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Fixed Point Theorems in General Probabilistic Metric Spaces

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Abstract. Contraction maps on probalistic metric spaces were first defined and studied by Sehgal [12]. They were subsequently studied by Sherwood [13], where firstly appeared the fundamental problem of "bounded orbits", This influenced many authors, and, consequently, a number of new results in this line followed (see, for example [3], [5],).
In the present paper, we give two new results which encompasses most of such generalization of the Sherwood theorem, further our result also extended many other results form metric spaces to general probalistic metric spaces. These results are of interest in view of analogous results in metric spaces (see, for example [14] and [8]) and in view of recent activity in fixed point theory of probalistic metric spaces and its applications (see, for example [1], [3-4], [5] and [6]).

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1. Introduction and Preliminaries

Our terminology and notation for probabilistic metric spaces are the same as those of Schweizer and Sklar [11]. A nonnegative real function f defined on $\Re^+ \cup \{\infty\}$ is called a distance distribution function (briefly, a d.d.f.) if it is nondecreasing, left continuous on $(0, \infty)$, with $f(0) = 0$ and $f(\infty) = 1$. The set of all d.d.f's will be denoted by Δ^+; while the set of all f in Δ^+ such that $\lim_{s\to\infty} f(s) = 1$ is denoted by $\mathcal{D}^+$.

Example 1.1. *For any a in $\Re^+ \cup \{\infty\}$ the unit step at a is the function ε_a belonging to Δ^+ defined by*

$$\varepsilon_a(x) = \begin{cases} 0, & \text{for } 0 \le x \le a, \\ 1, & \text{for } a < x \le \infty, \end{cases}$$

for $0 \le a < \infty$. While

$$\varepsilon_\infty(x) = \begin{cases} 0, & \text{for } 0 \le x < \infty, \\ 1, & \text{for } x = \infty. \end{cases}$$

Definition 1.1. *Consider f and g be in Δ^+, $h \in (0, 1]$, and let $(f, g; h)$ denotes the condition*

$$0 \le g(x) \le f(x + h) + h,$$

for all x in $(0, \frac{1}{h})$.
The modified Lévy distance is the function d_L defined on $\Delta^+ \times \Delta^+$ by

$$d_L(f, g) = \inf\{h : \text{ both conditions } (f, g; h) \text{ and } (g, f; h) \text{ hold}\}.$$

Note that for any f and g in Δ^+, both $(f, g; 1)$ and $(g, f; 1)$ hold, hence d_L is well-defined and $d_L(f, g) \le 1$.
Let us begin by recalling the following definitions and technical results from [11].

Lemma 1.1. [11] *The function d_L is a metric on Δ^+.*

Definition 1.2. *A sequence $\{F_n\}$ of d.d.f's is said to converge weakly to a d.d.f. F if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at each continuity point x of F.*

Lemma 1.2. [11] *Let $\{F_n\}$ be a sequence of functions in Δ^+, and let F be in Δ^+ Then $\{F_n\}$ converges weakly to F if and only if $d_L(F_n, F) \to 0$.*

Lemma 1.3. [11] *The metric spaces (Δ^+, d_L) is compact.*

Definition 1.3. *We say that τ is a triangle function on Δ^+ if it assigns a d.d.f. in Δ^+ to every pair of d.d.f's in $\Delta^+ \times \Delta^+$ and satisfies the following conditions:*

$$\begin{aligned} \tau(F, G) &= \tau(G, F), \\ \tau(F, G) &\le \tau(K, H) \quad \text{whenever} \quad F \le K, \quad G \le H, \\ \tau(F, \varepsilon_0) &= F, \\ \tau(\tau(F, G), H) &= \tau(F, \tau(G, H)). \end{aligned}$$

A commutative, associative and nondecreasing mapping $T : [0,1] \times [0,1] \to [0,1]$ *is called a t-norm if and only if*

$$\begin{aligned} (i) \quad T(a,1) &= a \text{ for all } a \in [0,1], \\ (ii) \quad T(0,0) &= 0. \end{aligned}$$

Example 1.2. *One can easily check that* $T(a,b) = Min(a,b)$ *is a t-norm, and that for any t-norm* T *we have* $T(a,b) \leq Min(a,b)$*. Moreover if* T *is left-continuous, then the operation* $\tau_T : \Delta^+ \times \Delta^+ \to \Delta^+$ *defined by*

$$\tau_T(F,G)(x) = \sup\{T(F(u), G(v)) : u + v = x\},$$

is a triangle function.

Lemma 1.4. [11] *If* T *is continuous, then* τ_T *is uniformly continuous on* (Δ^+, d_L)*.*

Definition 1.4. *A probabilistic metric space (briefly, a PM space) is a triple* (M, F, τ) *where* M *is a nonempty set,* F *is a function from* $M \times M$ *into* Δ^+*,* τ *is a triangle function, such that the following conditions are satisfied for all* p*,* q*,* r *in* M*:*
(i) $F_{pp} = \varepsilon_0$,
(ii) $F_{pq} \neq \varepsilon_0$ *if* $p \neq q$,
(iii) $F_{pq} = F_{qp}$,
(iv) $F_{pr} \geq \tau(F_{pq}, F_{qr})$.
If $\tau = \tau_T$ *for some t-norm* T*, then* (M, F, τ_T) *is called a Menger space.*

Definition 1.5. *Let* (M, F) *be a probabilistic semi-metric space (i.e.,* (i)*,* (ii) *and* (iii) *of Definition*1.4 *are satisfied). For* p *in* M *and* $t > 0$*, the strong t-neighborhood of* p *is the set*

$$N_p(t) = \{q \in M : \ F_{pq}(t) > 1 - t\},$$

and the strong neighborhood system for M *is*

$$\{N_p(t); \ p \in M, \ t > 0\}.$$

Lemma 1.5. [11] *Let* (M, F, τ) *be a PM space. If* τ *is continuous, then the family* $\Im$ *consisting of* $\emptyset$ *and all unions of elements of this strong neighborhood system for* M *determines a Hausdorff topology for* M*.*

An immediate consequence of Lemma 1.5 is that the family $\{N_p(t) : t > 0\}$ is a neighborhood system of p for the topology $\Im$.

Lemma 1.6. [11] *Let* $\{p_n\}$ *be a sequence in* M*. Then*

$$p_n \to p \text{ if and only if } d_L(F_{p_n p}, \varepsilon_0) \to 0 \text{ if and only if } \delta(p_n, p) \to 0.$$

Similarly, $\{p_n\}$ *is a strong Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} d_L(F_{p_n p_m}, \varepsilon_0) = 0.$$

Lemma 1.7. [11] *If $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n \to p$ and $q_n \to q$ (resp. are Cauchy sequences in M), then $d_L(F_{p_nq_n}, F_{pq}) \to 0$, i.e., $F_{p_nq_n}$ converges weakly to F_{pq} (resp. $\{F_{p_nq_n}\}$ is a Cauchy sequence in (Δ^+, d_L)).*

Here and in the sequel, when we consider a PM space (M, F, τ), we always assume that τ is continuous and that M is endowed with the topology $\Im$. Let us recall the definition of the probabilistic diameter of a nonempty set in a PM space introduced by Egbert [2].

Definition 1.6. *Let (M, F, τ) be a PM space and A a nonempty subset of M. The probabilistic diameter is the function D_A defined on $\Re^+ \cup \{\infty\}$ by*

$$D_A(x) = \begin{cases} \lim_{t\to x^-} \varphi_A(t), & \text{for } 0 \le x < \infty \\ 1, & \text{for } x = \infty, \end{cases}$$

where

$$\varphi_A(s) = \inf\{F_{pq}(s) : \ p, \ q \in A\}.$$

It is immediate that D_A is in Δ^+ for any $A \subset M$.

Lemma 1.8. [11] *The probabilistic diameter D_A has the following properties:*
(i) $D_A = \varepsilon_0$ iff A is a singleton set.
(ii) If $A \subset B$, then $D_A \ge D_B$.
(iii) For any $p, q \in A$, $F_{pq} \ge D_A$.
(iv) If $A = \{p, q\}$, then $D_A = F_{pq}$.
(v) If $A \cap B$ is nonempty, then $D_{A\cup B} \ge \tau(D_A, D_B)$.
(vi) $D_A = D_{\overline{A}}$, where $\overline{A}$ is the strong closure of A.

The diameter of a nonempty set A in a metric space is either finite or infinite. Accordingly, A is either bounded or unbounded. In a PM space, on the other hand, there are three distinct possibilities.

Definition 1.7. [11] *A nonempty set A in a PM space is*
(i) bounded if $\lim_{t\to\infty} D_A(t) = 1$, i.e., if D_A is in $\mathcal{D}^+$;
(ii) semibounded if $0 < \lim_{t\to\infty} D_A(t) < 1$;
(iii) unbounded if $\lim_{t\to\infty} D_A(t) = 0$, i.e., if $D_A = \varepsilon_\infty$.

Example 1.3. *Let (M, d) be a metric space. Define $F : M \times M \to \Delta^+$ by*

$$F_{pq} = \varepsilon_{d(p,q)}.$$

We can check easily that (M, F, τ_{Min}) is a PM (Menger) space, and

$$N_pt = \{q \in M : \ d(p, q) < t\},$$

for all t in $(0, 1)$. So, (M, F, τ_{Min}) is a complete PM space if and only if (M, d) is a complete metric space. Moreover, for a nonempty subset A of M, we have

$$D_A = \varepsilon_{diam(A)},$$

where

$$diam(A) = \sup\{d(p, q) : \ \ p, \ q \in A\}.$$

2. Main results

In all this note, (M, F, τ) denotes a complete PM space, and T is a map from M into itself. Powers of T are defined by $T^0x = x$ and $T^{n+1}x = T(T^nx)$, $n \geq 0$. When there is no risk of ambiguity, we will use the notation $x^k = T^kx$, in particular $x^0 = x$, $x^1 = Tx$, for the sake of brevity. The set $\mathcal{O}(x) = \{T^nx \ : \ n = 0, 1, 2, ...\}$ is called the orbit (starting at x), while the set $\mathcal{O}(x, y)$ is the union of two orbits starting at x and y. The letter Φ denotes the set of functions satisfying:

$(\mathcal{A}'_1)$ $\phi : [0, \infty] \to [0, \infty]$ is lower semi-continuous from the left, nondecreasing and $\phi(0) = 0$;
$(\mathcal{A}'_2)$ For each $t \in (0, \infty)$, $\phi(t) > t$ and $\phi(+\infty) = +\infty$.

Consider also the assertions:

(FP) T has one and only one fixed point.
(SA) There exists $z \in M$ such that $T^kx \to z$ as $k \to \infty$ for any $x \in M$, i.e., the successive approximations converge.

(C) For $x, y \in M$ and $s > 0$

$$F_{TxTy}(s) \geq D_{\mathcal{O}(x,y)}(\phi(s)).$$

The following lemma is obvious. We will need it below.

Lemma 2.1. *Let $\phi \in \Phi$. Then we have*
$(\mathcal{A}_1)$ *For every $s \in (0, \infty]$*

$$\lim_{n\to\infty} \phi^n(s) = \infty.$$

$(\mathcal{A}_2)$ *For any $G \in \mathcal{D}^+$ and $s \in (0, \infty]$*

$$\lim_{n\to\infty} G(\phi^n(s)) = 1.$$

Let us now state our main result.

Theorem 2.1. (C) implies (FP, SA), if for any $x, y \in M$ the set $\mathcal{O}(x, y)$ is bounded.

Note that the hypothesis, say $(\mathcal{B})$, "for any $x, y \in M$ the set $\mathcal{O}(x, y)$ is bounded" implies that the PM space (M, F, τ) has the property that $RanF \subset \mathcal{D}^+$, which is a necessary condition for the uniqueness of fixed points when they exist. Notice also that by Lemma 1.8-v, we can replace the hypothesis $(\mathcal{B})$ by "for any $x \in M$ the set $\mathcal{O}(x)$ is bounded" in the case when $\mathcal{D}^+$ is closed under τ. This is the case if M is a metric space, F and τ are as in Example 1.3. But in General, even if $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are bounded, it is not necessary that $\mathcal{O}(x, y)$ is bounded. For example, consider $M = \{p, q\}$ and $F_{pq} = \frac{1}{2}\varepsilon_0 + \frac{1}{2}\varepsilon_\infty$, then the identity function on M satisfies condition (C) with two fixed points and $\mathcal{O}(p, q)$ is only semi-bounded.

Proof. (C) $\Rightarrow$ (SA) Let $x, y \in M$ and $s > 0$. By (C) and Lemma 1.8 (ii) we have

$$\begin{aligned} F_{T^ixT^jy}(t) &\geq D_{\mathcal{O}(T^{i-1}x,T^{j-1}y)}(\phi(t)) \\ &\geq D_{\mathcal{O}(x,y)}(\phi(t)), \end{aligned}$$

for $i, j \geq 1$ and $s > t \geq 0$. This means that

$$F_{uv}(t) \geq D_{\mathcal{O}(x,y)}(\phi(t)), \quad \text{for} \quad u, v \in \mathcal{O}(Tx, Ty).$$

Taking the infimun over all $u, v \in \mathcal{O}(Tx, Ty)$, we obtain

$$\varphi_{\mathcal{O}(Tx,Ty)}(t) \geq D_{\mathcal{O}(x,y)}(\phi(t)). \tag{1}$$

Since $D_{\mathcal{O}(x,y)} \in \Delta^+$ and $\phi \in \Phi$, as $t \to s$ in (1), we obtain

$$D_{\mathcal{O}(Tx,Ty)}(s) \geq D_{\mathcal{O}(x,y)}(\phi(s)).$$

It follows that for any $k \geq 1$

$$D_{\mathcal{O}(T^kx,T^ky)}(s) \geq D_{\mathcal{O}(x,y)}(\phi^k(s)). \tag{2}$$

By Lemma 2.1 ($\mathcal{A}_2$), as $k \to \infty$ in (2), we obtain

$$\lim_{k\to\infty} D_{\mathcal{O}(T^kx,T^ky)}(s) = 1.$$

This clearly means that $D_{\mathcal{O}(T^kx,T^ky)} \rightharpoonup \varepsilon_0$, since s is an arbitrary positive number. So, in particular we have

$$D_{\mathcal{O}(x^k)} \rightharpoonup \varepsilon_0,$$

as $k \to \infty$. And since for any $m \geq k > 0$, we have $F_{x^mx^k} \geq D_{\mathcal{O}(x^k)}$. It follows that $\{x^k\}$ is a Cauchy sequence in a complete PM space (M, F, τ). Then, there exists a point $z \in M$ such that $x^n \to z$ as $n \to \infty$. By Lemma 1.8 (ii-vi), we have

$$\lim_{k\to\infty} F_{zy^k}(s) \geq \lim_{k\to\infty} D_{\overline{\mathcal{O}(T^kx,T^ky)}}(s) = \lim_{k\to\infty} D_{\mathcal{O}(T^kx,T^ky)}(s) = 1,$$

since s is an arbitrary positive number. Then, it follows from Definition 1.2 that

$$F_{zy^k} \rightharpoonup \varepsilon_0.$$

This means that $y^k \to z$ as $k \to \infty$, by Lemma 1.2 and Lemma 1.6.

(C) $\Rightarrow$ **(FP)** According to **(C)** $\Rightarrow$ **(SA)** there exists $z \in M$ such that $x^k \to z$ as $k \to \infty$ for all $x \in M$. We want to show that z is a fixed point of T. In order to do this, we shall show that $D_{\mathcal{O}(z)} = \varepsilon_0$, i.e., for any $\alpha > 0$, $D_{\mathcal{O}(z)}(\alpha) = 1$.

Assume that there is $\alpha > 0$ such that $D_{\mathcal{O}(z)}(\alpha) < 1$. Since $\phi \in \Phi$, it is easy to show by contradiction that there exists $t > 0$ such that

$$\alpha > t \text{ and } \phi(t) > \alpha.$$

Next, note that for $\epsilon = \frac{1-D_{\mathcal{O}(z)}(\alpha)}{3} > 0$, there is $\mu > 0$ such that for $t+\mu < t' < \alpha$

$$\frac{1 + 2D_{\mathcal{O}(z)}(\alpha)}{3} > \varphi_{\mathcal{O}(z)}(t').$$

From the definition of $\varphi_{\mathcal{O}(z)}(t')$, then there exist two sequences $\{p(k)\}$ and $\{q(k)\}$ such that $0 \le p(k) < q(k)$ and

$$F_{z^{p(k)}z^{q(k)}}(t') \to \varphi_{\mathcal{O}(z)}(t') \quad \text{as} \quad k \to \infty.$$

Since $\lim z^k = z$, we have that

$$F_{z^k z^l}(t') > \frac{1 + D_{\mathcal{O}(z)}(\alpha)}{2} > \varphi_{\mathcal{O}(z)}(t'),$$

for large $k,\ l$, say for $k, l \ge p_0$, and therefore $p(k) = p$ infinitely many values of k, where $0 \le p \le p_0$. Hence there is a subsequence $\{r(k)\}$ of $\{q(k)\}$ such that

$$F_{z^p z^{r(k)}}(t') \to \varphi_{\mathcal{O}(z)}(t') \quad \text{as} \quad k \to \infty.$$

If $r(k) = q$ for infinitely many values of k, then $F_{z^p z^q}(t') = \varphi_{\mathcal{O}(z)}(t')$; if not, a subsequence $\{v(k)\}$ of $\{r(k)\}$ converges to ∞. Since the points of discontinuity of $F_{z^p z}$ are countable, there is $t'' \in (t, t')$ a point of continuity of $F_{z^p z}$. Then,

$$\varphi_{\mathcal{O}(z)}(t') = \lim_{k\to\infty} F_{z^p z^{v(k)}}(t') \ge \lim_{k\to\infty} F_{z^p z^{v(k)}}(t'') = F_{z^p z}(t'').$$

In all cases, there exist $p, q \ge 0$ and $y \in (t + \mu, \alpha)$ such that

$$\varphi_{\mathcal{O}(z)}(t') \ge F_{z^p z^q}(y).$$

• If $p, q \ge 1$, then (C) implies

$$\varphi_{\mathcal{O}(z)}(t') \ge F_{z^p z^q}(y) \ge D_{\mathcal{O}(z^{p-1}, z^{q-1})}(\phi(y)) \ge D_{\mathcal{O}(z)}(\phi(t)).$$

• If $p = 0 \le q$, for the same reasons as above, there is $y' \in (t, y)$, a point of continuity of $F_{z^q z}$. Then,

$$\varphi_{\mathcal{O}(z)}(t') \ge F_{z z^q}(y) \ge F_{z z^q}(y') = \lim_{k\to\infty} F_{z^k z^q}(y') \ge D_{\mathcal{O}(z)}(\phi(y')) \ge D_{\mathcal{O}(z)}(\phi(t)).$$

In both cases, we have

$$\varphi_{\mathcal{O}(z)}(t') \ge D_{\mathcal{O}(z)}(\phi(t)).$$

As $t' \to \alpha$, we obtain

$$D_{\mathcal{O}(z)}(\alpha) \ge D_{\mathcal{O}(z)}(\phi(t)).$$

Since for $s \in (t, \alpha)$, we have $\phi(s) > \alpha$. A similar argument shows that for $s \in (t, \alpha)$ we have

$$D_{\mathcal{O}(z)}(\alpha) \ge D_{\mathcal{O}(z)}(\phi(s)).$$

As $s \to \alpha$, we obtain

$$D_{\mathcal{O}(z)}(\alpha) \ge D_{\mathcal{O}(z)}(\phi(\alpha)),$$

that is $D_{\mathcal{O}(z)}(\alpha) = D_{\mathcal{O}(z)}(\phi(\alpha)) < 1$. Continuing in this manner we can construct by induction a sequence $\{\alpha_i\}$ such that

$$\alpha_i = \phi^i(\alpha) \quad \text{and} \quad D_{\mathcal{O}(z)}(\phi^i(\alpha)) = D_{\mathcal{O}(z)}(\alpha) < 1,$$

with $\alpha_0 = \alpha$ and $\alpha_1 = \phi(\alpha)$. But this is impossible since $D_{\mathcal{O}(z)} \in \mathcal{D}^+$ and $\phi^i(\alpha) \to \infty$ as $i \to \infty$. Hence $D_{\mathcal{O}(z)}(\alpha) = 1$ for $\alpha > 0$, i.e, $D_{\mathcal{O}(z)} = \varepsilon_0$. Then from Lemma 1.8 (i), $\mathcal{O}(z)$ is a singleton, that is z is a fixed point of T. Now suppose that there is another fixed point $y (y \neq x)$, of T . By Lemma 1.8 (iv) and (C), for $s \in (0, \infty]$ we have

$$F_{xy}(s) \geq F_{xy}(\phi^n(s)), \tag{3}$$

for all $n \geq 1$. Using $F_{x,y} \in \mathcal{D}^+$, as $n \to \infty$ in (3) we get $F_{xy}(s) = 1$, then $F_{xy} = \varepsilon_0$, hence $x = y$ which is a contradiction. This completes the proof of the Theorem.

2.1. **Common Fixed Point Theorem in PM space.** Let S be a semigroup of selfmaps on PM space (M, F, τ). For any $x \in M$, the orbit of x under S starting at x is the set $\mathcal{O}(x)$ defined to be $\{x\} \cup Sx$, where Sx is the set $\{g(x) : \ g \in S\}$. For x, y in M, the set $\mathcal{O}(x, y)$ is the union of $\mathcal{O}(x)$ and $\mathcal{O}(y)$. Recall that a semigroup S is said to be left reversible if, for any f, g in S, there are a, b such that $fa = gb$. It is obvious that left reversibility is equivalent to the statement that any two right ideals of S have nonempty intersection.

Theorem 2.2. *Suppose that S is a left reversible semigroup of selfmaps on a complete PM space (M, F, τ) such that the following conditions $(\mathcal{B}_1')$ and $(\mathcal{B}_1')$ are satisfied:*

$(\mathcal{B}_1')$ *For any x, y in M, the orbit $\mathcal{O}(x, y)$ is bounded.*
$(\mathcal{B}_2')$ *There exists a function $\phi \in \Phi$ such that $F_{fxfy}(s) \geq D_{\mathcal{O}_f(x,y)}(\phi(s))$ for any f in S, x, y in M and $s \in (0, \infty]$.*

Then, S has a unique common fixed point z and, moreover, for any $f \in S$ and $x \in M$, the sequence of iterates $\{f^n(x)\}$ converges to z.

Proof. It follows from Theorem 2.1 that each $f \in S$ has a unique fixed point z_f in M and that for any $x \in M$, the sequence of iterates $\{f^n(x)\}$ converges to z_f. So, to complete the proof it suffices to show that $z_f = z_g$ for any $f, g \in S$. Let n be any positive integer. The left reversibility of S shows that there are a_n and b_n in S such that $f^n a_n = g^n b_n$. So,

$$F_{z_f z_g} \geq \tau(F_{z_f f^n a_n(x)}, F_{g^n b_n(x) z_g}). \tag{4}$$

Also, condition $(\mathcal{B}_2')$ implies that

$$F_{f^n(x) f^n a_n(x)} \geq D_{\mathcal{O}(x, a_n(x))}(\phi^n(j)) \geq D_{\mathcal{O}(x)}(\phi^n(j)),$$

where j is the identity function on $\Re^+$.
Letting $n \to \infty$ in the last inequality and using the fact that $D_{\mathcal{O}(x)}$ is in $\mathcal{D}^+$ we obtain that

$$F_{f^n(x) f^n a_n(x)} \rightharpoonup \varepsilon_0. \tag{5}$$

Since

$$F_{z_f f^n a_n(x)} \geq \tau(F_{z_f f^n(x)}, F_{f^n(x) f^n a_n(x)}). \tag{6}$$

Letting $n \to \infty$ in (6) and using $F_{z_f f^n(x)} \rightharpoonup \varepsilon_0$, we get

$$F_{z_f f^n a_n(x)} \rightharpoonup \varepsilon_0.$$

Likewise, we also have $F_{z_g g^n b_n(x)} \rightharpoonup \varepsilon_0$, which implies that, as $n \to \infty$ in (4) we obtain that $F_{z_f z_g} = \varepsilon_0$. i.e., $z_f = z_g$. This completes the proof of the Theorem.

3. Relative Results

Let (M, d) be a complete metric space. Define $F : M \times M \to \Delta^+$ by

$$F_{pq}(s) = \varepsilon_{d(p,q)}(s).$$

It is easy to see that (M, F, τ_{Min}) is a complete PM space and for $A \subset M$, $D_A = \varepsilon_{diam(A)}$ which implies that if A is bounded in (M, d) then it is in (M, F, τ_{Min}). Let T (S) be a self-map of (is a left reversible semigroup of selfmaps on) (M, d) and $\varphi : [0, \infty) \to [0, \infty)$ is a gauge function i.e., it is upper semi-continuous, increasing, $\varphi(0) = 0$ and $\varphi(s) < s$ for $s > 0$. Suppose that T (S) satisfies the following condition

$$d(Tx, Ty) \leq \varphi(diam(\mathcal{O}_T(x, y))) \quad x, y \in M. \quad (C_1)$$

$$(\ d(fx, fy) \leq \varphi(diam(\mathcal{O}_f(x, y))) \ \text{ for any } f \text{ in } S \text{ and } x, y \text{ in } M. \ (C_2) \ )$$

Then, there exists $\phi : (0, \infty] \to (0, \infty]$ with the property that T (S) satisfies (C) (($\mathcal{B}_2'$) of Theorem 2.2). Moreover, the function ϕ is in Φ. In fact, if $\varphi : [0, \infty) \to [0, \infty)$ is a gauge function, then Chang constructed [1] a strictly increasing continuous function $\alpha : [0, \infty) \to [0, \infty)$ such that $\alpha(0) = 0$ and $\varphi(s) \leq \alpha(s) < s$ for $s > 0$. For example, take ϕ as follows

$$\phi(s) = \begin{cases} \alpha^{-1}(s), & \text{if } 0 \leq s < \lim_{t\to\infty} \alpha(t), \\ +\infty, & \text{if } s \geq \lim_{t\to\infty} \alpha(t). \end{cases}$$

Define also the condition (C_3) by

$$d(Tx, Ty) \leq \varphi(diam(\{x, y, x^1, y^1\}) \quad x, y \in M.$$

Walter [14] has shown that under hypothesis (C_3) all orbits are bounded whenever φ satisfies $s - \varphi(s) \to \infty$ as $s \to \infty$.
Exploiting these observations yields the following results

Corollary 3.1. [14, Thm 2] (C_1) implies (FP, SA), if all orbits are bounded.

Corollary 3.2. [8, Thm 2.2] Suppose S is a left reversible semigroup of selfmaps on a complete metric space (M, d) such that the following conditions (C_1') and (C_2) are satisfied:

(C_1') For any x in M, the orbit $\mathcal{O}(x)$ is bounded.
(C_2) There exists a gauge function φ such that $d(fx, fy) \leq \varphi((diam(\mathcal{O}_f(x, y)))$ for any f in S, and x, y in M .

Then S has a unique common fixed point z. Moreover, for any $f \in S$ and $x \in M$, the sequence of iterates $\{f^n(x)\}$ converges to z.

Corollary 3.3. [14, Thm 4] (C_3) implies (FP, SA) whenever φ satisfies $s - \varphi(s) \to \infty$ as $s \to \infty$.

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