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# Weak and Strong Convergence Theorems for $k$-Strictly Pseudo-Contractive in Hilbert Space[1]

**Qinwei Fan[1], Minggang Yan[2], Yanrong Yu[1] and Rudong Chen[1]**

[1] Department of Mathematics, Tianjin Polytechnic University
Tianjin, P.R. China 300160
tjcrd@yahoo.com.cn (R. Chen)

[2] Adult Education College of Shangqiu Vocational and Technical College
Shangqiu, Henan, P.R. China, 476000

**Abstract.** Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$, and assume that $T_i : K \to H, i = 1, 2...N$ be a finite family of $k_i$-strictly pseudo-contractive mappings for some $0 \leq k_i \leq 1$ such that $\bigcap_{i=1}^{N} F(T_i) = \{x \in K : x = T_i x, i = 1, 2...N\} \neq \emptyset$. For the following iterative algorithm in $K$, for $x_1, x_1' \in K$ and $u \in K$,

$$\begin{cases} y_n = P_K[kx_n + (1-k)\Sigma_{i=1}^{N} \lambda_i T_i x_n] \\ x_{n+1} = \beta_n x_n + (1-\beta_n) y_n \end{cases}$$

and

$$\begin{cases} y_n' = P_K[\alpha_n' x_n' + (1-\alpha_n')\Sigma_{i=1}^{N} \lambda_i T_i x_n'] \\ x_{n+1}' = \beta_n' u + (1-\beta_n') y_n' \end{cases}$$

$P_K$ is the metric projection of $H$ onto $K$, $\{\alpha_n'\}$ and $\{\beta_n'\}$ are sequences in (0,1) satisfying appropriate conditions, we proved that $\{x_n\}$ and $\{x_n'\}$ respectively converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$. Our results improve and extend the results announced by Genaro L.A.and H.K.Xu [Iterative methods for strict pseudo-contractions in Hilbert spaces, Nonl.Anal.67(2007) 2258-2271], T.H.Kim and H.K.Xu [Strong convergence of modified Mann iterations, Nonlinear Anal.61(2005)51-60] and G.Marino and H.K.Xu [Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J.Math.Anal.Appl.329(2007)336-346].

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## 1. Introduction

Let $K$ be a nonempty closed convex subset of a Hilbert space $H$. We use $F(T)$ to denote the fixed point set of $T$ and $P_K$ to denote the metric projection of $H$ onto $K$. Recall that a mapping $T : K \to H$ is said to be a $k$-strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in K \tag{1.1}$$

Note that the class of $k$-strictly pseudo-contractions includes strictly the class of nonexpansive mappings which are mappings $T$ on $K$ such that

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in K.$$

When $k = 0$, $T$ is said to be nonexpansive, and it is said to be pseudo-contractive if $k = 1$. $T$ is said to be strongly pseudo-contractive if there exist a positive constant $\lambda \in (0, 1)$ such taht $T - \lambda I$ is pseudo-contractive. Clearly, the class of $k$ strict pseudo-contraction falls into the one between classes of nonexpansive mappings and pseudo-contractions.We remark also that the class of strongly pseudo-contractive mappings is independent of the class of $k$ strict pseudo-contraction (see[2, 3, 5]).

It is very clear that, in a real Hilbert space $H$, (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(x - Tx) - (y - Ty)\|^2, \forall x, y \in K. \tag{1.2}$$

$T$ is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 \tag{1.3}$$

$T$ is strongly pseudo-contractive if and only if there exists a positive constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq (1 - \lambda)\|x - y\|^2, \forall x, y \in K. \tag{1.4}$$

Recall that the normal Mann's iterative algorithm was introduced by Mann (see[1]) in 1953. Since then, construction of fixed points for nonexpansive mapping have been extensively investigated (see[4, 8, 9, 12, 14, 17, 18, 19, 20, 21]) and $k$ strict pseudo-contractions via the normal Mann's iterative algorithm has been extensively investigated by many authors (see[1, 7, 13, 15, 16, 22, 23]).

The normal Mann's iterative algorithm generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in K, x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, n \geq 1 \tag{1.5}$$

In 1967,Browder and Petryshyn [5] established the first convergence result for $k$-pseudo-contractive self mappings in real Hilbert spaces. They prove weak and strong convergence theorems by using algorithm (1.5) with a constant control sequence $\{\alpha_n\} \equiv \alpha$ for all $n$. Afterward, Rhoades [6] generalized in part the corresponding results in [5] in the sense that a variable control sequence $\{\alpha_n\}$ was taken into consideration. Under the assumption that the domain of mapping $T$ is compact convex, he established a strong convergence theorem by using algorithm (1.5) with a control sequence $\{\alpha_n\}$ satisfying the conditions $\alpha_1 = 1, 0 < \alpha_n < 1, \Sigma_{n=1}^{\infty}\alpha_n = \infty$ and the $\limsup_{n\to\infty} \alpha_n = \alpha < 1 - k$. However, without the compact assumption on the domain of mapping $T$, in general, one cannot expect to infer any weak convergence results from Rhoades' convergence theorem.

Very recently, G.L.Acedo and Xu [24] have proved a weak convergence theorem by using algorithm

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\Sigma_{i=1}^{N}\lambda_i T_i x_n \tag{1.6}$$

with certain control conditions.

In this paper, motivated by G.L.Acedo and Xu [24] and the above results, we study the following iteration process (1.7) and (1.8), for $x_1 \in K$,

$$\begin{cases} y_n = P_K[kx_n + (1-k)\Sigma_{i=1}^{N}\lambda_i T_i x_n] \\ x_{n+1} = \beta_n x_n + (1-\beta_n) y_n \end{cases} \tag{1.7}$$

and

$$\begin{cases} y'_n = P_K[\alpha'_n x'_n + (1-\alpha'_n)\Sigma_{i=1}^{N}\lambda_i T_i x'_n] \\ x'_{n+1} = \beta'_n u + (1-\beta'_n) y'_n \end{cases} \tag{1.8}$$

$P_K$ is the metric projection of $H$ onto $K$, $\{\alpha'_n\}$ and $\{\beta'_n\}$ are sequences in (0,1) satisfying appropriate conditions, we proved that $\{x_n\}$ and $\{x'_n\}$ respectively converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$. Our results extend and improve the corresponding results in [19, 23, 24].

We will use the following notation:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_\omega(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak $\omega$-limit set of $\{x_n\}$.

## 2. Preliminaries

We need some Lemmas and Propositions in real Hilbert space $H$, which are listed as follow:

**Lemma 2.1.** (Marino and Xu [23]) Let $H$ be a real Hilbert space, there hold the following identities.

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H$

(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2, \forall t \in [0, 1], \forall x, y \in H$

**Lemma 2.2.** (Demiclosedness Principle). If $T$ is $k$-strict pseudo-contraction on closed convex subset $K$ of a real Hilbert space $H$, then $I-T$ is demiclosed at any point $y \in H$.

**Lemma 2.3.** (Xu [23]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there satisfies $\langle x-y, y-z\rangle \geq 0 \ \forall z \in C$.

**Lemma 2.4.** (see,e.g. Liu [11]).Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfies the condition

$$a_{n+1} \leq (1-t_n)a_n + b_n + 0(t_n), n \geq 1,$$

where $\{t_n\}$ satisfies the restrictions:

(i) $t_n \to 0 (n \to \infty)$;

(ii) $\Sigma_{n=1}^{\infty} b_n < \infty$;

(iii) $\Sigma_{n=1}^{\infty} t_n = \infty$.

then $a_n \to 0$ as $n \to \infty$.

**Proposition 2.5.** Assume $K$ is closed convex subset of Hilbert space $H$.

(i) Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : K \to H$ is a $k_i$-strict pseudo-contraction for some $0 \leq k_i < 1$. Assume $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\Sigma_{i=1}^N \lambda_i = 1$ .Then $\Sigma_{i=1}^N \lambda_i T_i$ is a $k$-strict pseudo-contraction, with $k = max\{k_i : 1 \leq i \leq N\}$.

(ii) Let $\{T_i\}_{i=1}^N$ and $\{\lambda_i\}_{i=1}^N$ be given as in (i) above. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point. Then

$$Fix(\Sigma_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N Fix(T_i).$$

**Proof.** To prove (i), we only need to consider the case of $N = 2$. the general case can be proved by induction. Set $A = (1-\lambda)T_1 + \lambda T_2$, where $\lambda \in (0,1)$ and for $i = 1, 2$, $T_i$ is a $k_i$-strict pseudo-contraction. Set $k = max\{k_1, k_2\}$. We now to prove that $A$ is a $k$-strict pseudo-contraction, by lemma 2.1(ii) we have

$$\begin{aligned} &\|(I-A)x-(I-A)y\|^2 \\ &= \|(1-\lambda)[(I-T_1)x-(I-T_1)y] + \lambda[(I-T_2)x-(I-T_2)y]\|^2 \\ &= (1-\lambda)\|(I-T_1)x-(I-T_1)y\|^2 + \lambda\|(I-T_2)x-(I-T_2)y\|^2 \\ &-\lambda(1-\lambda)\|[(I-T_1)x-(I-T_1)y]-[(I-T_2)x-(I-T_2)y\|^2 \end{aligned} \tag{2.1}$$

and observe that $T : K \to H$ is a $k$-strict pseudo-contraction if and only if there holds the following

$$\langle x-y, (I-T)x-(I-T)y\rangle \geq \frac{1-k}{2}\|(I-T)x-(I-T)y\|^2 \tag{2.2}$$

Indeed, putting $V = I-T$, we see that (1.1) holds if and only if

$$\|(I-V)x-(I-V)y\|^2 \leq \|x-y\|^2 + k\|Vx-Vy\|^2, \forall x, y \in K \tag{2.3}$$

But by lemma 2.1(i) we have

$$\|(I-V)x-(I-V)y\|^2 = \|x-y\|^2 - 2\langle x-y, Vx-Vy\rangle + \|Vx-Vy\|^2 \quad (2.4)$$

substituting (2.4) into (2.3), we obtain (2.2). Noticing (2.1), we have

$$\begin{aligned}
&\langle x-y,(I-A)x-(I-A)y\rangle \\
&= (1-\lambda)\langle x-y,(I-T_1)x-(I-T_1)y\rangle + \lambda\langle x-y,(I-T_2)x-(I-T_2)y\rangle \\
&\geq \tfrac{1-k}{2}[(1-\lambda)\|(I-T_1)x-(I-T_1)y\|^2 + \lambda\|(I-T_2)x-(I-T_2)y\|^2] \\
&\geq \tfrac{1-k}{2}\|(I-A)x-(I-A)y\|^2
\end{aligned}$$

Hence $A$ is a $k$-strict pseudo-contraction

To prove (ii), we can assume $N = 2$. It suffices to prove that $Fix(A) \subset Fix(T_1)\cap Fix(T_2)$, where $A = (1-\lambda)T_1 + \lambda T_2$,with $0 < \lambda < 1$. Let $x \in Fix(A)$ and write $A_1 = I - T_1$ and $A_2 = I - T_2$.

Take $z \in Fix(T_1) \cap Fix(T_2)$ to deduce that

$$\begin{aligned}
\|z-x\|^2 &= \|(1-\lambda)(z-T_1x)+\lambda(z-T_2x)\|^2 \\
&= (1-\lambda)\|z-T_1x\|^2 + \lambda\|z-T_2x\|^2 - \lambda(1-\lambda)\|T_1x-T_2x\|^2 \\
&\leq (1-\lambda)(\|z-x\|^2 + k\|x-T_1x\|^2) \\
&\quad +\lambda(\|z-x\|^2 + k\|x-T_2x\|^2) - \lambda(1-\lambda)\|T_1x-T_2x\|^2 \\
&= \|z-x\|^2 + k[(1-\lambda)\|A_1x\|^2 + \lambda\|A_2x\|^2] - \lambda(1-\lambda)\|A_1x-A_2x\|^2.
\end{aligned}$$

It follows that

$$\lambda(1-\lambda)\|A_1x-A_2x\|^2 \leq k[(1-\lambda)\|A_1x\|^2 + \lambda\|A_2x\|^2] \quad (2.5)$$

Since $(1-\lambda)A_1x + \lambda A_2x = 0$, we have

$$(1-\lambda)\|A_1x\|^2 + \lambda\|A_2x\|^2 = \lambda(1-\lambda)\|A_1x-A_2x\|^2$$

This together with (2.5) implies that

$$(1-k)\lambda(1-\lambda)\|A_1x-A_2x\|^2 \leq 0$$

Since $0 < \lambda < 1$ and $k < 1$, we get $\|A_1x - A_2x\| = 0$ which implies $T_1x = T_2x$ which in turns implies that $T_1x = T_2x = x$ since $(1-\lambda)T_1x + \lambda T_2x = x$, Thus, $x \in Fix(T_1) \cap Fix(T_2)$. The general case can be proved by induction, this completes the proof.

**Proposition 2.6.** If $T : K \to H$ is a $k$-strict pseudo-contraction, then $T$ is L-Lipschitzian mapping.

**Proof.** By (1.2), for all $x, y \in K$, we have that

$$\begin{aligned}
\tfrac{1-k}{2}\|(I-T)x-(I-T)y\|^2 &\leq \langle (I-T)x-(I-T)y, x-y\rangle \\
&\leq \|(I-T)x-(I-T)y\|\|x-y\|
\end{aligned}$$

it follows that

$$\begin{aligned}
\|Tx-Ty\| - \|x-y\| &\leq \|(I-T)x-(I-T)y\| \\
&\leq \tfrac{2}{1-k}\|x-y\|,
\end{aligned}$$

i.e.,

$$\|Tx - Ty\| \leq L\|x - y\|, \ L = \frac{3-k}{1-k}.$$

**Proposition 2.7.** If $T$ is a $k$-strict pseudo-contraction on a closed convex subset $K$ of a real Hilbert space $H$, then the fixed point set $F(T)$ of $T$ is closed convex so that the projection $P_{F(T)}$ is well defined.

**Proof.** Since $T : K \to H$ is Lipschitzian, we see that $F(T)$ is closed. Thus, we only need to see that $F(T)$ is convex,take $p, q \in F(T)$, and $t \in (0, 1)$. Put $z = (1 - t)p + tq$. by using (1.2) we have

$$\langle z_t - Tz_t, z_t - p\rangle \geq \frac{1-k}{2}\|z_t - Tz_t\|^2 \tag{2.6}$$

and

$$\langle z_t - Tz_t, z_t - q\rangle \geq \frac{1-k}{2}\|z_t - Tz_t\|^2 \tag{2.7}$$

Noting that $z_t - p = t(q - p)$ and $z_t - q = (1 - t)(p - q)$, substituting these equalities into (2.6) and (2.7), respectively, we get

$$t\langle z_t - Tz_t, q - p\rangle \geq \frac{1-k}{2}\|z_t - Tz_t\|^2 \tag{2.8}$$

and

$$(1-t)\langle z_t - Tz_t, p - q\rangle \geq \frac{1-k}{2}\|z_t - Tz_t\|^2 \tag{2.9}$$

Multiplied by $(1-t)$ and $t$, and added up on the both sides of (2.8) and (2.9), respectively, we have

$$\frac{1-k}{2}\|z_t - Tz_t\|^2 \leq 0,$$

which implies that $z_t \in F(T)$. This completes the proof.

**Proposition 2.8.** Let $T : K \to H$ be a $k$-strict pseudo-contraction with $F(T) \neq \emptyset$. Then, $F(P_K T) = F(T)$.

**Proof.** Clearly, $F(T) \subset F(P_K T)$. Thus, we only need to show the converse inclusion. Assume that $x = P_K Tx$; then, by lemma 2.1 and lemma 2.3, we have for $p \in F(T)$ that

$$\begin{aligned} \|Tx - p\|^2 &= \|Tx - x + x - p\|^2 \\ &= \|Tx - x\|^2 + 2\langle Tx - x, x - p\rangle + \|x - p\|^2 \\ &= \|Tx - x\|^2 + 2\langle Tx - P_K Tx, P_K Tx - p\rangle + \|x - p\|^2 \\ &\geq \|Tx - x\|^2 + \|x - p\|^2. \end{aligned} \tag{2.10}$$

On the other hand, by (1.1), we have

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2. \tag{2.11}$$

Combining (2.10) and (2.11) yields

$$(1-k)\|x - Tx\|^2 \leq 0$$

Therefore, $x \in F(T)$. This completes the proof.

**Proposition 2.9.** Let $T : K \to H$ be $k$-strict pseudo-contraction. Define $S : K \to H$ by $Sx = \alpha x + (1-\alpha)Tx$ for each $x \in K$. Then, as $\alpha \in [k, 1)$, $S$ is nonexpansive such that $F(S) = F(T)$.

**Proof.** For all $x, y \in K$, by lemma2.1(ii) and (1.1) we have

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|\alpha(x-y) + (1-\alpha)(Tx - Ty)\|^2 \\
&= \alpha\|x-y\|^2 + (1-\alpha)\|Tx - Ty\|^2 \\
&\quad -\alpha(1-\alpha)\|(x-y) - (Tx - Ty)\|^2 \\
&\leq \alpha\|x-y\|^2 + (1-\alpha)\|x-y\|^2 + k(1-\alpha)\|(x-y) - (Tx-Ty)\|^2 \\
&\quad -\alpha(1-\alpha)\|(x-y) - (Tx - Ty)\|^2 \\
&= \|x-y\|^2 - (\alpha - k)(1-\alpha)\|(x-y) - (Tx-Ty)\|^2 \\
&\leq \|x-y\|^2
\end{aligned}$$

which proves that $S : K \to H$ is nonexpansive. By the definition of $S$, we have $x - Sx = (1-\alpha)(x - Tx)$, and this means that $p = Sp$ if and only if $p = Tp$. This completes the proof.

## 3. Main Results

**Theorem 3.1.** Let $K$ be a nonempty closed convex subset of a Hilbert space $H$ and $T_i : K \to H$ be a $k_i$-strictly pseudo-contractive non-self mapping, for some $0 \leq k_i < 1$, $k = max\{k_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^N Fix(T_i)$ is nonempty. Let $\{x_n\}$ be generated by (1.7), i.e.,

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_K[kx_n + (1-k)\Sigma_{i=1}^N \lambda_i T_i x_n]$$

where $\beta_n = \frac{\alpha_n - k}{1-k}$, $\{\lambda_i\}_{i=1}^N$ is a finite sequence of positive numbers, such that $\Sigma_{i=1}^N \lambda_i = 1$ for all $1 \leq i \leq N$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [k, 1]$ and $\Sigma_{n=1}^\infty (\alpha_n - k)(1-\alpha_n) = \infty$, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.

**Proof.** Let $T$ be defined by $T = \Sigma_{i=1}^N \lambda_i T_i$, by proposition 2.5 (i),(ii) we know that $Fix(T) = \bigcap_{i=1}^N Fix(T_i)$ and $T$ is a $k$-strict pseudo-contraction on $K$, with $k = \max\{k_i : 1 \leq i \leq N$, define $S : K \to H$ by $Sx = kx + (1-k)Tx$.

By proposition 2.9, we know that $S : K \to H$ is nonexpansive and $F(S) = F(T)$. By our assumption on $T$, we know $F(T) \neq \emptyset$ and hence $F(S) \neq \emptyset$.

Since $S : K \to H$ is nonexpansive, then $S : K \to H$ is $k$-strict pseudo-contraction on $K$, where $k = 0$. By proposition 2.8, we see that $F(P_K S) = F(S) \neq \emptyset$.

Since $P_K : H \to K$ is nonexpansive, we conclude that $P_K S : K \to K$ is nonexpansive.

From the control condition on $\{\alpha_n\}$, we have

$$\Sigma_{n=1}^{\infty}\beta_n(1-\beta_n) = \frac{1}{(1-k)^2}\Sigma_{n=1}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty.$$

Then, by Theorem 2 given by Reich in [7] to deduce that $\{x_n\}$ converges weakly to a fixed point of $P_K S$.

Notice that $F(P_K S) = F(S) = F(T)$, we have the conclusion.

The proof is completed.

From Theorem 3.1, we can deduce Theorem 3.2 of Marino and Xu [24].

**Corollary 3.2.**(Xu [24]) Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $T_i : K \to K$, be a $k_i$-strict pseudo-contraction for some $0 \leq k_i < 1$. Let $k = max\{k_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^{N} Fix(T_i)$ is nonempty. Assume also $\{\lambda_i\}_{i=1}^{N}$ is a finite sequence of positive numbers, such that $\Sigma_{i=1}^{N}\lambda_i = 1$. Given $x_0 \in K$, let $\{x_n\}_0^{\infty}$ be the sequence generated by Mann's algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\Sigma_{i=1}^{N}\lambda_i T_i x_n$$

Assume the control sequence $\{\alpha_n\}_0^{\infty}$ is chosen so that $k < \alpha_n < 1$ for all $n$ and

$$\Sigma_{n=1}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty.$$

Then $\{x_n\}$ converges weakly to a common fixed point $\{T_i\}_1^N$.

**Proof.** We observe first that, for all $x \in K$.

$$P_K[kI + (1-k)\Sigma_{i=1}^{N}\lambda_i T_i]x = [kI + (1-k)\Sigma_{i=1}^{N}\lambda_i T_i]x$$

Since $T_i : K \to K$, thus $kI + (1-k)\Sigma_{i=1}^{N}\lambda_i T_i : K \to K$ is a self-mapping.

For given $\{\alpha_n\}$, by the choice of $\{\beta_n\}$, we get

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)\Sigma_{i=1}^{N}\lambda_i T_i x_n \\ &= [k + (1-k)\beta_n]x_n + (1-k)(1-\beta_n)\Sigma_{i=1}^{N}\lambda_i T_i x_n \\ &= \beta_n x_n + (1-\beta_n)[kx_n + (1-k)\Sigma_{i=1}^{N}\lambda_i T_i x_n] \\ &= \beta_n x_n + (1-\beta_n)P_K[kx_n + (1-k)\Sigma_{i=1}^{N}\lambda_i T_i x_n] \end{aligned}$$

Consequently, we conclude that $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_1^N$ by Theorem 3.1.

The proof is completed.

**Remark 3.3.** Theorem 3.1 and its Corollary mainly improves Xu [24] in the following senses:

(i) relaxing the restriction on $\{\alpha_n\}$ from $(k, 1)$ to $[k, 1]$;

(ii) from $k$-strict pseudo-contraction self-mapping to non-self mapping.

In order to get a strong convergence theorem, we modify the iterative algorithm for $k$-strict pseudo-contraction. We have the following theorem.

**Theorem 3.4.** Let $K$ be a nonempty closed convex subset of a Hilbert space $H$ and $T_i : K \to H$ be a $k_i$-strictly pseudo-contractive nonself-mapping, for some $0 \leq k_i < 1$, let $k = \max\{k_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^N Fix(T_i)$ is nonempty. Assume also for each $n$, $\{\lambda_i\}_{i=1}^N$ is a finite sequence of positive numbers, such that $\Sigma_{i=1}^N \lambda_i = 1$ for all $1 \leq i \leq N$. Given $u \in K$ and sequences $\{\alpha'_n\}$ and $\{\beta'_n\}$ in (0,1), satisfying control conditions: (i)$\Sigma_{n=1}^\infty \beta'_n = \infty$; $\beta'_n \to 0$, (ii) $k \leq \alpha'_n \leq b < 1$ for all $n \geq 1$, and (iii)$\Sigma_{n=1}^\infty |\alpha'_{n+1} - \alpha'_n| < \infty$, $\Sigma_{n=1}^\infty |\beta'_{n+1} - \beta'_n| < \infty$, or $\frac{\beta'_n}{\beta'_{n+1}} \to 1$ as $n \to \infty$, let the sequence $\{x'_n\}$ be generated by (1.8), i.e.,

$$x'_{n+1} = \beta'_n u + (1 - \beta'_n) P_K[\alpha'_n x'_n + (1 - \alpha'_n)\Sigma_{i=1}^N \lambda_i T_i x'_n]$$

Then, $\{x'_n\}$ converges strongly to a common fixed point $z$ of $\{T_i\}_{i=1}^N$, where $z = P_{F(T)}u$ and $T = \Sigma_{i=1}^N \lambda_i T_i$.

**Proof.** 1. $\{x'_n\}$ is bounded. By Proposition 2.5, we know that $Fix(\Sigma_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$, take $p \in \bigcap_{i=1}^N Fix(T_i)$, from (1.8), we have

$$\begin{aligned}
\|x'_{n+1} - p\| &\leq \beta'_n \|u - p\| + (1 - \beta'_n)\|P_K[\alpha'_n x'_n + (1 - \alpha'_n)Tx'_n] - p\| \\
&\leq \beta'_n \|u - p\| + (1 - \beta'_n)\|\alpha'_n x'_n + (1 - \alpha'_n)Tx'_n - p\|^2 \\
&= \beta'_n \|u - p\| + (1 - \beta'_n)[\alpha'_n \|x'_n - p\|^2 + (1 - \alpha'_n)\|Tx'_n - p\|^2 \\
&\quad -\alpha'_n(1 - \alpha'_n)\|x'_n - Tx'_n\|^2] \\
&= \beta'_n \|u - p\| + (1 - \beta'_n)[\|x'_n - p\|^2 \\
&\quad -(1 - \alpha'_n)(\alpha'_n - k)\|x'_n - Tx'_n\|^2] \\
&\leq \beta'_n \|u - p\| + (1 - \beta'_n)\|x'_n - p\|^2 \\
&\leq \max\{\|u - p\|, \|x'_n - p\|\}
\end{aligned}$$

By induction, $\|x'_{n+1} - p\| \leq \max\{\|u - p\|, \|x'_1 - p\|\}$, $n \geq 0$, i.e., $\{x'_n\}$ is bounded.

2. $\limsup_{n\to\infty} \langle u - P_{F(T)}u, y'_n - P_{F(T)}u\rangle \leq 0$.

By Proposition 2.5, we also have $T$ is a $k$-strict pseudo-contraction on $K$ with $k = \max\{k_i : 1 \leq i \leq N\}$. Proposition 2.6 ensures that $P_{F(T)}u$ is well defined.

$P_K[\alpha'_n I + (1 - \alpha'_n)T] : K \to K$ is a nonexpansive mapping. Indeed, by using Lemma 2.1, the definition of strictly pseudocontraction and condition (ii), we have for all $x, y \in K$ that

$$\begin{aligned} & \|P_K[\alpha'_n I + (1-\alpha'_n)T]x - P_K[\alpha'_n I + (1-\alpha'_n)T]y\|^2 \\ \leq & \|\alpha'_n(x-y) + (1-\alpha'_n)(Tx-Ty)\|^2 \\ = & \alpha'_n\|x-y\|^2 + (1-\alpha'_n)\|Tx-Ty\|^2 \\ & -\alpha'_n(1-\alpha'_n)\|x-Tx-(y-Ty)\|^2 \\ \leq & \alpha'_n\|x-y\|^2 + (1-\alpha'_n)[\|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 \\ & -\alpha'_n(1-\alpha'_n)\|x-Tx-(y-Ty)\|^2 \\ = & \|x-y\|^2 - (1-\alpha'_n)(\alpha'_n-k)\|x-Tx-(y-Ty)\|^2 \\ \leq & \|x-y\|^2 \end{aligned}$$

which imply that $P_K[\alpha'_n I + (1-\alpha'_n)T]$ is nonexpansive.

Next we prove that $\|x'_{n+1} - x'_n\| \to 0$ as $n \to \infty$.

To this end, we first estimate $\|y'_n - y'_{n-1}\|$. Set $M_1 = sup\{\|x'_n - Tx'_{n-1}\|\}$ and $M_2 = \|u\| + sup\{\|y'_n\|\}$, then, by (1.8) and noting that $P_K[\alpha'_n I + (1-\alpha'_n)T]$ is nonexpansive, we have

$$\begin{aligned} \|y'_n - y'_{n-1}\| = & \|P_K[\alpha'_n I + (1-\alpha'_n)T]x'_n \\ & -P_K[\alpha'_{n-1} I + (1-\alpha'_{n-1})T]x'_{n-1}\| \\ = & \|P_K[\alpha'_n I + (1-\alpha'_n)T]x'_n - P_K[\alpha'_n I + (1-\alpha'_n)T]x'_{n-1} \\ & +P_K[\alpha'_n I + (1-\alpha'_n)T]x'_{n-1} - P_K[\alpha'_{n-1} I \\ & +(1-\alpha'_{n-1})T]x'_{n-1}\| \\ \leq & \|x'_n - x'_{n-1}\| + \|P_K[\alpha'_n I + (1-\alpha'_n)T]x'_{n-1} \\ & -P_K[\alpha'_{n-1} I + (1-\alpha'_{n-1})T]x'_{n-1}\| \\ \leq & \|x'_n - x'_{n-1}\| + M_1|\alpha'_n - \alpha'_{n-1}| \end{aligned} \tag{3.1}$$

then, from (3.1), we get

$$\begin{aligned} \|x'_{n+1} - x'_n\| \leq & \|(1-\beta'_n)\|y'_n - y'_{n-1}\| + M_2|\beta'_n - \beta'_{n-1}| \\ \leq & (1-\beta'_n)(\|x'_n - x'_{n-1}\| + M_1|\alpha'_n - \alpha'_{n-1}|) \\ & +M_2|\beta'_n - \beta'_{n-1}| \end{aligned} \tag{3.2}$$

By Lemma 2.4, we conclude that $\|x'_n - x'_{n-1}\| \to 0$ as $n \to \infty$.

Noting that $\|x'_{n+1} - y'_n\| = \beta'_n\|u - y'_n\| \to 0$ as $n \to \infty$, combining this and (3.2), we have $\|x'_n - y'_n\| \to 0$ as $n \to \infty$.

On the other hand, by condition (ii) and (iii), we have $\alpha'_n \to \alpha$ as $n \to \infty$, where $\alpha \in [k, 1)$. Define $S : K \to H$ by $Sx = \alpha x + (1-\alpha)Tx$.

Then, $S$ is nonexpansive mapping with $F(S) = F(T)$ by proposition 2.9, it follows from proposition 2.7 that $F(P_K S) = F(S) = F(T)$.

Set $M_3 = sup\{\|x'_n\| + \|Tx'_n\| : n \geq 1\}$. Since

$$\|P_K Sx'_n - y'_n\| \leq M_3|\alpha'_n - \alpha'_{n-1}| \to 0, \ as\ n \to \infty,$$

then we have

$$\|x'_n - P_K S x'_n\| \le \|x'_n - y'_n\| + \|y'_n - P_K S x'_n\| \to 0, \ as\ n \to \infty.$$

We now prove that $\limsup_{n\to\infty} \langle u - P_{F(T)}u, y'_n - P_{F(T)}u\rangle \le 0$.

To see this, assume that

$$\limsup_{n\to\infty} \langle u - P_{F(T)}u, y'_n - P_{F(T)}u\rangle = \lim_{j\to\infty} \langle u - P_{F(T)}u, y'_{n_j} - P_{F(T)}u\rangle.$$

Without loss of generality, assume that $y'_{n_j} \rightharpoonup p$ as $j \to \infty$,
then $x'_{n_j} \rightharpoonup p$ and $\|x'_{n_j} - P_K S x'_{n_j}\| \to 0$ as $j \to \infty$.
By Lemma 2.2 we have $p \in F(P_K S) = F(T)$.
By lemma 2.3, we have that

$$\langle u - P_{F(T)}u, p - P_{F(T)}u\rangle \le 0.$$

Hence,

$$\limsup_{n\to\infty} \langle u - P_{F(T)}u, y'_n - P_{F(T)}u\rangle \le 0.$$

3. we prove that $x'_n \to P_{F(T)}u$ as $n \to \infty$.
Putting $\gamma_n = max\{\langle u - P_{F(T)}u, y'_n - P_{F(T)}u\rangle, 0\}$, then $\gamma_n \to 0$ as $n \to \infty$.
By lemma 2.1, we have

$$\begin{aligned} \|x'_{n+1} - P_{F(T)}u\|^2 &= (1-\beta'_n)^2\|y'_n - P_{F(T)}u\|^2 + {\beta'_n}^2\|u - P_{F(T)}u\|^2 \\ &\quad +2\beta'_n(1-\beta'_n)\langle u - P_{F(T)}u, y'_n - P_{F(T)}u\rangle \\ &\le (1-\beta'_n)\|x'_n - P_{F(T)}u\|^2 + o(\beta'_n) \end{aligned}$$

which leads to $x'_n \to P_{F(T)}u$ as $n \to \infty$, by virtue of lemma 2.4.
This completes the proof.

## References


1. W.R.Mann, *Mean value methords in iterations*,Proc.Amer.Math.Soc.4(1953)506-510.
2. F.E.Browder, *Fixed point theorems for noncompact mappings in Hilbert spaces*, Proc.Natl.Acad.Sci.USA 53(1965)1272-1276.
3. F.E.Browder, *Convergence of approximants to fixed of nonexpansive nonlinear mappings in Banach spaces*, Arch.Ration.Mech.Anal.24(1967)82-90.
4. B.Halpern, *Fixed points of nonexpanding maps*,Bull.Amer.Math.Soc.73(1967)957-961
5. F.E.Browder,W.V.Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*,J.Math.Anal.Appl.20(1967)197-228.
6. B.E.Rhoades, *Fixed point iterations using infinite matrices*,Trans.Amer.Math.Soc.196(1974)162-176.
7. S.Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*,J.Math.Anal.Appl.75(1979)274-276.
8. K.Goebel,S.Reich, *Uniform Convexity,Hyperbolic Geometry,and Nonexpansive Mappings*,Marcel Dekker,1984.
9. K.Goebel and W.A.Kirk , *Topics in Metric Fixed Point Theory* (Cambridge Studies in Adanced Mathematics vol 28)(1990)(Cambridge:Cambridge University Press).

10. R.Wittmann, *Approximation of fixed points nonexpansive mappings*,Arch.Math.58(1992)486-491
11. L.S.Liu, *Ishikawa and Mann iteration process with errors for nonlinear strongly accretive mappings in Banach spaces*,J.Math.Anal.Appl.194(1995)114-125.
12. H.Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*,J.Math.Anal.Appl.202(1996)150-159.
13. T.Suzuki,W.Takahashi, *Weak and strong convergence theorems for nonexpansive mappings in Banach spaces*,Nonlinear Anal.47(2001)2805-2815.
14. H.K.Xu, *Iterative algorithms for nonlinear operators*,J.London Math.Soc.66(2002)240-256.
15. K.Nakajo,W.Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*,J.Math.Anal.Appl.279(2003)372-379.
16. W.Takahashi,M.Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*,J.Optim.Theory Appl.113(2003)417-428.
17. H.K.Xu, *Remarks on an iterative method for nonexpansive mappings*,Comm.Appl.Nonlinear Anal.10(2003)67-75.
18. C.Byren, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*,Inverse Problems.20(2004)103-120.
19. T.H.Kim,H.K.Xu, *Strong convergence of modified Mann iterations*,Nonlinear Anal.61(2005)51-60.
20. T.H.Kim,H.K.Xu, *Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups*,Nonliner Anal.64(2006)1140-1152.
21. H.K.Xu, *Strong convergence of an iterative method for nonexpansive mappings and accretive operatives*,J.Math.Anal.Appl.314(2006)631-643.
22. Rudong Chen and Huimin He, *Viscosity approximation of common fixed points of nonexpansive semigroups in Banach space*, Applied Mathematics Letters 20 (2007) 751C757.
23. G.Marino,H.K.Xu, *Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces*,J.Math.Anal.Appl.329(2007)336-346.
24. G.L.Acedo,H,K,Xu, *Iterative methods for strict pseudo-contractions in Hilbert spaces*,Nonlinear.Anal.67(2007)2258-2271


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