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Two Dimensional Free Surface Flow Computation

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Abstract

A two-dimensional free surface potential flow of a liquid poured from a container is calculated when the liquid runs along the underside of the spout. The effect of surface tension is taken into consideration and gravity is neglected. We computed numerically the solutions for the free-surface profiles using boundary–integral equation. The numerical computation shows that the solutions are found for different values of α and a train of capillary waves in the far field for $4.16 \le \alpha < 10$. No solutions were found for α < 4.16.

Mathematics Subject Classification: 76B07; 76D45; 76M40

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1. Introduction

Free surface flows of liquid poured from a container runs along the underside of the spout have been studied by many authors [1-4]. Scheidegger [1] took the idea of the teapot effect and explained the formation of hoodoos and made interesting calculation considering that the fluid is water. Keller [2] showed that it is explained by the Bernoulli principle and that the pressure is low where the velocity is high, so that the atmospheric pressure pushes the flowing fluid against the bottom of the spout. Keller gave an exact solution of the problem when neglecting gravity and surface tension. Vanden-Broeck and Keller [3] calculated the flow numerically for various configuration of the lip of the container.

In this paper, we approximate this natural phenomenon by considering a two-dimensional flow over a semi infinite horizontal plate. We neglect gravity and take into consideration the effect of surface tension. The fluid is assumed to be inviscid, incompressible and the flow is irrotational. The flow is bounded by the free surface ICJ and the horizontal wall IOI (see Fig.1). Far downstream the flow is uniform with a constant velocity U and a constant depth H and we define the Weber number by

$$\alpha = \frac{\rho U^2 H}{T} \tag{1}$$

Where T is the surface tension and ρ is the density of the fluid.

We compute accurate numerical solutions for the fully nonlinear problem via a boundary-integral procedure. This technique has been used successfully by many authors [6-11] where the mesh points are only on the free surface. The problem is first formulated as an integral equation for the unknown shape of the free surfaces then it is transformed to solve an algebraic non linear system. This algebraic system is solved by Newton's method [6-9]. Our results show that there is a unique solution for each value $\alpha \geq \alpha_0$. When the effects of surface tension and gravity are neglected, the classical exact solution can be found via the hodograph transformation .

The problem is formulated in section 2, the numerical procedure is described in section 3 and we conclude this work by a discussion of the results in section 4.

2. Mathematical formulation

Consider the steady two-dimensional flow over a semi infinite horizontal wall *IOI* and bounded by a free surface *ICJ* (see Fig.1a).

We choose cartesian coordinates with the origin at the edge of the plate, the x axis along the horizontal plate *IO* and the y axis perpendicularly to the x axis through the edge of the plate. Far downstream, as $x \to -\infty$, the flow approaches a uniform stream with a constant velocity U and a constant depth D. Here, we neglect gravity and study the effect of surface tension on the flow.

We choose the unit length and the unit velocity so that H =1. The fluid is assumed to be inviscid and incompressible, and the flow is irrotational. Therefore the velocity potential ϕ must satisfy Laplace's equation. We introduce the complex potential $f = \phi + i\,\psi$, which is a function of the complex variable $z = x + iy$ .The function ψ is the stream function. We denote by $u - iv$ the complex velocity. The function $u - iv$ is an analytic function of z everywhere in the fluid domain except at the origin.

We define the function

$$\xi = u - iv - 1 \tag{2}$$

which vanishes as $x \to -\infty$ since in the far field $u \to 1$ and $v \to 0$

Using Cauchy integral formula, we obtain

$$\oint_{\Gamma} \frac{\xi(\beta)d\beta}{\beta - z} = 2\pi i \sum Residues \tag{3}$$

Here the point z is on the free surface. The positively oriented contour consists of the free surface with the point z bypassed by a semicircle of vanishingly small radius and a vertical line joining the two branches of the free surface at $x = -\infty$ (Fig.1b). Following Forbes [9], we choose the arc length t along the free surface as the variable of integration. We denote by s the intrinseque coordinate corresponding to the point z. The contribution to the integral from the vertical line is 0. The contribution from the semicircle of the vanishingly small radius is $-i\pi\xi(z)$. Consequently, eq. (3) becomes

$$-i\pi\xi(z) = \int_{-\infty}^{0} \frac{\xi(z(t))\, z'(t)\, dt}{z(t) - z(s)} + \int_{-\infty}^{0} \frac{\overline{\xi}(z(t))\, \overline{z}'(t)\, dt}{\overline{z}(t) - z(s)} - 2\pi i \sum Residues \tag{4}$$

Here, the integrals are in the Cauchy principal value sense. To compute the residues, we evaluate the behaviour of $\xi(z)$ in the neighbourhood of the simple pole. $\xi(z) \approx \dfrac{1}{\pi\sqrt{z}}$, as $z \to 0$, consequently $Residu\,(\xi(z), 0) = 0$. Therefore,

$$\xi(z(s)) = \phi_s(s) x_s(s) - 1 \tag{5}$$

Taking the imaginary part of eq. (4) we obtain the integro-differential equation.

$$-i\pi(\phi_s(s)x_s(s) - 1) = -\int_{-\infty}^{0} \frac{\phi_t(t)(y(t) - y(s))dt}{(x(t) - x(s))^2 + (y(t) - y(s))^2} -$$

$$\int_{-\infty}^{0} \frac{\phi_t(t)(y(t) + y(s))dt}{(x(t) - x(s))^2 + (y(t) + y(s))^2} - 2\pi i \sum Residues \tag{6}$$

Using the arc length, we rewrite Bernoulli's equation as:

$$\phi_s^2 - \frac{2}{\alpha} \frac{|x'(s)y''(s) - x''(s)y'(s)|}{\left((x'(s))^2 + (y'(s))^2\right)} = 1 \tag{7}$$

The condition far downstream can be written as

$$\phi_s = 1 \quad \text{as } s \to -\infty \tag{8}$$

Finally, we must satisfy the arclength condition

$$x_s^2 + y_s^2 = 1 \tag{9}$$

This completes the formulation of the problem. We seek functions $x(s)$, y(s) and $\phi(s)$ satisfying eqs. (6)- (9).

3. Numerical method

The numerical procedure we use in this section essentially follows the procedure used by Forbes [9]. To solve the system of equations (6)-(9), we truncate the domain of integration in (6) at a finite value M and introduce the N mesh points:

$$s_i = (i\text{-}1)\, M/N, \quad I = 1,\ldots,N.$$

The values of the flow variables $x(s), y(s), x'(s), y'(s)$ and $\phi'(s)$ at the mesh point s_i are denoted by x_i, y_i, x'_i, y'_i, and ϕ'_i . We satisfy eq. (6) at all $(N\text{-}1)$ midpoints,

$$sh_i = (s_i - s_{i-1})/2, \qquad I = 2,\ldots, N$$

We use the trapezoidal rule with summation over the points s_i to approximate the integral.
We use Newton's method to solve for the unknowns $y_1',\ldots, y_n'$. We express the remaining flow variables in terms of $y_2',\ldots, y_n'$. Using the Bernoulli equation (7) we evaluate $\phi_2',\ldots.,\phi_n'$, and we calculate x_i' from the arclength condition (9).
Next, we use the trapezoidal rule to evaluate $y_2,., y_n$, and we use it again to compute $x_1,., x_n$. Finally, the values of variables at the midpoints sh_i are obtained in terms of the values at the points s_i by linear interpolation from the values at the whole-mesh points. The singularities in the Cauchy principal value integrals can now be ignored since they occur symmetrically between mesh points.
All the remaining flow variables in the integral equation (6) can be determined from the y derivation.
Now the integral equation (6) is evaluated at the N-1 mesh midpoints sh_i. This gives us a system of $N\text{-}1$ equations for the N-1 unknowns ($y'_2,\ldots, y'_N$). The system is solved using Newton's method. We start the calculation with a large value of α . For large α ($\alpha = 1000$) the initial guess is chosen as the exact solution in the absence of surface tension and gravity. Once a solution is obtained, it is used as an initial guess for a smaller value of α and so on.
Typically 200 points are used on the free surface. Usually five to nine iterations are required to get the maximum residual error less than 10^{-8}.

4. Results and discussion

In the absence of gravity and surface tension, the problem has an exact solution that can be computed using the hodograph transformation and the free stream line theory due to Birkhoff [5]. The turning point of the free surface $(y = 0)$ is at the ordinate $x = b = 0.429$ (Fig. 2).
In presence of surface tension and (or) force of gravity, there is no exact known solution. In our paper we only consider the surface tension to evaluate its effect on the flow. Applying the numerical procedure described in section 3, we compute solution for various values of the surface tension. We note that the surface tension is evaluated through the dimensionless parameter α (the Weber number).

When the effect of surface tension is included in the free surface condition, the numerical computation shows that there exists a critical value $\alpha = \alpha_0 = 4{,}16$ under which no solution exists.
Fig. 3a shows the free surface shapes for different values of Weber number $(\alpha), \alpha \geq 10$.
For $\alpha \geq \alpha^*$ $(\alpha^* = 150)$, all free surfaces for different values of $\alpha \geq \alpha^*$ are the same within graphical accuracy, and coincide with the graph of the exact solution without surface tension (Fig. 3b). This suggests that the surface tension can be neglected if $\alpha \geq \alpha^*$.
Fig. 2 shows that the free surface coincide with the graph of the exact solution for $\alpha \to +\infty$.
The effect of the surface tension is more apparent on the position of the turning point $C(b,0)$ as shown in Fig. 5.
Typical free surface profiles are shown in Fig. 4. It is observed that there is a train of waves in the far field for $\alpha < 10$. We define the amplitude Am of the waves as the difference elevation between crests and troughs in the far field. Fig. 6, shows that the amplitude Am increases as α decreases. As one expects with nonlinear waves, the troughs get broader and the crests get narrower.
We could not obtain solutions for all $\alpha < \alpha_0 = 4.16$, this is probably due to the fact that the surface tension tends to strengthen the surface as for the teapot effect tends to bend it at the turning point. Hence, there must be a value of the surface tension (the Weber number) where the two effects are of the same importance. This may explain the fact that the scheme diverges.

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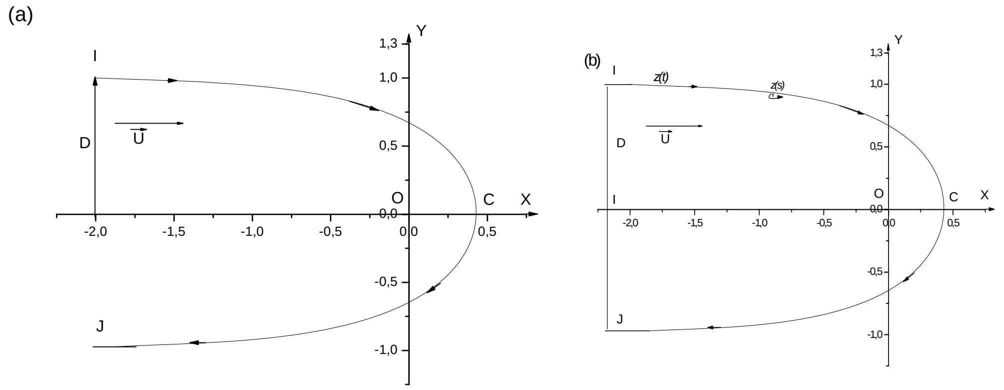


Figure 1: a- Sketch of the flow and of the system of coordinates.
b- Sketch of the flow and the contour of integration for the integral equation.

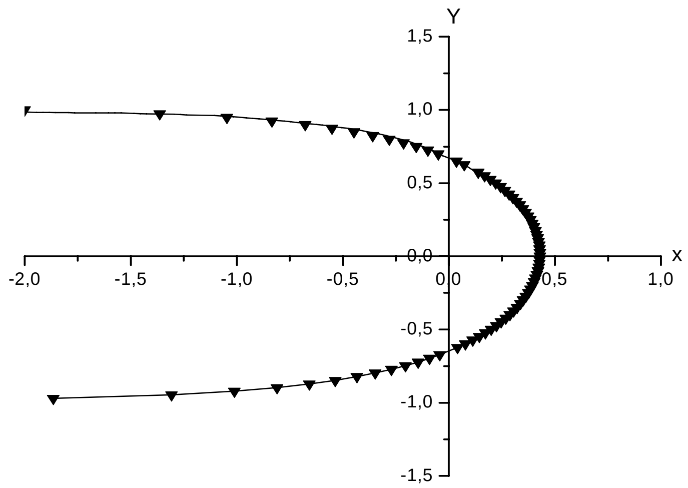
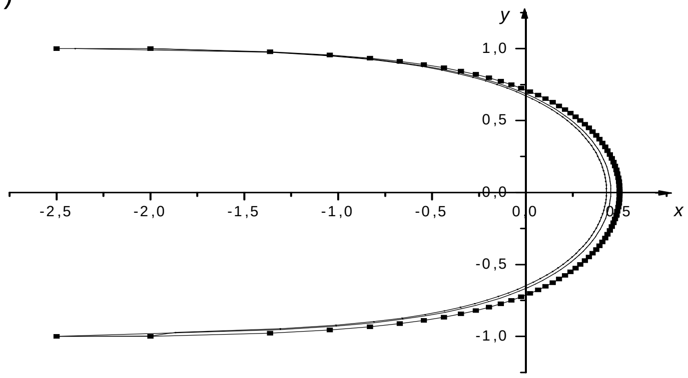


Figure 2: Free surface configuration without surface tension
▼ Via analytical computation by free streamline theory.
— Via boundary–integral equation method

(a)



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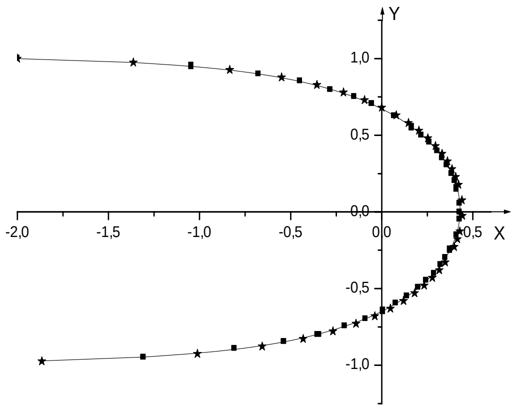

(**Figure 3:** a- Free surface shapes for different values of the Weber number (■ $\alpha = 20$, — $\alpha = 100$, ▼ $\alpha = +\infty$).

b- Free surface flow for $\alpha \geq 150$ (■ α=150, ★ α=500, — α=800)

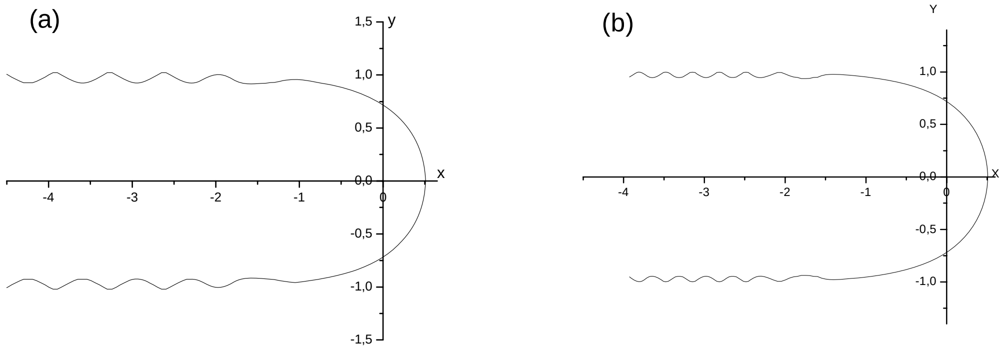

Figure 4: Free surface shapes, (a) for $\alpha = 4.16$, (b) for $\alpha = 7$.

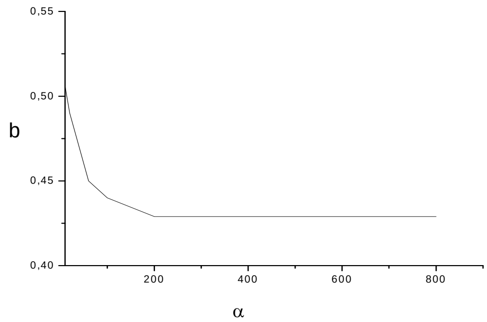

Figure 5: The position of the turning point "C" versus Weber number α

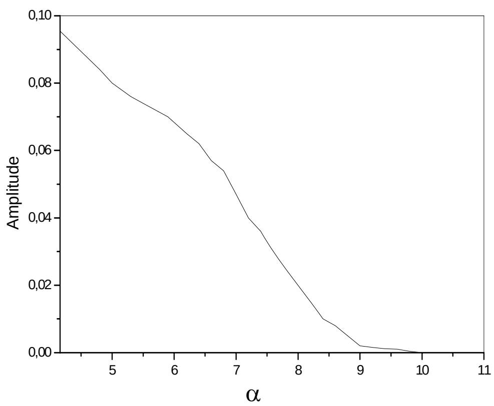

Figure 6: The amplitude of the waves versus α .