Applied Mathematical Sciences, Vol. 1, 2007, no. 45, 2231 - 2246

Numerical Solution of Fuzzy Differential Equations

Javad Shokri

Department of Mathematics
Urmia University
P.O. Box 165, Urmia, Iran
j.shokri@mail.urmia.ac.ir

Abstract

We present solution of first order fuzzy differential equations (FDEs) by modified Euler's method and it's iterative solution. The method is discussed in detail and this is followed by a complete error analysis. The algorithm is illustrated by solving some numerical tests.

Mathematics Subject Classification: 34A12; 65L05

Keywords: Fuzzy differential equations, Iterative Modified Euler's method, Ordinary differential equations

1 Introduction

The study of fuzzy differential equations (FDEs) forms a suitable setting for model dynamical systems in which uncertainties or vagueness pervades. First order linear and nonlinear FDEs are one of the simplest FDEs which appear in many applications.

In the recent years, the topics of FDEs have been investigated extensively. The concept of a fuzzy derivative was first introduced by S. L. Chang , L. A. Zadeh in [4]. It was followed up by D. Dubois, H. Prade in[5]. Other methods in this subject have been studied by R. Goetschel, W. Voxman in[7] and by M. L. Puri, D. A. Ralescu in [11]. O. Kaleva and S. Seikkala in [8, 12] studied simultaneously the fuzzy differential equations and initial values problem. O. Kaleva in [9] has solved FDEs by using the standard Euler method, our main aim in this paper is study of FDEs by iterative solution of Modified Euler's method.

The organized of the paper is as follows. In the first three sections below, we recall some concepts and introductory materials to deal with the fuzzy initial value problem. In section five, we present modified Euler's method and it's iterative solution for solving of Fuzzy differential equations and the corresponding convergence theorems are presented. The proposed algorithm is illustrated by some examples in section 6 and conclusion is in section 7.

2 Preliminary Notes

A triangular fuzzy number u is defined by three real number $a < b < c$ where the base of the triangle is the interval $[a, c]$ and its vertex is at $x = b$. We specify u as $(a/b/c)$. The membership function for the triangular fuzzy number $u = (a/b/c)$ id defined as the following:

$$u(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b \\ \frac{x-c}{b-c}, & b \le x \le c \end{cases} \tag{1}$$

we will have : (1) $u > 0$ if $a > 0$; (2) $u \ge 0$ if $a \ge 0$; (3) $u < 0$ if $c < 0$; and (4) $u \le 0$ if $c \le 0$.

Let denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy set subsets of the real axis (i.e. $u : \mathbb{R} \to [0, 1]$) satisfying the following properties:

(i) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$;

(ii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is convex fuzzy set (i.e. $u(tx + (1 - t)y) \ge \min\{u(x), u(y)\}$, $\forall t \in [0, 1]$, $x, y \in \mathbb{R}$);

(iii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is upper semicontinuous on $\mathbb{R}$;

(iv) $\{x \in \mathbb{R}; u(x) > 0\}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers (see e.g. [9]). Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ is understood as $\mathbb{R} = \{\chi_{\{x\}}; x \text{ is usual real number}\}$. We define the r-level set,

$$\begin{aligned} [u]_r &= \{x \in \mathbb{R}; u(x) \ge r\}, \quad 0 < r \le 1; \\ [u]_0 &= \{x \in \mathbb{R}; u(x) > 0\} \quad \textit{is compact}. \end{aligned} \tag{2}$$

Then it is well-known that for each $r \in [0, 1]$, $[u]_r$ is bounded closed interval. We denote by $[u]_r = [u_1(r), u_2(r)]$. It is clear that the following statements are true.

- $u_1(r)$ is a bounded left continuous non decreasing function over $[0, 1]$,
- $u_2(r)$is a bounded right continuous non increasing function over $[0, 1]$,

- $u_1(r) \leq u_2(r)$ for all $r \in (0,1]$,

for more details see [2],[3].
Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$, $D(u,\, v) = \sup_{r\in[0,1]} \max\{|u_1(r) - v_1(r)|,\, |u_2(r) - v_2(r)|\}$, be *Hausdorff distance* between fuzzy numbers, where $[u]_r = [u_1(r), u_2(r)]$, $[v]_r = [v_1(r), v_2(r)]$. The following properties are well-known (see e.g. [11]):

$D(u + w, v + w) = D(u, v),\ \forall u,\ v,\ w \in \mathbb{R}_{\mathcal{F}}$,
$D(k.u, k.v) = |k| D(u, v),\ \forall k \in \mathbb{R},\ u, v \in \mathbb{R}_{\mathcal{F}}$,
$D(u + v, w + e) \leq D(u, w) + D(v, e),\ \forall u,\ v,\ w,\ e \in \mathbb{R}_{\mathcal{F}}$
and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

3 Fuzzy Initial Value Problem

Here, we introduce fuzzy initial value problem in the following form:

$$\begin{cases} y'(t) = f(t, y(t))\,, \quad t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{3}$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number. Therefore we have a fuzzy cauchy problem.
We denote the fuzzy function y by $y = [y_1, y_2]$. It means that the r-level set of $y(t)$ for $t \in [t_0, T]$ is

$$[y(t_0)]_r = [y_1(t_0; r), y_2(t_0; r)],\ \ [y(t)]_r = [y_1(t; r), y_2(t; r)] \quad r \in (0, 1].$$

By using the extension principle of Zadeh, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) | s = f(t, \tau)\}, \quad s \in \mathbb{R} \tag{4}$$

so $f(t, y(t))$ is a fuzzy number. From this it follows that

$$[f(t, y(t))]_r = [f_1(t, y(t); r), f_2(t, y(t); r)],\ r \in (0, 1] \tag{5}$$

where

$$\begin{aligned} f_1(t, y(t); r) &= \min\{f(t, u) | u \in [y_1(t; r), y_2(t; r)]\}, \\ f_2(t, y(t); r) &= \max\{f(t, u) | u \in [y_1(t; r), y_2(t; r)]\}. \end{aligned} \tag{6}$$

We define

$$\begin{aligned} f_1(t, y(t); r) &= F[t, y_1(t; r), y_2(t; r)], \\ f_2(t, y(t); r) &= G[t, y_1(t; r), y_2(t; r)]. \end{aligned} \tag{7}$$

Definition 3.1 *A function $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is called a fuzzy function. If for arbitrary fixed $t_0 \in \mathbb{R}$ and $\epsilon > 0$, a $\delta > 0$ such that*

$$|t - t_0| < \delta \Rightarrow D[f(t), f(t_0)] < \epsilon$$

exist, is said to be continuous.

Throughout this work we also consider fuzzy functions which are continuous in metric D. Then the continuity of $f(t, y(t); r)$ guarantees the existence of the Definitionnite of $f(t, y(t); r)$ for $t \in [t_0, T]$ and $r \in [0, 1]$, [6]. Therefore the functions G and F can be definite too.

4 Modified Euler's Method

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t))\,, \quad t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{8}$$

It is known that, the sufficient conditions for the existence of a unique solution to (8) are that f to be continuous function satisfying the Lipschitz condition of the following form:

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad L > 0.$$

We replace the interval $[t_0, T]$ by a set of discrete equally spaced grid points

$$t_0 < t_1 < t_2 < \ldots < t_N = T,\ h = \frac{T - t_0}{N},\ t_i = t_0 + ih,\ i = 0, 1, \ldots, N.$$

to obtain the Euler method for the system (8), we apply Trapezoidal numerical integration method. Integrate the differential equation $y'(t) = f(t, y(t))$ over $[t_n, t_{n+1}]$ to obtain

$$\int_{t_n}^{t_{n+1}} y'(t)\, dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt.$$

Therefore

$$\begin{aligned} y(t_{n+1}) = \quad & y(t_n) + \tfrac{h}{2}\Big[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\Big] \\ & -\tfrac{h^3}{12} f^{(2)}(\xi_1, y(\xi_1)) \end{aligned} \tag{9}$$

for some $t_n \leq \xi_1 \leq t_{n+1}$. Equation (9) is an implicit equation in term of $y(t_{n+1})$. To avoid of solving such implicit equation we will substitute $y(t_{n+1})$

by $y(t_n)+hf(t_n,y(t_n))+\frac{h^2}{2}f'(\xi_2,y(\xi_2))$ in right hand of (9), where $\xi_2\in[t_n,t_{n+1}]$. Therefore,

$$\begin{aligned}&y(t_{n+1})\\&=y(t_n)+\tfrac{h}{2}f(t_n,y(t_n))+\tfrac{h}{2}f\left(t_{n+1},y(t_n)+hf(t_n,y(t_n))+\tfrac{h^2}{2}f'(\xi_2,y(\xi_2))\right)\\&-\tfrac{h^3}{12}f^{(2)}(\xi_1,y(\xi_1)),\ t_n\le\xi_1\le t_{n+1},\ t_n\le\xi_2\le t_{n+1}.\end{aligned}\tag{10}$$

But we have

$$\begin{aligned}&f\left(t_{n+1},y(t_n)+hf(t_n,y(t_n))+\tfrac{h^2}{2}f'(\xi_2,y(\xi_2))\right)\\&=f\Big(t_{n+1},y(t_n)+hf(t_n,y(t_n))\Big)+\tfrac{h^2}{2}f'(\xi_2,y(\xi_2))f_y(t_{n+1},\xi_3)\end{aligned}\tag{11}$$

where ξ_3 is in between $y(t_n)+hf(t_n,y(t_n))$ and $y(t_n)+hf(t_n,y(t_n))+\frac{h^2}{2}f'(\xi_2,y(\xi_2))$.
As the result of above we will have

$$\begin{aligned}y(t_{n+1})\ &=y(t_n)+\tfrac{h}{2}\Big[f(t_n,y(t_n))+f(t_{n+1},y(t_n)+hf(t_n,y(t_n))\Big]\\&+\tfrac{h^3}{4}f'(\xi_2,y(\xi_2))f_y(t_{n+1},\xi_3)-\tfrac{h^3}{12}f''(\xi_1,y(\xi_1))\end{aligned}\tag{12}$$

Thus we have the following one-step explicit equation for calculation $y(t_{n+1})$ using $y(t_n)$:

$$y(t_{n+1})=y(t_n)+\frac{h}{2}\Big[f(t_n,y(t_n))+f(t_{n+1},y(t_n)+hf(t_n,y(t_n))\Big]\tag{13}$$

with initial value $y_0=y(t_0)$.

By dropping the remainder term in (9), we obtain an equivalent equation with (13), modified Euler's method as following,

$$y(t_{n+1})=y(t_n)+\frac{h}{2}[f(t_n,y(t_n))+f(t_{n+1},y(t_{n+1}))]\quad n\ge 0.\tag{14}$$

Let $y^{(0)}(t_{n+1})=y(t_n)+hf(t_n,y(t_n))$ be a good initial guess of the solution $y(t_{n+1})$, and define

$$y^{(j+1)}(t_{n+1})=y(t_n)+\frac{h}{2}[f(t_n,y(t_n))+f(t_{n+1},y^{(j)}(t_{n+1}))],\quad j=0,1,\dots\tag{15}$$

which (15) is known as iterative solution of modified Euler's method relation. To analyze the iteration and to determine conditions under which it will converge, subtract (15) from (14) to obtain

$$y(t_{n+1})-y^{(j+1)}(t_{n+1})=\frac{h}{2}[f(t_n,y(t_n))+f(t_{n+1},y^{(j)}(t_{n+1}))].\tag{16}$$

Use the Lipschitz condition in problem (8) to bound this with

$$|y(t_{n+1}) - y^{(j+1)}(t_{n+1})| \leq \frac{hK}{2}|y(t_{n+1}) - y^{(j)}(t_{n+1})| \quad j \geq 0, \tag{17}$$

thus

$$|y(t_{n+1}) - y^{(j+1)}(t_{n+1})| \leq \left(\frac{hK}{2}\right)^{j+1}|y(t_{n+1}) - y^{(0)}(t_{n+1})|. \tag{18}$$

If

$$\frac{hK}{2} \leq 1$$

then the iterates $y^{(j)}(t_{n+1})$ will converge to $y(t_{n+1})$ as $j \to \infty$, and the computation of y_{n+1} from y_n contains a truncation error of $O(h^3)$, for more details see [1].

5 Modified Euler's Method for Numerical Solution of FDEs

Let $Y = [Y_1, Y_2]$ be the exact solution and $y = [y_1, y_2]$ be the approximated solution of the initial value equation(3) by using the one-step modified method. Let,

$$[Y(t)]_r = [Y_1(t;r), Y_2(t;r)]\ ,\ [y(t)]_r = [y_1(t;r), y_2(t;r)].$$

Also we note that throughout each integration step, the value of r is unchanged. The exact and approximated solution at t_n are denoted by

$$[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)]\ ,\ [y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]\ (0 \leq n \leq N),$$

respectively. The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N}, \quad t_i = t_0 + ih \quad 0 \leq i \leq N.$$

By using the modified Euler method we obtain:

$$\begin{aligned} Y_1(t_{n+1};r) &= Y_1(t_n;r) + \tfrac{h}{2}F[t_n, Y_1(t_n;r), Y_2(t_n;r)] \\ &+\tfrac{h}{2}F\Big[t_{n+1}, Y_1(t_n;r) + hF[t_n, Y_1(t_n;r), Y_2(t_n;r)] \\ &, Y_2(t_n;r) + hG[t_n, Y_1(t_n;r), Y_2(t_n;r)]\Big] \\ &+h^3A_1(r) \end{aligned} \tag{19}$$

and

$$\begin{aligned} Y_2(t_{n+1};r) &= Y_2(t_n;r) + \tfrac{h}{2}G[t_n, Y_1(t_n;r), Y_2(t_n;r)] \\ &+\tfrac{h}{2}G\Big[t_{n+1}, Y_1(t_n) + hF[t_n, Y_1(t_n;r), Y_2(t_n;r)] \\ &, Y_2(t_n;r) + hG[t_n, Y_1(t_n;r), Y_2(t_n;r)]\Big] \\ &+h^3A_2(r) \end{aligned} \tag{20}$$

where $A = [A_1, A_2]$, $[A]_r = [A_1(r), A_2(r)]$ and

$$[A]_r = [\frac{1}{4} f'(\xi_2, Y(\xi_2)).f_y(t_{n+1}, \xi_3) - \frac{1}{12} f''(\xi_1, Y(\xi_1)]_r. \tag{21}$$

Also we have

$$\begin{aligned} y_1(t_{n+1}; r) &= y_1(t_n; r) + \tfrac{h}{2} F[t_n, y_1(t_n; r), y_2(t_n; r)] \\ &+ \tfrac{h}{2} F\Big[t_{n+1}, y_1(t_n; r) + hF[t_n, y_1(t_n; r), y_2(t_n; r)] \\ &, y_2(t_n; r) + hG[t_n, y_1(t_n; r), y_2(t_n; r)]\Big] \end{aligned} \tag{22}$$

and

$$\begin{aligned} y_2(t_{n+1}; r) &= y_2(t_n; r) + \tfrac{h}{2} G[t_n, y_1(t_n; r), y_2(t_n; r)] \\ &+ \tfrac{h}{2} G\Big[t_{n+1}, y_1(t_n) + hF[t_n, y_1(t_n; r), y_2(t_n; r)] \\ &, y_2(t_n; r) + hG[t_n, y_1(t_n; r), y_2(t_n; r)]\Big] \end{aligned} \tag{23}$$

Next, we will show that $y_1(t; r)$ and $y_2(t; r)$ mentioned in the previous method converge to $Y_1(t; r)$ and $Y_2(t; r)$, respectively whenever $h \to 0$. In order to prove these assertions, we first recall the following lemmas.

Lemma 5.1 *Let the sequence of numbers* $\{W_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \le A|W_n| + B, \; 0 \le n \le N - 1,$$

for the given positive constants A *and* B*. Then*

$$|W_n| \le A^n |W_0| + B\frac{A^n - 1}{A - 1}, \; 0 \le n \le N.$$

Proof. See[10]. □

Lemma 5.2 *Let the sequence of numbers* $\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \le |W_n| + A \max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \le |V_n| + A \max\{|W_n|, |V_n|\} + B,$$

for the given positive constants A *and* B*. Then, denoting*

$$U_n = |W_n| + |V_n|, \quad 0 \le n \le N,$$

we have

$$U_n \le \bar{A}^n U_0 + \bar{B}\frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \le n \le N,$$

where $\bar{A} = 1 + 2A$ *and* $\bar{B} = 2B$*.*

Proof. See[10]. □

Our next result determined the point wise convergence of the modified Euler approximations to the exact solution. Let $F[t,u,v]$ and $G[t,u,v]$ be the functions which are given by the equations (6), (7) where u and v are constants and $u \leq v$. Thus the domain of F and G are defined as the following:

$$K = \{(t,u,v)|t_0 \leq t \leq T, -\infty < u \leq v, -\infty < v < \infty\}.$$

With the above notations in the following we will present the convergence theorem.

Theorem 5.3 *Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^1(\mathbb{R}_\mathcal{F})$ and the partial derivatives of F and G be bounded over $\mathbb{R}_\mathcal{F}$. Then for arbitrarily fixed r, $0 \leq r \leq 1$, the numerical solutions of* (22) *and* (23) *converge to the exact solutions $Y_1(t;r)$ and $Y_2(t;r)$ uniformly in t.*

Proof. It is sufficient to show

$$\lim_{h\to 0} y_1(t_N;r) = Y_1(t_N;r), \quad \lim_{h\to 0} y_2(t_N;r) = Y_2(t_N;r)$$

where $t_N = T$. Let $W_n = Y_1(t_n;r) - y_1(t_n;r)$, $V_n = Y_2(t_n;r) - y_2(t_n;r)$, by using the equations (19), (20), (22) and (23), we get:

$$|W_{n+1}| \leq |W_n| + Lh\max\{|W_n|,|V_n|\} + Lh[2Lh\max\{|W_n|,|V_n|\} + \max\{|W_n|,|V_n|\}] + h^3M_1,$$

$$|V_{n+1}| \leq |V_n| + Lh\max\{|W_n|,|V_n|\} + Lh[2Lh\max\{|W_n|,|V_n|\} + \max\{|W_n|,|V_n|\}] + h^3M_2,$$

where M_1, M_2 are upper bound for $A_1(r)$, $A_2(r)$ respectively. Hence,

$$|W_{n+1}| \leq |W_n| + Lh\{1 + (1+2Lh)\}\max\{|W_n|,|V_n|\} + h^3M,$$

$$|V_{n+1}| \leq |V_n| + Lh\{1 + (1+2Lh)\}\max\{|W_n|,|V_n|\} + h^3M,$$

where $M = \max\{M_1, M_2\}$, and $L > 0$ is a bound for the partial derivatives of F and G. Therefore from Lemma 5.2, we obtain

$$|W_n| \leq (1+2Lh)^{2n}|U_0| + 2h^3M\frac{(1+2Lh)^{2n}-1}{(1+2Lh)^2-1},$$

$$|V_n| \leq (1+2Lh)^{2n}|U_0| + 2h^3M\frac{(1+2Lh)^{2n}-1}{(1+2Lh)^2-1},$$

where $|U_0| = |W_0| + |V_0|$. In particular,

$$|W_N| \leq (1+2Lh)^{2N}|U_0| + 2h^3M\frac{(1+2Lh)^{\frac{2(T-t_0)}{h}}-1}{(1+2Lh)^2-1},$$

$$|V_N| \leq (1+2Lh)^{2N}|U_0| + 2h^3 M \frac{(1+2Lh)^{\frac{2(T-t_0)}{h}} - 1}{(1+2Lh)^2 - 1},$$

since $W_0 = V_0 = 0$, we have

$$|W_N| \leq M\, \frac{e^{4L(T-t_0)} - 1}{2L(1+hL)} h^2, \quad |V_N| \leq M\, \frac{e^{4L(T-t_0)} - 1}{2L(1+hL)} h^2,$$

Thus, if $h \to 0$, we conclude $W_N \to 0$ and $V_N \to 0$, which completes the proof. □

By using modified Euler method (14), we obtain:

$$\begin{aligned}
&y_1(t_{n+1};r) \\
&= y_1(t_n;r) + \tfrac{h}{2}\Big[F(t_n, y_1(t_n;r), y_2(t_n;r)) + F(t_{n+1}, y_1(t_{n+1};r), y_2(t_{n+1};r))\Big], \\
&y_2(t_{n+1};r) \\
&= y_2(t_n;r) + \tfrac{h}{2}\Big[G(t_n, y_1(t_n;r), y_2(t_n;r)) + G(t_{n+1}, y_1(t_{n+1};r), y_2(t_{n+1};r))\Big],
\end{aligned} \tag{24}$$

and from (13), we have $y_1(t_{n+1};r)$ and $y_2(t_{n+1};r)$ in right side of above equations as follows:

$$\begin{aligned}
y_1(t_{n+1};r) &= y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)], \\
y_2(t_{n+1};r) &= y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)].
\end{aligned} \tag{25}$$

From section 5, we consider initial guesses,

$$\begin{aligned}
y_1^{(0)}(t_{n+1};r) &= y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)], \\
y_2^{(0)}(t_{n+1};r) &= y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)],
\end{aligned} \tag{26}$$

for the iterative solutions below, respectively:

$$\begin{aligned}
&y_1^{(j+1)}(t_{n+1};r) \\
&= y_1(t_n;r) + \tfrac{h}{2}\Big[F(t_n, y_1(t_n;r), y_2(t_n;r)) + F(t_{n+1}, y_1^{(j)}(t_{n+1};r), y_2^{(j)}(t_{n+1};r))\Big], \\
&y_2^{(j+1)}(t_{n+1};r) \\
&= y_2(t_n;r) + \tfrac{h}{2}\Big[G(t_n, y_1(t_n;r), y_2(t_n;r)) + G(t_{n+1}, y_1^{(j)}(t_{n+1};r), y_2^{(j)}(t_{n+1};r))\Big].
\end{aligned} \tag{27}$$

Following lemma is needed to prove next important theorem.

Lemma 5.4 *Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^1(\mathbb{R}_{\mathcal{F}})$ and the partial derivatives of F and G be bounded over $\mathbb{R}_{\mathcal{F}}$. Then for arbitrarily fixed r, $0 \leq r \leq 1$,*

$$D(y(t_{n+1}), y^{(0)}(t_{n+1})) \leq h^2 L(1+2C),$$

where L is a bound of partial derivatives of F and G, and $C = \max\{|G[t_N, y_1(t_N;r), y_2(t_{N-1};r)]| r \in [0,1]\} < \infty$.

Proof. By substituting (25) in (24) and subtraction obtained equation from (26), we get,

$$
\begin{aligned}
& y_1(t_{n+1};r) - y_1^{(0)}(t_{n+1};r) \\
& = \tfrac{h}{2}\Big\{F\Big[t_{n+1}, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)], y_2(t_n;r) \\
& + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] - F[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\}, \\
& y_2(t_{n+1};r) - y_2^{(0)}(t_{n+1};r) \\
& = \tfrac{h}{2}\Big\{G\Big[t_{n+1}, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)], y_2(t_n;r) \\
& + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] - G[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\},
\end{aligned}
$$

and from those, we can get,

$$
\begin{aligned}
& y_1(t_{n+1};r) - y_1^{(0)}(t_{n+1};r) \\
& = \tfrac{h}{2}\Big\{F\Big[t_{n+1}, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)], y_2(t_n;r) \\
& + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
& - F\Big[t_n, y_1(t_n;r) + hF[t_n, y_1(t_;r), y_2(t_n;r)], y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
& + F\Big[t_n, y_1(t_n;r) + hF[t_n, y_1(t_;r), y_2(t_n;r)], y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
& - F[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\},
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
& y_2(t_{n+1};r) - y_2^{(0)}(t_{n+1};r) \\
& = \tfrac{h}{2}\Big\{G\Big[t_{n+1}, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)], y_2(t_n;r) \\
& + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
& - G\Big[t_n, y_1(t_n;r) + hF[t_n, y_1(t_;r), y_2(t_n;r)], y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
& + G\Big[t_n, y_1(t_n;r) + hF[t_n, y_1(t_;r), y_2(t_n;r)], y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
& - G[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\}.
\end{aligned} \tag{29}
$$

Let $L > 0$ is a bound for the partial derivatives of F and G, following relations are obtained from applying the mean value theorem to (28) and

(29):

$$\begin{aligned}
&|y_1(t_{n+1};r) - y_1^{(0)}(t_{n+1};r)| \\
&\le \tfrac{h^2L}{2}\Big\{1 + |F[t_n, y(t_n;r), y_2(t_n;r)]| + |G[t_n, y_1(t_n;r), y_2(t_n;r)]|\Big\} \\
&\le \tfrac{h^2L}{2}\{1 + 2|G(t_n, y_1(t_n;r), y_2(t_n;r)]|\}, \\
&|y_2(t_{n+1};r) - y_2^{(0)}(t_{n+1};r)| \\
&\le \tfrac{h^2L}{2}\Big\{1 + |F[t_n, y(t_n;r), y_2(t_n;r)]| + |G[t_n, y_1(t_n;r), y_2(t_n;r)]|\Big\} \\
&\le \tfrac{h^2L}{2}\{1 + 2|G(t_n, y_1(t_n;r), y_2(t_n;r)]|\}.
\end{aligned} \tag{30}$$

In particular,

$$\begin{aligned}
|y_1(t_N);r) - y_1^{(0)}(t_N;r)| \le \tfrac{h^2L}{2}(1+2C), \\
|y_2(t_N;r) - y_2^{(0)}(t_N;r)| \le \tfrac{h^2L}{2}(1+2C),
\end{aligned}$$

by adding two inequalities, one obtains,

$$|y_1(t_N);r) - y_1^{(0)}(t_N;r)| + |y_2(t_N;r) - y_2^{(0)}(t_N;r)| \le h^2L(1+2C).$$

Hence

$$D(y(t_N), y^{(0)}(t_N)) \le h^2L(1+2C), \tag{31}$$

This completes the proof. □

Theorem 5.5 *Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^1(\mathbb{R}_{\mathcal{F}})$ and the partial derivatives of F and G be bounded over $\mathbb{R}_{\mathcal{F}}$ and $2Lh < 1$. Then for arbitrarily fixed $0 \le r \le 1$, the iterative numerical solutions of $y_1^{(j)}(t_n;r)$ and $y_2^{(j)}(t_n;r)$ converge to the numerical solutions $y_1(t_n;r)$ and $y_2(t_n;r)$ in $t_0 \le t_n \le t_N$, when $j \to \infty$.*

Proof. It is sufficient to show

$$\lim_{j\to\infty} y_1^{(j)}(t_N;r) = y_1(t_N;r), \quad \lim_{j\to\infty} y_2^{(j)}(t_N;r) = y_2(t_N;r)$$

where $t_N = T$. For $n = 0, 1, \ldots, N-1$, By using the equations (24) and (27), we get:

$$\begin{aligned}
&y_1(t_{n+1};r) - y_1^{(j+1)}(t_{n+1};r) \\
&= \tfrac{h}{2}\{F[t_{n+1}, y_1(t_{n+1};r), y_2(t_{n+1};r)] - F[t_{n+1}, y_1^{(j)}(t_{n+1};r), y_2^{(j)}(t_{n+1};r)]\}, \\
&y_2(t_{n+1};r) - y_2^{(j+1)}(t_{n+1};r) \\
&= \tfrac{h}{2}\{G[t_{n+1}, y_1(t_{n+1};r), y_2(t_{n+1};r)] - G[t_{n+1}, y_1^{(j)}(t_{n+1};r), y_2^{(j)}(t_{n+1};r)]\}.
\end{aligned} \tag{32}$$

Let $L > 0$ is a bound for the partial derivatives of F and G, following relations are obtained from applying the mean value theorem to (32):

$$\begin{aligned} &|y_1(t_{n+1};r) - y_1^{(j+1)}(t_{n+1};r)| \\ &\leq \tfrac{Lh}{2}\{|y_1(t_{n+1};r) - y_1^{(j)}(t_{n+1};r)| + |y_2(t_{n+1};r) - y_2^{(j)}(t_{n+1};r)|\}, \\ &|y_1(t_{n+1};r) - y_1^{(j+1)}(t_{n+1};r)| \\ &\leq \tfrac{Lh}{2}\{|y_1(t_{n+1};r) - y_1^{(j)}(t_{n+1};r)| + |y_2(t_{n+1};r) - y_2^{(j)}(t_{n+1};r)|\}. \end{aligned} \tag{33}$$

Thus, from Definition D, Hausdroff distance, in section 2, we will have:

$$|y_1(t_{n+1};r) - y_1^{(j+1)}(t_{n+1};r)| \leq LhD(y(t_{n+1}), y^{(j)}(t_{n+1})),$$

$$|y_2(t_{n+1};r) - y_2^{(j+1)}(t_{n+1};r)| \leq LhD(y(t_{n+1}), y^{(j)}(t_{n+1})).$$

Hence, adding two inequalities gives,

$$\begin{aligned} D(y(t_{n+1}), y^{(j+1)}(t_{n+1})) &\leq 2LhD(y(t_{n+1}), y^{(j)}(t_{n+1})) \\ &\vdots \\ D(y(t_{n+1}), y^{(j+1)}(t_{n+1})) &\leq (2Lh)^{j+1}D(y(t_{n+1}), y^{(0)}(t_{n+1})). \end{aligned}$$

Using lemma 5.4 in special case, we get:

$$D(y(t_N), y^{(j+1)}(t_N)) \leq \frac{1}{2}(2Lh)^{j+2}h(1 + 2C).$$

The desired result finally follows from condition $2Lh \leq 1$,

$$\lim_{j\to\infty} D([y(t_N)]_r, [y^{(j)}(t_N)]_r) = 0.$$

The proof is complete. □

6 Numerical Results

In this section we present two numerical examples. In order to see the rate of accuracy between theoretical exact solution and our numerical solution we have devoted error table for each of examples, the errors are obtained from $D[Y(t;r), y(t;r)]$ for $t = t_N$, $r \in [0, 1]$.

As well as the convergence theorem shows, the numerical results also show that for smaller stepsize h we obtain smaller errors. The exact solutions and approximated solutions by Euler method and presented method(Mod.Euler method) are plotted in figure 1 and figure 2 respectively for example 1 and example 2.

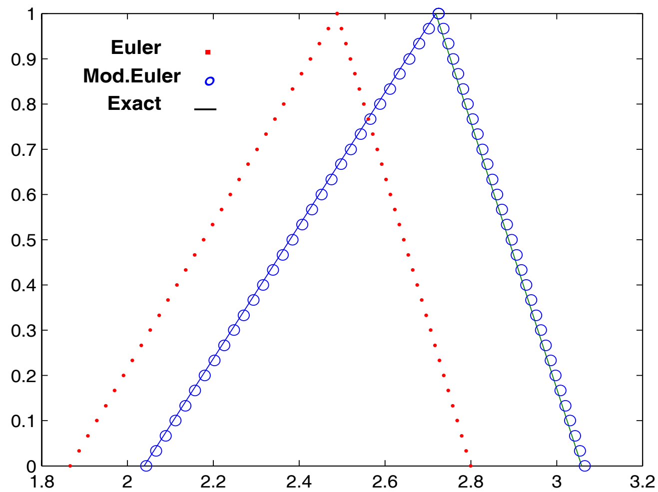

Figure 1: h=0.2

Example 6.1 *Consider the initial value problem* [10]

$$\begin{cases} y'(t) = y(t), & t \in [0,1] \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r) \end{cases}$$

The exact solution at $t = 1$ *is given by*

$$Y(1;r) = [(0.75 + 0.25\,r)e,\ (1.125 - 0.125\,r)e], \quad 0 \le r \le 1.$$

Using iterative solution of modified Euler's method, we have

$$y_1(0;r) = 0.25 + 0.25\,r,\ \ y_2(0;r) = 1.125 - 0.125\,r,$$

and by

$$\begin{aligned} y_1^{(0)}(t_{i+1};r) &= y_1(t_i;r) + hy_1(t_i;r), \\ y_2^{(0)}(t_{i+1};r) &= y_2(t_i;r) + hy_2(t_i;r), \end{aligned}$$

where $i = 0, 1, \ldots, N-1$ and $h = \frac{1}{N}$. Now, using these equations as an initial guess for following iterative solutions, respectively,

$$\begin{aligned} y_1^{(j)}(t_{i+1};r) &= y_1(t_i;r) + \tfrac{h}{2}[y_1(t_i;r) + y_1^{(j-1)}(t_{i+1};r)], \\ y_2^{(j)}(t_{i+1};r) &= y_2(t_i;r) + \tfrac{h}{2}[y_2(t_i;r) + y_2^{(j-1)}(t_{i+1};r)], \end{aligned}$$

where $j = 1, 2, 3$. Thus we have $y_1(t_i;r) = y_1^{(3)}(t_i;r)$ and $y_2(t_i;r) = y_2^{(3)}(t_i;r)$, for $i = 1, \ldots, N$.
Therefore, $Y_1(1;r) \approx y_1^{(3)}(1;r)$ and $Y_2(1;r) \approx y_2^{(3)}(1;r)$ are obtained. Table 1 shows estimation of error for different values of $r \in [0,1]$ and h.

Table 1

h / r	0.1	0.01	0.001	.0001
0	0.0025350660	0.0000254824	0.0000002548	0.0000000025
0.2	0.0024787331	0.0000249162	0.0000002491	0.0000000024
0.4	0.0024223964	0.0000243499	0.0000002435	0.0000000024
0.6	0.0023660616	0.0000237836	0.0000002378	0.0000000023
0.8	0.0023097268	0.0000232173	0.0000002321	0.0000000023
1	0.0022533920	0.0000226510	0.0000002265	0.0000000022

Example 6.2 *Consider the fuzzy initial value problem*

$$y'(t) = k_1 y^2(t) + k_2, \quad y(0) = 0,$$

where $k_j > 0(j = 1, 2)$ *are triangular fuzzy numbers.*

The exact solution is given by

$$Y_1(t;r) = l_1(r)\tan(w_1(r)t),$$
$$Y_2(t;r) = l_2(r)\tan(w_2(r)t),$$

with

$$l_1(r) = \sqrt{k_{2,1}(r)/k_{1,1}(r)}, \qquad l_2(r) = \sqrt{k_{2,2}(r)/k_{1,2}(r)},$$
$$w_1(r) = \sqrt{k_{1,1}(r)k_{2,1}(r)}, \qquad w_2(r) = \sqrt{k_{1,2}(r)k_{2,2}(r)},$$

where

$$[k_1]_r = [k_{1,1}(r), k_{1,2}(r)] \text{ and } [k_2]_r = [k_{2,1}(r), k_{2,2}(r)],$$

$$k_{1,1}(r) = 0.5 + 0.5r, \qquad k_{1,2}(r) = 1.5 - 0.5r,$$
$$k_{2,1}(r) = 0.75 + 0.25r, \qquad k_{2,2}(r) = 1.25 - 0.25r.$$

Now by using equations below

$$y_1(0;r) = y_2(0;r) = 0,$$
$$y_1^{(0)}(t_{i+1};r) = y_1(t_i;r) + h(k_{11}y_1^2(t_i;r) + k_{21}),$$
$$y_2^{(0)}(t_{i+1};r) = y_2(t_i;r) + h(k_{12}y_2^2(t_i;r) + k_{22}),$$

for $i = 0, 1, \ldots, N-1$ and $h = \frac{1}{N}$, as an initial guess for following iterative solutions, respectively,

$$y_1^{(j)}(t_{i+1};r) = y_1(t_i;r) + \frac{h}{2}[k_{11}y_1^2(t_i;r) + k_{11}(y_1^{(j-1)}(t_{i+1};r))^2 + 2k_{21}]$$
$$y_2^{(j)}(t_{i+1};r) = y_2(t_i;r) + \frac{h}{2}[k_{12}y_2^2(t_i;r) + k_{12}(y_2^{(j-1)}(t_{i+1};r))^2 + 2k_{22}]$$

where $j = 1, 2, 3$. Similar to example 6.1,we have $y_1(t_i;r) = y_1^{(3)}(t_i;r)$ and $y_2(t_i;r) = y_2^{(3)}(t_i;r)$, for $i = 1, \ldots, N$.

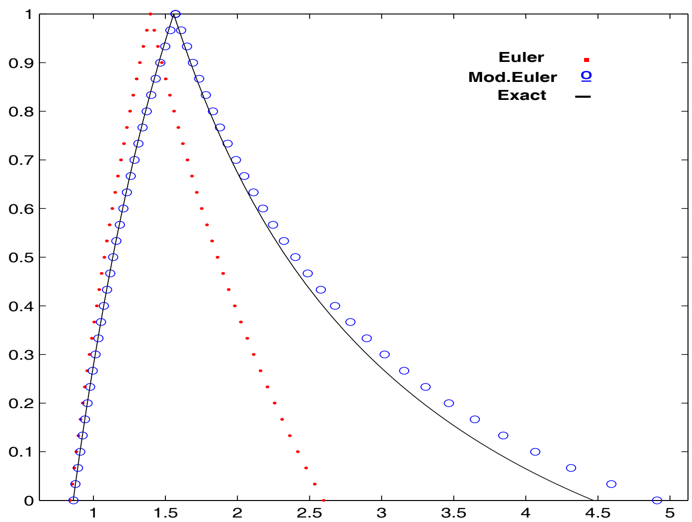

Figure 2: h=0.1

Therefore,$Y_1(1;r) \approx y_1^{(3)}(1;r)$ and $Y_2(1;r) \approx y_2^{(3)}(1;r)$. Table 2 shows estimation of error for different values of $r \in [0,1]$ and h.

Table 2

h / r	0.1	0.01	0.001	.0001
0	0.4417099428	0.0079388827	0.0000845005	0.0000008504
0.2	0.18106314062	0.0027073124	0.0000282513	0.0000001172
0.4	0.0847937757	0.0011335243	0.0000116861	0.0000001172
0.6	0.0433920492	0.0005375096	0.0000054962	0.0000000550
0.8	0.0235983073	0.0002766096	0.0000028118	0.0000000281
1	0.0133874352	0.0001505383	0.0000015235	0.0000000152

7 Conclusion

In this work we have applied iterative solution of modified Euler's method for numerical solution of fuzzy differential equations. It is obvious that the method introduced in this paper with $O(h^3)$ performs better than Euler's method with $O(h)$ in [10].

References

[1] K. E. Atkinson, *An Introduction to Numerical Analysis,* second ed., John Wiley and Sons, 1989.

[2] J. J. Buckley and E. Eslami, Introduction to Fuzzy Logic and Fuzzy Sets, *Physica-Verlag, Heidelberg, Germany.* 2001.

[3] J. J. Buckley and E. Eslami and T. Feuring, Fuzzy Mathematics in Economics and Engineering,*Physica-Verlag, Heidelberg, Germany.* 2002.

[4] S. L. Chang and L. A. Zadeh, On Fuzzy Mapping and Control, IEEEE Trans. *Systems Man Cybernet.,* **2** (1972) 30-34.

[5] D. Dubois and H. Prade, Towards Fuzzy Differential Calculus: Part 3, Differentiation, *Fuzzy Sets and Systems,* **8** (1982) 225-233.

[6] R. Goetschel and W. Voxman, Elementary Calculus, *Fuzzy Sets and Systems,* **18** (1986) 31-43.

[7] R. Goetschel and W. Voxman, Elementary Fuzzy Calculus, *Fuzzy Sets and Systems,* **24** (1987) 31-43.

[8] O. Kaleva, Fuzzy Differential Equations, *Fuzzy Sets Systems,* **24** (1987) 301-317.

[9] O. Kaleva, The Cauchy Problem for Fuzzy Differential Equations, *Fuzzy Sets Systems,* **35** (1990) 389-396.

[10] M. Ma M.Friedman, and A. Kandel, Numerical Solutions of Fuzzy Differential Equations, *Fuzzy Sets Systems,* **105** (1999) 133-138.

[11] M. L. Puri and D. A. Ralescu, Differentials of Fuzzy Functions, *J. Math. Anal. Appl.,* **91** (1983) 321-325.

[12] S. Seikkala, On the Fuzzy Initial Value Problem, *Fuzzy Sets Systems,* **24** (1987) 319-330.

Received: March 18, 2007