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*Some Dynamic Propagation Problems Concerning Mode III Crack

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Abstract. By the methods of the theory of complex functions, dynamic propagation problems concerning mode Ⅲ crack were investigated. Analytical solutions can be obtained by the approaches of self-similar functions. The problems dealt with can be readily transformed into Riemann-Hilbert problems and their closed solutions are attained rather straightforward by this technique. By application of the gained those solutions and superposition theorem, the solutions of discretionary complex problems can be acquired.

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Introduction

In a few decades, many researchers studied statics problems on mode III crack[1-4]. Because of the difficulty in mathematics, researches concerning dynamics problems are not enough thoroughly[5-9]. In this paper fracture dynamics problems on mode propagation crack are lucubrated, a general expression of

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solution is represented by the methods of the theory of complex functions. The problems discussed can be facilely transformed into a Riemann-Hilbert problem which is very readily resolved by the usual Muskhelishvili's measure[10-11]. Analytical solution of propagation crack problems under the different loading conditions such as center, concentration, unit step etc., was given in this article.

1. Anti-plane Problem on Elastodynamics for An Orthotropic Anisotropic Body

For an orthotropic body, let the Cartesian co-ordinates be accordant with the axes of elastic symmetry. The anti-plane equation of motion for an orthotropic anisotropic body is given as:

$$C_{55}\partial^2 w/\partial x^2 + C_{44}\partial^2 w/\partial y^2 = \rho\partial^2 w/\partial t^2 \qquad (1)$$

where C_{44}, C_{55} are the elastic constants, ρ is the mass density and w is the displacement component along z. Applying Atkinson transform[12], it is found that

$$\xi = x - \eta t + T y \qquad (2)$$

where η is to be understood as a complex variable and T is a function of η.

A solution of equation of motion can be written as follows

$$w = \mathrm{Re}\left[\int_{-\infty}^{\infty}\phi(\xi)d\eta\right] \qquad (3)$$

where the integral is the real part of η-axis.

Now substituting Eq. (3) into (1), there results

$$C_{55} + C_{44}T^2 - \rho\eta^2 = 0 \qquad (4)$$

Eq.(1) will become identical equation, hence $\phi(\xi)$ is an arbitrary function to be determined from the boundary conditions.

Assuming that Eq. (4) has two complex roots, then we only take the imaginary part with positive sign, i. e. positive square root. The result is:

$$T(\eta) = i\sqrt{(C_{55} - \rho\eta^2)/C_{44}} \qquad (5)$$

Then putting Eq. (3) into Eq.(1) for an orthotropic body, the results are

$$\tau_{yz} = \mathrm{Re}\int_{-\infty}^{\infty} C_{44}T\frac{\partial\phi(\xi)}{\partial\xi}d\eta \ , \qquad \tau_{xz} = \mathrm{Re}\int_{-\infty}^{\infty} C_{55}\frac{\partial\phi(\xi)}{\partial\xi}d\eta \qquad (6)$$

At $y = 0$, Eq.(2) will become:

$$\xi = x - \eta t \qquad (7)$$

1.1 Displacements are homogeneous functions

When the displacements are homogeneous function (in the following, homogeneous functions of zeroth dimension are called homogeneous), take

$$\phi'(\xi) = f(\eta)/\xi \qquad (8)$$

Put it into Eqs. (3) and (6), and apply Cauchy's theorem for $y = 0$, there gives

$$\tau_{yz} = \mathrm{Re}[-2\pi i \cdot \frac{C_{44}}{t} T(\frac{x}{t}) f(\frac{x}{t})] \ , \qquad \tau_{xz} = \mathrm{Re}[-2\pi i \cdot \frac{C_{55}}{t} f(\frac{x}{t})]$$

$$\frac{\partial w}{\partial t} = \mathrm{Re}[2\pi i \cdot \frac{x}{t^2} f(\frac{x}{t})] \tag{9}$$

In terms of self- similar methods[7, 12-13] with $\tau = x/t$, it is found that

$$F(\tau) = -2\pi i C_{44} T(\tau) f(\tau)$$

Eq. (9) can thus be rewritten as follows:

$$\tau_{xz} = \frac{C_{55}}{C_{44}t} \mathrm{Re}[\frac{F(\tau)}{T(\tau)}], \qquad \tau_{yz} = \frac{1}{t} \mathrm{Re}\, F(\tau)$$

$$\frac{\partial w}{\partial \tau} = \frac{1}{C_{44}} \mathrm{Re}[\frac{F(\tau)}{T(\tau)}] \tag{10}$$

1.2 Stresses are homogeneous functions

When stresses are homogeneous functions, take

$$\phi''(\xi) = f(\eta)/\xi \tag{11}$$

Substitute Eq. (11) into Eqs. (3) and (6) and apply Cauchy's theorem for $y = 0$, it is found that

$$\frac{\partial \tau_{yz}}{\partial t} = \mathrm{Re}[2\pi i C_{44} \cdot \frac{\tau}{t} T(\tau) f(\tau)] \ , \qquad \frac{\partial \tau_{xz}}{\partial t} = \mathrm{Re}[2\pi i C_{55} \cdot \frac{\tau}{t} f(\tau)]$$

$$\frac{\partial^2 w}{\partial t^2} = \mathrm{Re}[-2\pi i \cdot \frac{\tau^2}{t} f(\tau)] \tag{12}$$

This suggests that

$$F(\tau) = 2\pi i \tau C_{44} T(\tau) f(\tau)$$

Using Eq. (12), the results are

$$\frac{\partial \tau_{yz}}{\partial t} = \frac{1}{t} \mathrm{Re}\, F(\tau) \ , \qquad \frac{\partial \tau_{xz}}{\partial t} = \frac{C_{55}}{C_{44}t} \mathrm{Re}[\frac{F(\tau)}{T(\tau)}]$$

$$\frac{\partial^2 w}{\partial t \partial \tau} = \frac{1}{C_{44}} \mathrm{Re}[\frac{F(\tau)}{T(\tau)}] \tag{13}$$

1.3 The problem with arbitrary self-similarity index

Postulate an infinite elastic semi-space $y = 0$ initially at rest. It has any number of loaded segments as well as displacement segments, the ends of these segments are propagating with the unlike constant velocity. The loads and displacements on these sections are arbitrary linear combinations of the following functions:

$$\frac{d^k f_{k_1}(x)}{dx^k} \cdot \frac{d^s f_{s_1}(t)}{dt^s} \ , \qquad f_i(\xi) = \begin{cases} 0 & for \quad \xi < 0 \\ \xi^i & for \quad \xi > 0 \end{cases} \tag{14}$$

Here k, k_1, s and s_1 are arbitrary integer positive numbers. A discretionary sequential function of two variables x and t may be expressed as a linear superposition of Eq.(14). If a solution on load and displacement problems with the modality of Eq.(14) can be found, the solution of intricate problem is obtained by superposition. Let's introduce the linear differential operator as well as inverse:

$$L = \frac{\partial^{m+n}}{\partial x^m \partial t^n} \quad , \qquad \text{inverse} \qquad L^- = \frac{\partial^{-m-n}}{\partial x^{-m} \partial t^{-n}} \tag{15}$$

where *+m+n*, *-m-n* and 0 represent the (*m+n*)th order derivative, the (*m+n*)th order integral and function's self, respectively. It is easy to prove that there exist constants m and n. When putting L into Eq.(14); one will gain functions that are homogeneous functions of x and t of zeroth dimension (homogeneous), the couple m, n will be called an index of self-similarity[3, 14]. Utilizing the same methods as the above, one can attain:

The function Lw is homogeneous. In this case, Eqs.(8)—(10) will apparently hold, in these relationships one must substitute Lw , $L\tau_{xz}$ and $L\tau_{yz}$ instead of displacements w and stresses τ_{xz}, τ_{yz} respectively.

The functions $L\tau_{xz}$ and $L\tau_{yz}$ are homogeneous. In this case Eqs. (11)—(13) will be still hold in which $Lw, L\tau_{xz}$ and $L\tau_{yz}$ instead of displacements w and stresses τ_{xz}, τ_{yz} respectively.

At $y = 0$, one attains the universal conclusions[14, 7]:

When functions Lw is homogeneous, there results

$$w^0 = Lw \quad , \qquad {\tau_{xz}}^0 = L\tau_{xz} \quad , \qquad {\tau_{yz}}^0 = L\tau_{yz} \tag{16}$$

When $L\tau_{xz}$ and $L\tau_{yz}$ are homogeneous, there results

$$w^0 = \frac{\partial}{\partial t} Lw \quad , \quad {\tau_{xz}}^0 = \frac{\partial}{\partial t} L\tau_{xz} \quad , \quad {\tau_{yz}}^0 = \frac{\partial}{\partial t} L\tau_{yz} \tag{17}$$

With the help of the notation introduced all the general representations can be written in the following modality:

$$\begin{gathered} {\tau_{yz}}^0 = \frac{1}{t}\operatorname{Re} F(\tau) , \qquad {\tau_{xz}}^0 = \frac{C_{55}}{C_{44} t}\operatorname{Re}[\frac{F(\tau)}{T(\tau)}] \\ \frac{\partial w^0}{\partial \tau} = \frac{1}{C_{44}}\operatorname{Re}[\frac{F(\tau)}{T(\tau)}] \end{gathered} \tag{18}$$

Presuming $f(\tau) = F(\tau)/T(\tau)$, Eq. (18) can be rewritten [6] as

$$\begin{gathered} {\tau_{yz}}^0 = \frac{1}{t}\operatorname{Re}[f(\tau)T(\tau)] , \qquad {\tau_{xz}}^0 = \frac{C_{55}}{C_{44} t}\operatorname{Re} f(\tau) \\ \frac{\partial w^0}{\partial \tau} = \frac{1}{C_{44}}\operatorname{Re} f(\tau) \end{gathered} \tag{19}$$

2. Illustration of A Dynamic Model of Mode Ⅲ Crack

The crack is presumed to nucleate from an infinitesimally small micro-crack which runs with high speed along the $x-$axis in the manner of self-similarity, that is to say, it moves symmetrically in the positive and negative $x-$axis from initial zero length with the constant velocity V. The sketch of a dynamic model of mode Ⅲ crack of anti-plane problem is depicted in Fig.1. The model is symmetry both in geometry and mechanics with respect to $x-$axis and $y-$axis. As displayed in Fig.1, the crack lies in the realm of $y=0, -Vt<x<Vt$ in the matrix; moreover closed force acts on this interval, whose magnitude is P. The force shows shear stress τ locating in the segment of the crack. When the crack extends with high speed, its dimension must correlate to variables x and t, then the edges of the crack subjected to loads also have relation to variables x and t.

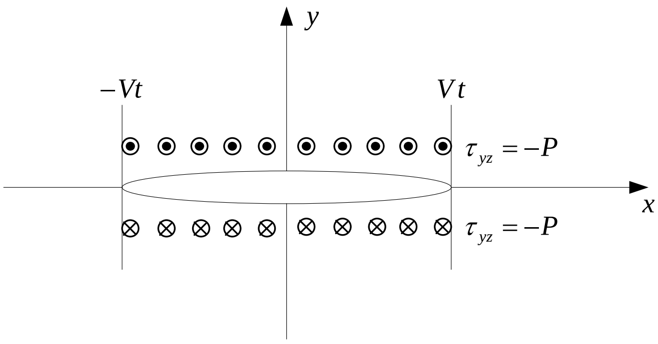

Fig.1. Sketch of dynamic model of mode Ⅲ crack of anti-plane problem

3. Frondose Solutions of Some Problems

In order to settle effectually fracture dynamics problems concerning an orthotropic anisotropic body, analytic solutions will be found under the action of point loads for mode Ⅲ moving crack. According to the theorem of generalized functions, the unlike boundary condition problems studied will be changed into Keldysh-Sedov mixed boundary value problem by the approaches of self-similar functions, and the reciprocal solutions will be acquired. Supposing at $t<0$, the half-plane was at rest. Let at the initial moment t=0 a cut occur at the origin of coordinates and let it begin spreading in both directions of x-axis with a constant velocity V (subsonic velocity) under the state of anti-plane.

3.1 Stresses are homogeneous

3.1.1 Assuming at the initial moment $t=0$ a crack appears at the origin of the coordinates and spreads symmetrically in both directions of $x-$axis with a constant velocity V (subsonic velocity); moreover the central zone of the edges of the crack is subjected to homogeneous loads P propagating at velocity $\alpha<V$. At the half-plane of $y=0$, the boundary conditions can be stated as:

$$\tau_{yz} = -P\,, \qquad 0 \le |x| < \alpha t$$
$$\tau_{yz} = 0\,, \qquad \alpha t \le |x| < Vt$$
$$w = 0\,, \qquad |x| > Vt \tag{20}$$

In the light of the theory of generalized functions[15-16], the boundary conditions Eq. (20) can be rewritten as follows:

$$\tau_{yz} = -P\,[H(x+\alpha t) - H(x-\alpha t)]\,, \qquad |x| < Vt$$
$$w = 0\,, \qquad |x| > Vt \tag{21}$$

In which $L = 1$, applying Eqs. (17) and (19), the first of Eq. (21) will be written as:

$$\mathrm{Re}[T(\tau)\, f(\tau)] = -P\alpha t[\delta(x+\alpha t) + \delta(x-\alpha t)] = -P\alpha[\delta(x-\alpha) + \delta(x-\alpha)] \qquad |\tau| < V \tag{22}$$

The behavior of δ (Dirac) functions[15-16] gives:

$$\delta(\tau+\alpha) + \delta(\tau-\alpha) = \mathrm{Re}\left[\frac{2\tau i}{\pi(\tau^2 - \alpha^2)}\right] \tag{23}$$

Because $T(\tau)$ is purely imaginary for the subsonic speeds, $f(\tau)$ of Eq. (22) must be purely real in the domain of $|\tau| < V$, just $f(\alpha) = -P\alpha$. At $|x| < Vt$, τ_{yz} takes two different values, consequently $f(\tau)$ must have two unknown real constants. Moreover, the displacements are bound at the origin of the coordinates, i.e. $\tau \to 0$, $f(\tau) = o(1)$. In terms of symmetry and conditions of the infinite point of the plane as well as singularities of the stress at the crack tip[17-19], the unique solution of $f(\tau)$ ascertained must content the following shape:

$$f(\tau) = \tau^n (V^2 - \tau^2)^{-3/2}\, \{A + 2B/[\pi(\tau^2 - \alpha^2)]\} \tag{24}$$

where A and B are unknown constants with n being an unknown index.

Substituting Eq. (24) into (20), $n = 1$ is determined:

At $\tau \to \alpha$, from Eqs. (22), (23), (24) and (5) constant B will be confirmed:

$$B = \frac{-P\alpha(V^2 - \alpha^2)^{3/2}}{\sqrt{(C_{55} - \rho\alpha^2)/C_{44}}} \tag{25}$$

Then Eq. (24) may be inserted into Eqs. (19) and (17) for $y = 0$, the stress τ_{yz} and the stress intensity factor $K_3(t)$ are found, respectively:

$$\tau_{yz}(x,0,t) = \mathrm{Re}\int_{C_d}^{x/t} -\frac{\sqrt{(C_{55} - \rho\tau^2)/C_{44}}}{(\tau^2 - V^2)^{3/2}}\left(A + \frac{2B}{\pi(\tau^2 - \alpha^2)}\right) d\tau \tag{26}$$

$$K_3(t) = \lim_{x \to Vt} \sqrt{2\pi(x - Vt)}\ \mathrm{Re}\int_{C_d}^{x/t} -\frac{\sqrt{(C_{55} - \rho\tau^2)/C_{44}}}{(\tau^2 - V^2)^{3/2}}\left(A + \frac{2B}{\pi(\tau^2 - \alpha^2)}\right) d\tau$$

$$=-\frac{\sqrt{\pi t(C_{55}-\rho V^2)/C_{44}}}{V^{3/2}}\left(A+\frac{2B}{\pi(V^2-\alpha^2)}\right) \tag{27}$$

The limit of the above belongs to the type of $0\cdot\infty$, which should be changed into the format of ∞/∞, then its limiting value is derived by the method of L'Hospital theorem[20].

In an orthotropic isotropic body, the disturbance range of elastic wave can be represented by the circular area of radius $c_1 t$ and $c_2 t$. Where c_1 and c_2 are the velocities of longitudinal and transverse waves ($c_1 > c_2$) of elastic body respectively. In an orthotropic anisotropic body, the disturbance range of elastic wave is not the circular area and can not surpass threshold value $C_d=\sqrt{C_{55}/\rho}$ (sonic velocity) of elastic body, here C_{55} is elastic constant of the material. At $|x|>C_d t$, with $\mathrm{Im}[T(\tau)]=0$, thus the stresses and the displacements are zero , which are coincident with the initial conditions; and this illuminates that at $y=0$ disturbance of elastic wave cannot overrun $C_d t$.

Then substituting Eq. (26) into (20), the constant A is ascertained*:*

$$A=-2B\pi^{-1}J_2/J_1$$

$$J_1=\mathrm{Re}\int_M^{C_d}-\frac{\sqrt{(C_{55}-\rho\tau^2)/C_{44}}}{(\tau^2-V^2)^{3/2}}\,d\tau$$

$$J_2=\mathrm{Re}\int_M^{C_d}-\frac{\sqrt{(C_{55}-\rho\tau^2)/C_{44}}}{(\tau^2-\alpha^2)(V^2-\tau^2)^{3/2}}\,d\tau \tag{28}$$

where $\alpha<M<V$. The integral is carried out in the sense of principal value.

Then putting Eq. (24) into (19), w^0 can be gained by literature [21]:

$$w^0=\frac{1}{C_{44}}\int f(\tau)d\tau=\frac{1}{C_{44}}\int\frac{\tau}{(\tau^2-V^2)^{3/2}}[A+\frac{2B}{\pi(\tau^2-\alpha^2)}]d\tau=\frac{A}{C_{44}\sqrt{V^2-\tau^2}}+\frac{B}{\pi C_{44}}\cdot$$

$$[\frac{2}{(V^2-\alpha^2)\sqrt{V^2-\tau^2}}+\frac{1}{\sqrt{(V^2-\alpha^2)^3}}\ln\left|\frac{\sqrt{V^2-\alpha^2}-\sqrt{V^2-\tau^2}}{\sqrt{V^2-\alpha^2}+\sqrt{V^2-\tau^2}}\right|]+C \tag{29}$$

The crack runs along the *x*-axis, therefore w^0 can be computed in the definite integral, we take constant $C=0$. Then substituting Eq. (29) into (17), the displacement w on the upper edge of the crack is acquired by literature [21]:

$$w=\mathrm{Re}\int_0^t w^0\,dt=\mathrm{Re}\int_\infty^{\frac{x}{t}} w^0\cdot(\frac{-x}{\tau^2})d\tau=\mathrm{Re}\int_{C_d}^{x/t}(-\frac{x}{\tau^2})\{\frac{A}{C_{44}\sqrt{V^2-\tau^2}}$$

$$+\frac{B}{\pi C_{44}}[\frac{2}{(V^2-\alpha^2)\sqrt{V^2-\tau^2}}+\frac{1}{\sqrt{(V^2-\alpha^2)^3}}\ln\left|\frac{\sqrt{V^2-\alpha^2}-\sqrt{V^2-\tau^2}}{\sqrt{V^2-\alpha^2}+\sqrt{V^2-\tau^2}}\right|]\}d\tau$$

$$=\mathrm{Re}[A+\frac{2B}{\pi(V^2-\alpha^2)}]\frac{x\sqrt{V^2-\tau^2}}{C_{44}V^2\tau}\Bigg|_{C_d}^{x/t}-\mathrm{Re}\int_{C_d}^{x/t}\frac{Bx}{C_{44}\pi\tau^2\sqrt{(V^2-\alpha^2)^3}}\times$$

$$\ln\left|\frac{\sqrt{V^2-\alpha^2}-\sqrt{V^2-\tau^2}}{\sqrt{V^2-\alpha^2}+\sqrt{V^2-\tau^2}}\right|d\tau=[A+\frac{2B}{\pi(V^2-\alpha^2)}]\frac{\sqrt{V^2t^2-x^2}}{C_{44}V^2}-\mathrm{Re}\{\frac{Bx}{C_{44}\pi\sqrt{(V^2-\alpha^2)^3}}$$
$$\times[\frac{t}{x}\ln\left|\frac{\sqrt{V^2-\alpha^2}+\sqrt{V^2-\tau^2}}{\sqrt{V^2-\alpha^2}-\sqrt{V^2-\tau^2}}\right|+\frac{1}{\alpha}\ln\left|\frac{\alpha\sqrt{V^2-\tau^2}-\sqrt{V^2-\alpha^2}\tau}{\alpha\sqrt{V^2-\tau^2}+\sqrt{V^2-\alpha^2}\tau}\right|]\Big|_{C_d}^{x/t}\}$$
$$=\frac{1}{C_{44}}\{[A+\frac{2B}{\pi(V^2-\alpha^2)}]\frac{\sqrt{V^2t^2-x^2}}{V^2}-\frac{B}{\pi\sqrt{(V^2-\alpha^2)^3}}\cdot[t\ln\left|\frac{\sqrt{V^2-\alpha^2}t+\sqrt{V^2t^2-x^2}}{\sqrt{V^2-\alpha^2}t-\sqrt{V^2t^2-x^2}}\right|$$
$$+\frac{x}{\alpha}\ln\left|\frac{\alpha\sqrt{V^2t^2-x^2}-\sqrt{V^2-\alpha^2}x}{\alpha\sqrt{V^2t^2-x^2}+\sqrt{V^2-\alpha^2}x}\right|]\},\qquad |x|<Vt \tag{30}$$

3.1.2 With all conditions remaining the same as that considered in the previous example except that the applied loads become a unit step load Pt.The boundary conditions will be as follows:

$$\tau_{zy}=-PH(x),\qquad |x|<Vt$$
$$w=0,\qquad |x|>Vt \tag{31}$$

where $H(x)$ is a unit step (Heavyside) function, with $H'(x)=\delta(x)$.

In which $L=1$. By application of Eqs. (17) and (19), the first of Eq. (31) can be rewritten as:

$$\mathrm{Re}[T(\tau)f(\tau)]=-PtH'(x)=P\delta(\tau)\ ,\qquad |\tau|<V \tag{32}$$

From the above formula, a unique solution of $f(\tau)$ can be deduced in the following form[7]:

$$f(\tau)=A\tau^n(V^2-\tau^2)^{-3/2} \tag{33}$$

where A is an unknown constant with n being an unknown index.

Putting Eq. (33) into (32), $n=-1$ is determined:

At $\tau\to 0$, from Eqs. (32), (33) and (5) constant A will be determined:

$$A=\frac{-PV^3}{\pi\sqrt{C_{55}/C_{44}}} \tag{34}$$

Substituting Eq. (33) into (19) and (17) for $y=0$ to render the stress τ_{yz}, the displacement w and dynamic stress intensity factor $K_3(t)$, respectively.

$$\tau_{yz}(x,0,t)=\mathrm{Re}\int_{\infty}^{x/t}-\frac{A\sqrt{(C_{55}-\rho\tau^2)/C_{44}}}{\tau^2(\tau^2-V^2)^{3/2}}d\tau\ ,\qquad |x|>Vt \tag{35}$$

$$w=\frac{1}{C_{44}}\mathrm{Re}\int_0^t dt\int_{\infty}^{\tau}f(\tau)d\tau=\frac{A}{C_{44}}\mathrm{Re}\int_0^t[\int_{\infty}^{\tau}\frac{1}{\tau\cdot(V^2-\tau^2)^{3/2}}d\tau]dt$$
$$=\mathrm{Re}\int_{\infty}^{\frac{x}{t}}\frac{A}{C_{44}V^2}\cdot[\frac{1}{\sqrt{V^2-\tau^2}}-\frac{1}{V}\ln\frac{V+\sqrt{V^2-\tau^2}}{\tau}]\cdot(\frac{-x}{\tau^2})d\tau$$

$$= \frac{-Ax}{C_{44}V^2}\mathrm{Re}[\frac{-\sqrt{V^2-\tau^2}}{V^2\tau} - \frac{1}{V\tau}(\frac{\sqrt{V^2-\tau^2}}{V} + \ln\frac{V+\sqrt{V^2-\tau^2}}{\tau})]\Big|_{\infty}^{x/t}$$

$$= \frac{2A}{C_{44}V^4}\sqrt{V^2t^2-x^2} + \frac{At}{C_{44}V^3}\ln\left|\frac{Vt+\sqrt{V^2t^2-x^2}}{x}\right|] \ , \qquad |x| < Vt \qquad (36)$$

$$K_3(t) = \lim_{x\to Vt}\sqrt{2\pi(x-Vt)}\ \mathrm{Re}\int_{\infty}^{x/t} -\frac{A\sqrt{(C_{55}-\rho\tau^2)/C_{44}}}{\tau^2(\tau^2-V^2)^{3/2}}d\tau$$

$$= \frac{-A\sqrt{\pi t(C_{55}-\rho V^2)/C_{44}}}{V^{7/2}} \qquad (37)$$

The limiting value of the above is also be gained by means of literature[20].

3.2 Displacements are homogeneous

3.2.1 Postulating at the initial moment $t=0$, a crack abruptly occurs under the action of concentrated load P situating at the origin of the coordinates and begins to propagate symmetrically in both directions of $x-$axis with a constant velocity V. At the half-plane of $y=0$, the boundary conditions are given by

$$\tau_{yz} = -P\delta(x)\,, \qquad |x| < Vt$$

$$w = 0\,, \qquad |x| > Vt \qquad (38)$$

In which $L=1$, utilizing Eqs. (16) and (19), the first of the boundary conditions can be rewritten as follows:

$$\mathrm{Re}[T(\tau)f(\tau)] = -Pt\delta(x) = -P\delta(\tau) \qquad |\tau| < V \qquad (39)$$

In terms of the above formula a unique solution of $f(\tau)$ can be deducted as:

$$f(\tau) = A_1\tau^n(V^2-\tau^2)^{-1/2} \qquad (40)$$

where A_1 is an unknown constant, and n is an unknown index.

Substituting Eq. (40) into (39), the constant $n=-1$ can be ensured.

At $\tau\to 0$, applying Eqs. (39), (40) and (5) the constant A_1 is also found as:

$$A_1 = \frac{PV}{\pi\sqrt{C_{55}/C_{44}}} \qquad (41)$$

Putting Eq (40) into Eqs.(19) and (16) for $y=0$, there results the stress τ_{yz}, the displacement w and dynamic stress intensity factor $K_3(t)$ respectively.

$$\tau_{yz}(x,0,t) = \frac{A_1\sqrt{(C_{55}-\rho\tau^2)/C_{44}}}{\tau\sqrt{x^2-V^2t^2}}\,, \qquad |x| > Vt \qquad (42)$$

$$w = \frac{1}{C_{44}}\cdot\mathrm{Re}\int_{\infty}^{x/t}\frac{A_1}{\tau(V^2-\tau^2)^{1/2}}d\tau = \frac{A_1}{C_{44}}\cdot\mathrm{Re}[\frac{1}{V}\ln\left|\frac{V+\sqrt{V^2-\tau^2}}{\tau}\right|]\Big|_{\infty}^{x/t}$$

$$= \frac{A_1}{VC_{44}}\ln\left|\frac{Vt+\sqrt{V^2t^2-x^2}}{x}\right| \ , \qquad |x| < Vt \qquad (43)$$

$$K_3(t) = \frac{A_1\sqrt{\pi(C_{55}-\rho V^2)/C_{44}}}{V\sqrt{Vt}} \tag{44}$$

3.2.2 With all conditions remaining the same as that considered in the previous example, the point load P which extends with a constant velocity $\beta < V$ along the positive directions of $x-$axis.The boundary conditions will be as follows:

$$\tau_{yz} = -P\delta(x-\beta t), \qquad |x| < Vt$$
$$w = 0, \qquad |x| > Vt \tag{45}$$

In which $L=1$, using Eqs. (16) and (19), the first formula of Eq. (45) can be rewritten as:

$$\mathrm{Re}[T(\tau)f(\tau)] = -Pt\delta(x-\beta t) = -P\delta(\tau-\beta), \qquad |\tau| < V \tag{46}$$

From the above formula a unique solution of $f(\tau)$ must suffice the following modality:

$$f(\tau) = A_1\tau^n(\tau-\beta)^{-1}(V^2-\tau^2)^{-1/2} \tag{47}$$

where A_1 is an unknown constant and n is an unknown index.

Now substituting Eq. (47) into (46), the constant $n=0$ is ascertained

At $\tau \to \beta$, utilizing Eqs. (39), (40) and (5) the constant A_1 will be gained:

$$A_1 = \frac{-P(V^2-\beta^2)^{1/2}}{\pi\sqrt{(C_{55}-\rho\beta^2)/C_{44}}}\,. \tag{48}$$

Then substituting Eq (40) into Eqs.(19) and (16) for $y=0$, the stress τ_{yz}, displacement w and the stress intensity factor $K_3(t)$ are obtained, respectively:

$$\tau_{yz}(x,0,t) = \frac{A_1\sqrt{(C_{55}-\rho\tau^2)/C_{44}}}{(\tau-\beta)\sqrt{x^2-V^2t^2}}, \qquad |x| > Vt \tag{49}$$

$$w = w^0 = \frac{1}{C_{44}}\mathrm{Re}\int_\infty^{x/t}\frac{A_1}{(\tau-\beta)(V^2-\tau^2)^{1/2}}d\tau$$
$$= \frac{A_1}{C_{44}}[\mathrm{Re}\frac{1}{\sqrt{(V^2-\beta^2)}}\ln\left|\frac{V^2-\beta\tau-\sqrt{(V^2-\beta^2)(V^2-\tau^2)}}{V(\beta-\tau)}\right|\cdot]\Bigg|_\infty^{\frac{x}{t}}$$
$$= \frac{A_1}{C_{44}\sqrt{(V^2-\beta^2)}}\ln\left|\frac{V^2t-\beta x+\sqrt{(V^2-\beta^2)(V^2t^2-x^2)}}{V(x-\beta t)}\right|, \qquad |x| < Vt \tag{50}$$

$$K_3(t) = \frac{A_1\sqrt{\pi(C_{55}-\rho V^2)/C_{44}}}{(V-\beta)\sqrt{Vt}}, \tag{51}$$

4. Conclusion

Applying the relevant representation: $f(x, y, t) = t^n f(x/t, y/t)$,just n is an integer number; the problem dealt with will be transformed into homogeneous functions of zeroth dimension, i.e. homogeneous functions. All satisfying this function relationship are settled by Eqs. (16), (17) and (19) according to the type of homogeneous functions corresponding to variable τ. This method is not only utilized in elastodynamics, but also in elastostatics, so much as in the else domain.

The concrete solutions of dynamic propagation problems concerning mode III crack were attained by the approaches of the self-similar functions. This is regarded as the analogous class of dynamic problem of the elasticity theory. The method of solution is based exclusively on techniques of analytical-function theory and is straightforward and compendious. This has comparatively reduced the amount of the calculative work needed to resolve such a crack problem.

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