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An Approximation for the Activity Duration Distribution, Supporting Original PERT

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Abstract

The Program Evaluation and Review Technique (PERT) has been widely used as a tool for project management. In PERT analysis the activity time distribution is assumed to be a beta distribution, and the mean and variance of the activity time are estimated. The activity mean and variance are very useful to find the expected project duration and variance of the critical path. By generalizing the assumption on parameters in original PERT an approximation for the mean and variance of a PERT activity duration is proposed and by comparison with numerical case it is shown that the mean and variance of PERT activity duration in this proposed method and original PERT are approximately equal. This supports that the original PERT estimates are valid in generalized case also.

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1. Introduction

A project is classically defined as a set of activities which must be performed according to some precedence constraints requiring that some activities can not start before the completion of some others. When duration of the activities are well known, critical path method [15], provide the minimal project duration and identify the critical paths. In real world, the durations of particular project activities can not be precisely defined. This is the way the original PERT has been developed [9]. PERT is the most widely used management technique for planning and coordinating large scale projects [2,6,20,21,22,23,26]. Since estimation of operation times of activities in a project network is difficult, therefore it is important to compute the variance of the project completion time in a network [2,3]. The Beta distribution has been applied in simulation[1] and in PERT analysis to model variable activity times [9,18]. The creators of PERT [9] considered beta distribution

$$f_y(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(y-a)^{\alpha-1}(b-y)^{\beta-1}}{(b-a)^{\alpha+\beta-1}}, a < y < b, \alpha, \beta > 0. \quad (1)$$

as an adequate distribution of the activity duration y where α and β are parameters of the beta distribution and the interval (a,b) is the domain of y. They suggested the estimates of the mean and variance values

$$\mu = \frac{1}{6}(a + 4m + b), \quad (2)$$

$$\sigma^2 = \frac{1}{36}(b-a)^2, \quad (3)$$

where a,m and b are the optimistic, most likely and pessimistic activity duration estimates respectively. In PERT, when a little sample information is available to ‘fit’ the distribution a,m and b are subjectively determined. In 1986 Sasieni [19] pointed out where the mean equation (2) came from. T.K. Little field and P.H. Randolph [24] answered that using four assumptions, that the activity duration distribution is beta , that the estimates of a,m and b are good, that equation (3) holds and so equation (2) holds good. C. Gallagher [5] also answered by approximating the same equation assuming the duration has a Beta distribution with α+β=4. Further Farnum and Stanton [12] and Golenko-Ginzburg [7] developed further steps to refine the approximations. Since then, numerous authors have participated in extending the work as Beta [8], some using extreme value theory [4], some defending particular distribution [25], and some giving accurate estimates of mean and variance [11],[16].

In this paper, an improvement to Ginzberg [14] approximation for the mean and variance of a PERT activity time is proposed by means of reasonable assumption , and it is practically shown by taking Milwaukee General hospital project [13].

2 Original PERT and Ginzburg's PERT approximations

2.1 Traditional PERT approximation

In PERT, when a little formal sample information is available to fit the distribution a,m,b are subjectively determined. Therefore, by using

$$x = \frac{y-a}{b-a}.$$

We can transform the density function (1) to a standard form

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha, \beta > 0, \tag{4}$$

When this 'standardization' is done, then the following simple relations holds.

$$\mu_x = \frac{\mu_y - a}{b-a}, \quad \sigma_x = \frac{\sigma_y}{b-a}, m_x = \frac{m_y - a}{b-a} \tag{5}$$

By letting α-1 = p , β-1 = q. The density function (4) becomes

$$f(x) = \frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)} x^{p}(1-x)^{q}, 0 < x < 1, p, q > -1, \tag{6}$$

with the mean, variance and mode as follows :

$$\mu_x = \mu_x(p,q) = \frac{p+1}{p+q+2}, \tag{7}$$

$$\sigma_x^2 = \sigma_x^2(p,q) = \frac{(p+1)(q+1)}{(p+q+2)^2(p+q+3)}, \tag{8}$$

$$m_x = m_x(p,q) = \frac{p}{p+q}. \tag{9}$$

From (6) and (9) we obtain

$$f(x) = \frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)} x^{p}(1-x)^{p(1/m_x - 1)}. \tag{10}$$

Thus value m_x, being obtained from the analyst's subjective knowledge, indicates the density function. On the basis of statistical analysis and some other intuitive arguments, the creators of PERT assumed [18] that

$$\alpha + \beta = 4 \tag{11}$$

It is from that assertion that estimates (2) and (3) were finally obtained, according to (6) – (9).

2.2 Ginzburg's [14] PERT approximation

Little Field et.al [24] showed that using the least square regression technique, that the mean activity duration can be approximated by

$$\hat{\mu}_x = \frac{1}{6}(4m_x + 1) \qquad (12)$$

Here the value m_x determined by a specialist is transformed into an estimated mean value of x, $\hat{\mu}_x$,using the relation(11),(12) and standard deviation of x namely

$$\sigma_x = \frac{1}{6} \qquad (13)$$

On the basis of (13) Farnum et.al [12] argued that σ_x is not much affected by α and β and therefore

$$\sigma_x(\alpha, \beta) \cong \sigma_x(\alpha - 1, \beta - 1) \cong 1/6 \qquad (14)$$

Ginzberg [14] showed that the PERT assumption (11) is poor in the tails of the distribution and proposed the following modified formula for the mean and the variance of activity duration assuming

$$p + q = z(\text{constant}) \qquad (15)$$

obtained

$$\mu_x = \frac{9m_x + 2}{13}, \qquad (16)$$

$$\sigma_x^2 = \frac{1}{1268}(22 + 81m_x - 81m_x^2) \qquad (17)$$

For the general beta distribution of the activity time, estimates (16) and (17) are transformed to

$$\mu_y = \frac{2a + 9m + 2b}{13} \qquad (18)$$

$$\sigma_y^2 = \frac{(b-a)^2}{1268}\left[22 + 81\frac{m-a}{b-a} - 81\left(\frac{m-a}{b-a}\right)^2\right]. \qquad (19)$$

Thus estimates (2) and (3) are replaced by estimates (18) and (19).

By assuming that p =1, q = 2 and m= $\frac{2a+b}{3}$, he further improved these estimates when the estimated mode of the activity time is located in the tail of the distribution as follows:

$$\mu_y = \frac{8a + 5b}{13} \qquad (20)$$

$$\sigma_y^2 = \frac{10(b-a)^2}{317} \cong (b-a)^2/32 \qquad (21)$$

3. Proposed approximation

From Farnum et al [12] and Ginzberg [14] it can be seen that the PERT approximations are based on various assumptions on beta parameters. In this paper, we propose a new approximation for estimating the mean and variance without imposing any restrictions on the value of beta parameters α and β and this proposal does not violate the PERT assumptions. In order to make the assumption more flexible, Ginzberg [14] assumed that the sum p+q in (6) is approximately constant but not predetermined; i.e., relation

$$p + q \cong k \text{ (constant)} \tag{22}$$

By standard PERT approximation, we have

$$\sigma_x(p,q) \cong \frac{1}{6} \tag{23}$$

Farnum et al. [12] argued that on the basis of PERT assumption (1) the standard deviation σ_x is not affected much by p,q.
Assume that

$$\sigma_x(p+1,q+1) \cong \frac{1}{6} \text{ or } \sigma_x^2(p+1,q+1) \cong \frac{1}{36} \tag{24}$$

where

$$\sigma_x^2(p+1,q+1) = \frac{(p+2)(q+2)}{(p+q+4)^2(p+q+5)} \tag{25}$$

is the variance of x with beta parameters α and β.
From (9) we obtain

$$p = km_x\,, \tag{26}$$

and values μ_x and σ_x^2 are

$$\mu_x(m_x) = \frac{km_x+1}{k+2}, \tag{27}$$

$$\sigma_x^2(m_x) = \frac{(km_x+2)(k-km_x+2)}{(k+4)^2(k+5)} \tag{28}$$

Since the average value of variance $\sigma_x^2(m_x)$ for $0 < m_x < 1$ has to be equal to 1/36; i.e.,

$$\int_0^1 \sigma_x^2(m_x)\,dm_x = \frac{1}{36}. \tag{29}$$

Substituting (15) in (16), integrating and solving (16) for k, we get k = 3.4. Approximating k to 3.4 and getting

$$p = 3.4\, m_x,\ \ q = 3.4\,(1-m_x) \tag{30}$$

from (12) and (13) , finally obtain

$$f(x) = \frac{\Gamma(6.5)}{\Gamma(4.5m_x+1)\Gamma(5.5-4.5m_x)}\, x^{4.5m_x}(1-x)^{4.5(1-m_x)}\cdot, \tag{31}$$

with the mean and variance

$$\mu_x = \frac{17m_x + 5}{27} \tag{32}$$

$$\sigma_x^2 = \frac{(17m_x + 10)(27 - 17m_x)}{2300} \tag{33}$$

For the general beta distribution of the activity time, estimates (18) and (19) are transformed to

$$\mu_y = \frac{5a + 17m + 5b}{27}, \tag{34}$$

$$\sigma_y^2 = \frac{(17m - 27a + 10b)(27b - 10a - 17m)}{2300}. \tag{35}$$

Analyzing over a large number of activities selected from different projects [10,17,18], showed that the most likely time can be taken as $\frac{2a+b}{3}$.

Substituting m = $\frac{2a+b}{3}$ in eq (35) and simplifying

$$\sigma_y^2 = \frac{(b-a)^2}{35} \tag{36}$$

Thus estimates (2) and (3) are replaced by estimates (34) and (36).

Numerical Example:

Milwaukee General hospital project [13] is considered in table I. The data for activities is represented in table II including mean and variance estimates for original , Ginzburg and proposed approximations and its network diagram as in Fig.1. The estimated project duration has approximately same value by using the original and proposed methods.

Table I . Milwaukee General hospital project

Activity	Description	Immediate Predecessor	*a*	m	b
A	Build inernal components	---	1	2	3
B	Modify roof and floor	---	2	3	4
C	Construct collection stack	A	1	2	3
D	Pour concrete and install frame	A,B	2	4	6
E	Build high-temperature burner	C	1	4	7
F	Install pollution control system	C	1	2	9
G	Install air pollution device	D,E	3	4	11
H	Inspect and test	F,G	1	2	3

Table II. Mean and variance estimates

Activity	a	m	b	Original approximation		Ginzburg approximation		New approximation	
				μ	σ^2	μ	σ^2	μ	σ^2
A	1	2	3	2.0	0.1111	2.0	0.125	2.0	0.1143
B	2	3	4	3.0	0.1111	3.0	0.125	3.0	0.1143
C	1	2	3	2.0	0.1111	2.0	0.125	2.0	0.1143
D	2	4	6	4.0	0.4444	4.0	0.500	4.0	0.4571
E	1	4	7	4.0	1.0000	4.0	1.125	4.0	1.0286
F	1	2	9	3.0	1.7778	2.9	2.000	3.1	1.8286
G	3	4	11	5.0	1.7778	4.9	2.000	5.1	1.8286
H	1	2	3	2.0	0.1111	2.0	0.125	2.0	0.1143

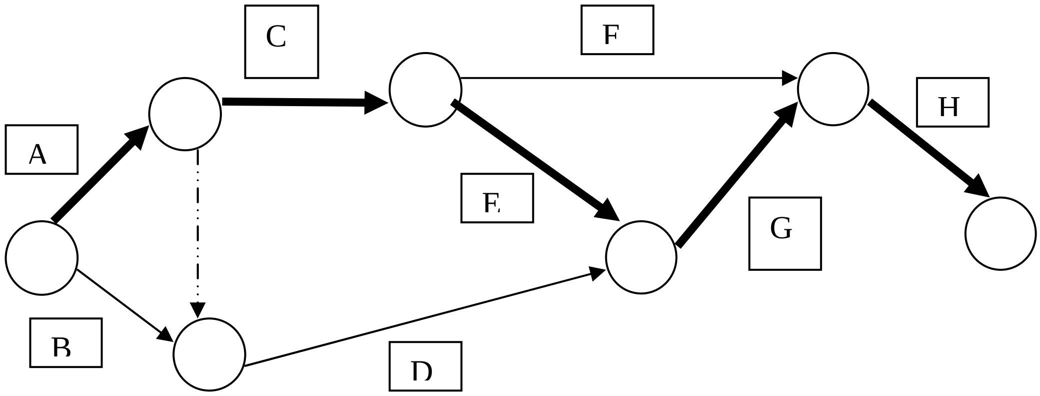

Fig.1 : Network diagram

Conclusions

By comparison with actual values, it was shown that the proposed approximations are accurate compared to the existing ones. Our improvement is free from the assumption on parameters α ,β. The PERT estimates of the mean and variance in PERT analysis can be replaced by our estimates. The value of mean and variance in original PERT are not only valid for restrictive condition p + q = 4 but also for generalized condition p + q = k (k is a constant) and it was observed by considering a numerical case.

References

[1] A.M.Law and W.D.Keton, Simulation Modelling and Analysis, Mc Graw Hill, New York, (1982).

[2] A.Azaran, C.Perkgoz and M.Sakawa, A genetic algorithm approach in the time-cost trade off in PERT Networks, Applied Mathematics and computation 168 (2005), 1317-1339.

[3] A. Azaron, H. Katagiri, M.Sakawa, K.Kato and A.Memariani. A , Multi-objective resource allocation problem in PERT networks, European Journal of Operational Research , 172 (2006), 838 – 854.

[4] B.Dodin and M.Sirvanci, Stochastic networks and the extreme value distribution, Comp. J. Oprs. Res. ,17 (1990), 397-409.

[5] C.Gallagher, A note on PERT assumptions, Management Science, 33 (1987), 1360.

[6] D.Dubois, H.Fargier and V.Galvaganon, On latest starting times and floats in task networks with ill-known duration, European Journal of operational Research, 107 (2003), 266-280.

[7] D.Golenko- Ginzburg , PERT assumptions revisited, Omega ,17 (1989), 393-396.

[8] D.Sculli and K.L,Wong , The maximum and sum of two Beta variables and the analysis of PERT networks. Omega, 13 (1985). 233-240.

[9] D.G.Malcolm, J.H. Roseboom and C.E.Clark, Application of a technique of research and development program evaluation, operations research, 7 (1959), 646-669.

[10] D.J.Welsh , Errors introduced by a PERT assumptions , Operations Research, 13 (1965) , 141-143.

[11] D.M. Keefer and W.A.Verdini, Better estimation of PERT activity time parameters, Mgmt. Sci. 39 (1993), 1086-1091.

[12].Farnum N R, Stanton LW., Some results concerning the estimation of beta distribution parameters in PERT. Journal of the Operations Research Society, 38 (1987), 287-290.

[13] F. E.Williams,PERT Completion Times Revisited, INFORMS Transactions on Education, 6 (2005) .

[14] Golenko DI. (Ginzburg), On the distribution of activity time in PERT, Journal of the Operations Reseach Society, 39 (1988), 767-771.

[15] J.E.Kelley and M.R.Walker, Critical path planning and scheduling, Proc. Eastern Joint computer corp. 16 (1959), 160 -172.

[16] I.M.Premachandra, An approximation of the activity duration distribution in PERT, Computers and operations Research, 28 (2001), 443-452.

[17] J.Lukaszewicz, On the estimation of errors introduced by standard assumptions concerning the distribution of activity duration in PERT caluculation, Operations Research , 13 (1965), 326-327.

[18] K.R. Maccrimmon and C.A.Ryaveck , An analytical study of the PERT assumptions, Operations Research ,12 (1964) , 16-37.

[19] M.W. Sasieni, A note on PERT time, Management Science, 32 (1986), 1652- 1653.

[20] K.N. Jha AND K.C.Iyer, Critical determinants of project duration, International Journal of Project management , 24 (2006) , 314-322

[21] K.R.Graham, Critical chain, the theory of constraints applied to project management, International Journal of Project management 18 (2000), 173-177.

[22] S.M. Chen, T.H. Chang, Finding multiple possible critical paths using fuzzy PERT, IEEE Transactions on systems, man and cybernetics – part B : Cybernetics 31 (2001), 930-937.

[23] S.M.T.Falemi Ghomi and E.Teimouri, Path critical index and task critical index in PERT networks, European Journal of operations research, 141 (2002), 147-152.

[24] T.K. Little field and P.H. Randolph, Reply : An answer to sasieni's question on PERT times. Management Science, 33 (1987) , 1357-1359

[25] T.M. Williams, Practical use of distributions in network analysis, J. Opl. Res. Soc., 43(1992) , 265-270.

[26] W.C. Wang, Impact of soft logic on the probabilistic duration of construction projects, International Journal of Project management, 23 (2005), 600-610.

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