y=0}} = \alpha \frac{pb^4}{D}, \tag{2.15}$$

In the calculation for the coefficient α, the value of Poisson's ratio has been taken as $\nu = 0.3$. From Eq.(2.15), Table 1 of coefficients for various values of b/a of the sides of the plate was compiled. This table aggress essentially with that compiled by Henkcy based on his method of solving Eq.(2.1).

Table 1. Deflections for clamped rectangular plates with uniform load ($\nu = 0.3\ ,\ b \geq a$)

b/a	$\dfrac{w(0,0)}{pb^4/D}$	Evans pb^4/D	Taylor & Govindjee pb^4/D
1.0	0.00126725	0.00126	0.00126532
1.2	0.00172833	0.00172	0.00172487
1.4	0.00207217	0.00207	0.00206814
1.6	0.00230399	0.00230	0.00229997
1.8	0.00244989	0.00245	0.00244616
2.0	0.00253625	0.00254	0.00253297
∞	0.00260417	0.00260	0.00260417

It is observed that the results of the presented study are in good agreement with others. The results show that about twice the accuracy can be obtained. In comparison to the other methods, presented method has the advantage that the series representation satisfies the partial differential equation exactly. However, it is clearly seen that the method in this paper is simple and straightforward. The results presented in this paper can be compared with those of the previous papers.

4 Conclusions

A method for treating uniformly loading on thin rectangular plates has been presented. By using this method, a rectangular plate having four edges fixed and subjected to uniformly distributed loading has been modeled, the results of calculations for maximum deflection for several ratios of the sides of the plate have been computed, and the results have been compared with those in the literature. The result shows close agreement with other analysis methods. However, it is clearly seen that the method in this paper is simple and straightforward.

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