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A Tauberian Theorem for $(C, 1)$ Summability Method

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Abstract

In this paper, we retrieve slow oscillation of a real sequence $u = (u_n)$ out of $(C, 1)$ summability of the generator sequence $(V_n^{(0)}(\Delta u))$ of (u_n) under some additional condition. Consequently, we recover convergence or subsequential convergence of (u_n) out of $(C, 1)$ summability of (u_n) under certain additional conditions that control oscillatory behavior of the sequence (u_n).

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1 Introduction

Let $u = (u_n)$ be a sequence of real numbers. The classical control modulo of the oscillatory behavior of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n = n(u_n - u_{n-1})$. The general control modulo of the oscillatory behavior of order 1 of (u_n) is defined by

$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n^{(1)}(\omega^{(0)}(u))$$

where $\sigma_n^{(1)}(u) = \frac{1}{n+1}\sum_{k=0}^{n} u_k$. The identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u)$$

where $V_n^{(0)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k$ is known as Kronecker identity. It is now clear that $\omega_n^{(1)}(u) = n\Delta V_n^{(0)}(\Delta u)$. Since $\sigma_n^{(1)}(u) = \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0$, Kronecker identity can be rewritten as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0 \tag{1}$$

in terms of $(V_n^{(0)}(\Delta u)$, the generator sequence of (u_n). A sequence (u_n) is said to be subsequentially convergent [1] if there exists a finite interval $I(u)$ such that all accumulation points of (u_n) are in $I(u)$ and every point of $I(u)$ is an accumulation point of (u_n). A sequence (u_n) is $(C, 1)$ summable to s if $\lim_n \sigma_n^{(1)}(u) = s$. A sequence (u_n) is said to be slowly oscillating [5] if

$$\lim_{\lambda \to 1^+} \limsup_n \max_{n+1 \le k \le [\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| = 0,$$

where $[\lambda n]$ denotes the integer part of λn. A sequence (u_n) is said to be $|C, 1|_p$ summable [3] if for $p > 1$

$$\sum_{j=1}^{\infty} j^{p-1} |\Delta \sigma_j^{(1)}(u)| < \infty.$$

A sequence (u_n) is said to be slowly varying [4] if

$$\lim_n \frac{u([\lambda n])}{u(n)} = 1$$

for $\lambda > 1$.

We now establish the main result and its consequences. As a corollary to the main result, we recover classical convergence or subsequential convergence of the sequence (u_n).

Theorem 1.1 *Let $(V_n^{(0)}(\Delta u))$ be $(C, 1)$ summable to s. If for some $p > 1$*

$$(\lambda - 1)^{p-1} \limsup_n \sum_{j=n+1}^{[\lambda n]} \frac{|\omega_j^{(1)}(u)|^p}{j} = o(1), \quad \lambda \to 1^+, \tag{2}$$

then (u_n) is slowly oscillating.

Theorem 1.2 *Let (u_n) be $(C,1)$ summable to s. If for some $p > 1$*

$$(\lambda-1)^{p-1}\limsup_n \sum_{j=n+1}^{[\lambda n]} \frac{|\omega_j^{(0)}(u)|^p}{j} = o(1), \quad \lambda \to 1^+, \tag{3}$$

then (u_n) is convergent.

The proofs are based on the following Lemma.

Lemma 1.3 *[5] For $\lambda > 1$*

$$u_n - \sigma_n^{(1)}(u) = \frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta u_j, \tag{4}$$

where $[\lambda n]$ denotes the integer part of λn.

2 Proofs of Theorems

Proof of Theorem 1.1

Applying Lemma 1.3 to $(V_n^{(0)}(\Delta u))$ we have

$$\begin{aligned}\left|V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)\right| &\le \frac{[\lambda n]+1}{[\lambda n]-n}\left|V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)\right| \\ &+ \max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_j^{(0)}(\Delta u)\right|. \end{aligned}\tag{5}$$

Since $(V_n^{(0)}(\Delta u))$ is $(C,1)$ summable, the first term on the right-hand side of (5) is o(1) as $n \to \infty$ and (5) becomes

$$\limsup_n \left|V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)\right| \le \limsup_n \max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_j^{(0)}(\Delta u)\right|. \tag{6}$$

For the second term on the right-hand side of (5) we have

$$\begin{aligned}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_j^{(0)}(\Delta u)\right| &\le \sum_{j=n+1}^{[\lambda n]}|\Delta V_j^{(0)}(\Delta u)| \\ &\le ([\lambda n]-n)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(1)}(u)|^p}{j^p}\right)^{\frac{1}{p}}, \text{where } \tfrac{1}{p}+\tfrac{1}{q}=1 \\ &\le ([\lambda n]-n)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(1)}(u)|^p}{j^{p-1}j}\right)^{\frac{1}{p}}\end{aligned}$$

$$
\begin{aligned}
&\le ([\lambda n]-n)^{\frac{1}{q}}\left(\frac{1}{n^{p-1}}\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(1)}(u)|^p}{j}\right)^{\frac{1}{p}}\\
&\le \frac{([\lambda n]-n)^{\frac{1}{q}}}{n^{\frac{1}{q}}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(1)}(u)|^p}{j}\right)^{\frac{1}{p}}\\
&\le (\lambda-1)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(1)}(u)|^p}{j}\right)^{\frac{1}{p}}. \qquad (7)
\end{aligned}
$$

From (6) and (7) we have

$$
\limsup_n\left|V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u)\right|\le(\lambda-1)^{\frac{1}{q}}\limsup_n\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(1)}(u)|^p}{j}\right)^{\frac{1}{p}}. \qquad (8)
$$

Letting $\lambda\to 1^+$ in (8) and taking (2) into account, we deduce that

$$
\limsup_n\left|V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u)\right|\le 0. \qquad (9)
$$

From (9) we have $\lim_n V_n^{(0)}(\Delta u)=s$. Since $\sigma_n^{(1)}(u)=\sum_{k=1}^n\frac{V_k^{(0)}(\Delta u)}{k}+u_0$, it follows by (1) that (u_n) is slowly oscillating. Furthermore, for some slowly varying sequence (B_n), we have $u_n=O(B_n)$, $n\to\infty$.

Since $u_n=O(B_n)$, $n\to\infty$, it follows that there exists a finite interval I such that for every $r\in I$, there is a subsequence $\left(\frac{u_{n(r)}}{B(n(r))}\right)$ such that $\lim_{n(r)}\frac{u_{n(r)}}{B(n(r))}=r$. (See [1, 2]).

Notice that since (u_n) is slowly oscillating, for all nonnegative integers m, the sequence $(V_n^{(m)}(\Delta u))$ is subsequentially convergent [2].

As a corollary we have the following.

Corollary 2.1 *Let $(V_n^{(0)}(\Delta u))$ be $(C,1)$ summable to s. If $(V_n^{(0)}(\Delta u))$ is $|C,1|_p$ summable, then (u_n) is slowly oscillating.*

Proof of Theorem 1.2

Applying Lemma 1.3 to (u_n) we have

$$
\left|u_n-\sigma_n^{(1)}(u)\right|\le\frac{[\lambda n]+1}{[\lambda n]-n}\left|\sigma_{[\lambda n]}^{(1)}(u)-\sigma_n^{(1)}(u)\right|+\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right|. \qquad (10)
$$

Since (u_n) is $(C,1)$ summable to s, the first term on the right-hand side of (10) is o(1) as $n\to\infty$ and (10) becomes

$$\limsup_n \left|u_n-\sigma_n^{(1)}(u)\right| \le \limsup_n \max_{n+1\le k\le[\lambda n]} \left|\sum_{j=n+1}^{k}\Delta u_j\right|. \tag{11}$$

For the second term on the right-hand side of (10) we have

$$\begin{aligned}
\max_{n+1\le k\le[\lambda n]} \left|\sum_{j=n+1}^{k}\Delta u_j\right| &\le \sum_{j=n+1}^{[\lambda n]}|\Delta u_j| \\
&\le ([\lambda n]-n)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(0)}(u)|^p}{j^p}\right)^{\frac{1}{p}}, \text{where } \frac{1}{p}+\frac{1}{q}=1 \\
&\le ([\lambda n]-n)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(0)}(u)|^p}{j^{p-1}j}\right)^{\frac{1}{p}} \\
&\le ([\lambda n]-n)^{\frac{1}{q}}\left(\frac{1}{n^{p-1}}\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(0)}(u)|^p}{j}\right)^{\frac{1}{p}} \\
&\le \frac{([\lambda n]-n)^{\frac{1}{q}}}{n^{\frac{1}{q}}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(0)}(u)|^p}{j}\right)^{\frac{1}{p}} \\
&\le (\lambda-1)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(0)}(u)|^p}{j}\right)^{\frac{1}{p}}.
\end{aligned} \tag{12}$$

From (11) and (12) we have

$$\limsup_n \left|u_n-\sigma_n^{(1)}(u)\right| \le (\lambda-1)^{\frac{1}{q}}\limsup_n\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(0)}(u)|^p}{j}\right)^{\frac{1}{p}}. \tag{13}$$

Letting $\lambda\to 1^+$ in (13) and taking (3) into account, we deduce that

$$\limsup_n \left|u_n-\sigma_n^{(1)}(u)\right| \le 0. \tag{14}$$

From (14) we have $\lim_n u_n = s$.

As a corollary we have the following.

Corollary 2.2 *Let (u_n) be $(C,1)$ summable. If (v_n), where $u_n=\sigma_n^{(1)}(v)$, is $|C,1|_p$ summable, then (u_n) is convergent.*

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