Applied Mathematical Sciences, Vol. 1, 2007, no. 46, 2269 - 2275

Fixed Point by Approximated Sequences for Nonexpansive Mappings in Banach Spaces

A. Ouahab [1], S. Lahrech, S. Rais

Dept of Mathematics, Faculty of Science, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

A. Mbarki

National School of Applied Sciences, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

A. Jaddar

National School of Managment, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

Abstract. We give a strong convergence theorem for the iteration process defined by

$$x_0 \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n,\ n = 0, 1, \dots, \tag{0.1}$$

where $0 \leq \alpha_n \leq 1$ and T is a non expansive self mapping from a closed convex subset of a Banach space.

Mathematics Subject Classification: 47H09, 49M05

Keywords: Accretive operators, duality mapping, nonexpansive mappings, strong convergence, fixed point

1. Introduction

Let X be a Banach space with norm $\| . \|$ and C a closed convex subset of X. We denote by X^* the dual of X, that is the Banach space of the continuous functionals on X.
Let T be a nonexpansive mapping on C into itself, that is

$$\| Tx - Ty \| \leq \| x - y \| \ \ for\ all\ \ x, y \in C.$$

[1] ouahab05@yahoo.fr

We should show that the sequence defined in (0.1) converges strongly to a fixed point of T nearest to x.

The process defined in (0.1) were studied by many authors, Reich [4],[5] proved the convergence in the case that C is a weakly compact, convex subset of a uniformly smooth Banach space and $\alpha_n = n^{-a}$ with $0 < a < 1$, Wittmann [7] showed that the sequence guiven by (0.1) converges in Hilbert spaces and $\{\alpha_n\}$ satisfies

$$0 \leq \alpha_n \leq 1, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} = \infty \ and \ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{1.1}$$

Shioji and Takahashi [6] showed that (0.1) converges strongly to a fixed point nearest to x while the norm of X is a uniformly Gateaux differentiable and $\{\alpha_n\}$ satisfies (1.1).

2. Preleminaries

Let $x \in X$, and $f \in X^*$, as usually we use the pairing $\langle x, f\rangle$ to denote $f(x)$. Suppose φ is a continuous strictly increasing real-valued function on $\mathbb{R}^+$ satisfying $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$.

Definition 2.1. *A mapping $J_\varphi : X \to X^*$ is called a duality mapping with gauge function φ if for every $x \in X$,*

$$\langle x, J_\varphi x\rangle = \| J_\varphi x \| \| x \| = \varphi(\| x \|) \| x \| .$$

We say that J_φ is weakly sequentially continuous if J_φ is sequentially continuous relative to the weak topologies on both X and X^*.

If we set $\Phi(t) = \displaystyle\int_0^t \varphi(\tau)d\tau$, $t \geq 0$, then we obtain

$$J_\varphi(x) = \partial\Phi(\| x \|), \forall x \in X,$$

Definition 2.2. *Let D be a subset of a Banach space X. A mapping $T : D \to X$ is said to be accretive if for all $x, y \in D$ and some $j \in J(x - y)$,*

$$\langle Tx - Ty, j\rangle \geq 0.$$

Here J denotes the normalized duality mapping from X into 2^{X^*}, that is

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \| x \|^2 = \| x^* \|^2\}, \ x \in X.$$

There a connection between nonexpansive mappings and accretive opeartors, in fact, if $T : D \subset X \to X$ is a nonexpansive mapping, then for $U = I - T$, $x, y \in D$ and $j \in J(x - y)$,

$$\begin{aligned}
\langle Ux - Uy, j\rangle &= \langle x - y - (Ux - Uy), j\rangle \\
&= \| x - y \|^2 - \langle Tx - Ty, j\rangle \\
&\geq \| x - y \|^2 - \| Tx - Ty \| \| x - y \| \\
&\geq 0.
\end{aligned}$$

And thus U is accretive.

Definition 2.3. *A Banach space X is said to satisfy Opial's condition if whenever a sequence $\{x_n\}$ in X converges weakly to x, then for $y \neq x$,*

$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - y \| .$$

Gossez and Lami Dozo [2] have shown that Banach spaces wich are reflexive and having a weakly sequentially continuous duality mapping with gauge function satisfy Opial's condition.
The first part of the following lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [3], see also [8]

Lemma 2.1. *Let X be a Banach space which has a weakly sequentially continuous duality mapping J_φ with gauge function φ, then the following occur:*

(i) *For all $x, y \in X$, there holds the inequality*

$$\Phi(\| x + y \|) \leq \Phi(\| x \|) + \langle y, J_\varphi(x + y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ of elements of X is weakly convergent to a point x. then*

$$\limsup_{n\to\infty} \Phi(\| x_n - y \|) = \limsup_{n\to\infty} \Phi(\| x_n - x \|) + \Phi(\| x - y \|),$$

for all $x, y \in X$. In particular X satisfies Opial's condition.

A mapping f on a subset D of a Banach space X is said to be demiclosed if for any sequence $\{a_n\}$ in D, the following implication occurs:

$$w - \lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} \| f(a_n) - b \| = 0$$

implies

$$a \in D \text{ and } f(a) = b.$$

Where $w - \lim$ denotes the limit relative to the weak topology.
The proof of the following can be found in [1]

Proposition 2.1. *Let X be a reflexive Banach space which satisfies the Opial's condition, les K be a closed, convex subset of X, and let $T : K \to X$ be a nonexpansive mapping. Then the mapping given by $f = I - T$ is demiclosed on K.*

3. Main Result

For a mapping T we denote by $FixT$ the set of fixed points of T.

Proposition 3.1. *Let X be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge φ, C a closed, convex subset of X and let $T : C \to C$ be a nonexapnsive mapping such that $FixT \neq \emptyset$. Let*

x be a fixed element in C.
Denote by z_t the unique element of C which satisfy

$$z_t = tx + (1-t)Tz_t,\ 0 < t < 1.$$

Then $\lim_{t\to 0} z_t = z$ exists and $z \in FixT$.

Proof:
Since $FixT \neq \emptyset, \{z_t\}$ and $\{Tz_t\}$ are bounded, in fact we have for $\omega \in FixT$

$$\begin{array}{rcl} \| z_t - \omega \| & = & \| tx + (1-t)Tz_t - t\omega - (1-t)T\omega \| \\ & \leq & t \| z_t - \omega \| + (1-t) \| Tz_t - T\omega \| \\ & \leq & t \| x - \omega \| + (1-t) \| z_t - \omega \| . \end{array}$$

Thus, we obtain $\| Tz_t - \omega \| \leq \| z_t - \omega \| \leq \| x - \omega \|$.
Let $\{z_{t_n}\}$ be a subsequence of $\{z_t\}$ which converges weakly to z, we should show that $z \in FixT$.
$\| z_t - Tz_t \| = t \| x - Tz_t \| \to 0$ as $t \to 0$.
Since X is reflexive and satisfy the Opial's condition, T satisfy the demiclosedness principle and $z \in FixT$.
Now we have to show that

$$\langle x - z_t, J_\varphi(\omega - z_t)\rangle \leq 0,\ \omega \in FixT.$$

We have $x - z_t = (\frac{1}{t} - 1)(z_t - Tz_t)$, then for $\omega \in FixT$, we get

$$\begin{array}{rcl} \langle x - z_t, J_\varphi(\omega - z_t)\rangle & = & (\frac{1}{t} - 1)\langle (I-T)z_t, J_\varphi(\omega - z_t)\rangle \\ & = & -(\frac{1}{t} - 1)\langle (I-T)\omega - (I-T)z_t, J_\varphi(\omega - z_t)\rangle \\ & \leq & 0. \end{array}$$

Because $I - T$ is accretive.
Now, applying **lemma 2.1**(i), we get for $\omega \in FixT$,

$$\begin{array}{rcl} \Phi(\| z_t - \omega \|) & = & \Phi(\| t(x-\omega) + (1-t)(Tz_t - \omega) \|) \\ & \leq & \Phi((1-t) \| Tz_t - \omega \|) + t\langle x - \omega, J_\varphi(z_t - \omega)\rangle \\ & \leq & (1-t)\,\Phi(\| z_t - \omega \|) + t\langle x - \omega, J_\varphi(z_t - \omega)\rangle. \end{array}$$

Thus,

$$\Phi(\| z_t - \omega \|) \leq \langle x - \omega, J_\varphi(z_t - \omega)\rangle. \tag{3.1}$$

Let $\{z_{t_n}\}$ be an arbitrary subsequence of $\{z_t\}$ and assume that $\{z_{t_n}\}$ converges weakly to some $z \in C$. As it was seen above we have $u \in FixT$.
In fact, $\{z_{t_n}\}$ converges strongly to u. Indeed, from (3.1) we get,

$$\Phi(\| z_{t_n} - z \|) \leq \langle x - z, J_\varphi(z_{t_n} - z)\rangle. \tag{3.2}$$

But since J_φ is weakly continuous, then the second term in (3.2) tends to 0 as $t_n \to 0$. Thus, we deduce that $\{z_{t_n}\}$ converges strongly to z.

We claim that $\{z_t\}$ converges strongly to z. To show that it suffices to prove that for any other subsequence $\{z_{s_n}\}$ and which converges strongly to some $u \in C$, we have $u = z$.
As it was already seen, we know that $u, z \in FixT$. Then we have

$$\langle x - z_{t_n}, J_\varphi(u - z_{t_n})\rangle \le 0, \tag{3.3}$$

and

$$\langle x - z_{s_n}, J_\varphi(z - z_{s_n})\rangle \le 0. \tag{3.4}$$

Letting $n \to \infty$ we obtain

$$\langle x - z, J_\varphi(u - z)\rangle \le 0, \tag{3.5}$$

and

$$\langle x - u, J_\varphi(z - u)\rangle \le 0. \tag{3.6}$$

By (3.5) and (3.6) we get that $\| u - z \| \varphi(\| u - z \|) \le 0$, then we deduce that $u = z$. ∎

Theorem 3.1. *Let X be a reflexive Banach space which has a weakly sequentially duality mapping J_φ with gauge function φ and C a closed, convex subset of X. Let T be a nonexpansive mapping from C onto it self such that $FixT \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence which satisfy (1.1). and let $x \in C$ be a fixed element. Then the sequence $\{x_n\}$ given by (0.1) converges strongly to z where z is the limit of the sequence $\{z_t\}$ given in proposition 3.1.*

Before to prove the theorem, we need the following lemmas which we shall make use..

Lemma 3.1. [7] $\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0.$

Lemma 3.2. *Let J be the normalized duality mapping of X, then we have*

$$\limsup_{n\to\infty} \langle x - z, J(x_n - z)\rangle \le 0.$$

Proof:
One can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x - z, J(x_n - z)\rangle = \lim_{k\to\infty} \langle x - z, J(x_{n_k} - z)\rangle,$$

and

$$w - \lim_{k\to\infty} x_{n_k} = \omega.$$

We know already that $\omega \in FixT$. In other hand, in view of [2], J is single valued and weakly continuous, then we have

$$\limsup_{n\to\infty} \langle x - z, J(x_n - z)\rangle = \lim_{k\to\infty} \langle x - z, J(x_{n_k} - z)\rangle = \langle x - z, J(\omega - z)\rangle.$$

Now, to complete the proof we have to show that $\langle x - z_t, J(\omega - z_t)\rangle \leq 0$. Indeed, we have

$$\begin{aligned} \langle x - z_t, J(\omega - z_t)\rangle &= \langle (\frac{1}{t} - 1)(z_t - Tz_t, J(\omega - z_t)\rangle \\ &= -(\frac{1}{t} - 1)\langle (I - T)\omega - (I - T)z_t, J(\omega - z_t)\rangle \\ &\leq 0. \end{aligned}$$

Because $I - T$ is accretive, now by letting $t \to 0$ we get that

$$\langle x - z, J(\omega - z)\rangle \leq 0.$$

∎

Now we give the proof of the theorem.

Proof:

We can follow the same arguments as in [6].

Since $(1 - \alpha_n)(Tx_n - z) = (x_{n+1} - z) - \alpha_n(x - z)$, we obtain

$$\| (1 - \alpha_n)(Tx_n - z) \|^2 \geq \| x_{n+1} - z \|^2 - 2\alpha_n \langle x - z, J(x_{n+1} - z)\rangle,$$

then,

$$\| x_{n+1} - z \|^2 \leq (1 - \alpha_n) \| x_n - z \|^2 + 2(1 - (1 - \alpha_n))\langle x - z, J(x_{n+1} - z)\rangle$$

for all $n \in \mathbb{N}$.

Now, let $\varepsilon > 0$, then by lemma 3.1, there are some $m \in \mathbb{N}$ such that

$$\langle x - z, J(x_n - z)\rangle \leq \varepsilon,$$

for every $n \geq m$.

Thus, we get that

$$\| x_{n+m} - z \| \leq \left(\prod_{k=m}^{m+n-1} (1 - \alpha_k)\right) \| x_m - z \|^2 + \left(1 - \prod_{k=m}^{m+n-1} (1 - \alpha_k)\right) \varepsilon,$$

for every $n \in \mathbb{N}$.

But, since $\sum_{k=0}^{\infty} \alpha_k = \infty$, we get

$$\limsup_{n\to\infty} \| x_n - z \|^2 = \limsup_{n\to\infty} \| x_{n+m} - z \|^2 \leq \varepsilon.$$

Thus $\{x_n\}$ converges strongly to z. ∎

Acknowledgements

Much of the work of the third author was done when he was in National school of Applied Sciences of Oujda. The second authors and third authors were grateful to Professor Larbi ROUBI Directeur of ENSA-Oujda for their supports.

References

[1] K. Goebel and W. A. Kirk, *Topics in metric fied point theory*, Cambridge University Press. (1990).

[2] J. P. Gossez and Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. **40**, (1972), 565-573.

[3] T. C. Lim and H. K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear Anal. **22**(1994), 1345-1355.

[4] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75**(1980), 287-292. MR **82a:**47050.

[5] S. Reich, *Some problems and results in fixed point theory*, Contemp. Math. **21** (1983), 179-187. MR **85e:**47082.

[6] N. Shioji and W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc. Vol **125**, N 12, (1997), 3641-3645.

[7] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58**,(1992), 486-491. MR **93c:**47069.

[8] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), 240-256.

Received: April 4, 2007