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# Solow Model, An Economic Dynamical System of Growth

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**Abstract**

In this paper we construct a new nonlinear ODE system in the economic variables capital $K(t)$, and production $Y(t)$, based on the neoclassical Solow model. We call this system the Dynamical System of Solow Model, DSSM.

We evaluate the stability analysis results obtained by two different ways: the neoclassical Solow model and the DSSM. We conclude that the most important advantage achieved with the construction of the DSSM is to allow the system to have instability occurrences which is not possible to observe in the original model.

We also show how the following economic elements are crucial in the stability analysis: depreciation $\delta$, saving rate $s$, and the relative capital intensity $\alpha$.

**Keywords:** Dynamical systems; Solow growth model

## 1 Introduction

In the 1950s, Solow constructed a model which main purpose was to fit together short-run macroeconomics, with long run factors, as Solow wrote himself in a Addendum to his Nobel Lecture, from August 2001 [4]. His paper [3] remains a very useful approach even today on the explanation of several important aspects of macroeconomics, as well as it provides a simple example of a dynamic system model.

In this paper we start from the original Solow's model, OSM, and propose a different study of this model, based on the methods of Dynamical Systems Theory. In

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section 2 we describe the Solow model and present the main results in a systematic way. In section 3 we define the Dynamical System of Solow Model, DSSM, and present the new approach. In section 4 we evaluate the results obtained by considering both, the OSM and the DSSM. The main results achieved thanks to DSSM are summarised in section 5.

# 2 The Solow Model

The Solow Model consists on three equations modelling changes over time in inputs and outputs. These equations are concerned with a production function, a capital accumulation and a labour force growth.

The production function describes how inputs combine to produce output. We denote output as $Y(t)$ and the inputs are capital, $K(t)$, labour, $L(t)$, and technological progress, $A(t)$.

The capital accumulation equation is the most important equation in Solow model and it describes how capital accumulates.

The labour force and the technological progress growth equations are basically assumptions of the model, they give their own growth rates.

## 2.1 The Solow Model without Technological Progress

**Production Function**

Generally, the production function, $Y(t) = F(K(t), L(t))$, has the Cobb-Douglas form,

$$Y = F(K, L) = K^{\alpha} L^{\beta}, \text{ with } \alpha,\ \beta > 0$$

and it exhibits constant returns to scale, that is $\beta = 1 - \alpha$,

$$Y = F(K, L) = K^{\alpha} L^{1-\alpha}, \text{ with } 0 < \alpha < 1 \tag{1}$$

**Capital Accumulation**

The capital accumulation equation is given by

$$K'(t) = sY(t) - \delta K(t). \tag{2}$$

The change in the capital stock, $K'$, is the difference between gross investment, $sY$, and the amount of depreciation, $\delta K$. The rate of depreciation of capital, $\delta > 0$, and the saving rate, $s > 0$, are generally consider to be constant.

**Labour Force**

An important assumption is that the labour force grows at a constant rate $n$, so its growth rate, $L'/L$, is equal to $n$. We have

$$L(t) = L_0 e^{nt} \tag{3}$$

as the solution of $L' = nL$.

**Analysis of the model**

Rewritting the capital accumulation equation with $Y$ and $L$ given above, we obtain a differential Bernoulli equation in $K$. The following initial value problem is then

$$\left\{ \begin{array}{c} K' = sK^{\alpha} L_0^{1-\alpha} e^{n(1-\alpha)t} - \delta K \\ K(0) = K_0 \end{array} \right. \tag{4}$$

which solution is given by

$$K(t) = e^{-\delta t} \left\{ K_0^{1-\alpha} + \frac{sL_0^{1-\alpha}}{\delta + n} \left[ e^{(1-\alpha)(\delta+n)t} - 1 \right] \right\}^{\frac{1}{1-\alpha}}. \tag{5}$$

In order to get the long-run growth rate of capital,

$$\lim_{t \to +\infty} \frac{K'}{K}$$

we divide by $K$ the differential equation in (4), and then we replace the value of $K$ given by (5) into the second member. So, the long-run growth rate of capital is

$$\left( \frac{K'}{K} \right)_{\infty} = n.$$

The same happens with the output $Y$,

$$\left( \frac{Y'}{Y} \right)_{\infty} = n.$$

According to [1] a *balanced growth path* is a path on which some economic variable is growing at a constant rate. We can say that, under the above conditions, output, capital and labour force follow a *long-run balanced growth path* at rate $n$, usually called the *natural rate of Harrod*.

Analysing the next three ratios, the *capital-output ratio* $K/Y$, the *capital per capita* $K/L$ and the *output per capita* $Y/L$, we can state that the long-run behaviour of the model is accordingly with

$$\frac{K}{Y} \underset{t\to\infty}{\to} \frac{s}{\delta+n} \tag{6}$$

$$\frac{K}{L} \underset{t\to\infty}{\to} \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}} \tag{7}$$

$$\frac{Y}{L} \underset{t\to\infty}{\to} \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}. \tag{8}$$

## 2.2 The Solow Model with Technological Progress

Similarly with labour force, it is assumed that the technological progress, t.p., grows at rate $g$, which means that

$$A(t) = A_0 e^{gt} \tag{9}$$

is the solution of $A' = gA$.

As Solow wrote in [3], "An especially easy kind of technological change is that which simply multiplies the production function by an increasing scale factor." This was the beginning of the Technological Progress Age. Since then, t.p. has been introduced in the model in three different ways, depending on the techonological variable $A$ is associated with $F(K,L)$, $K$, or $L$.

**Hicks-neutral technology**

$$Y = A(t)F(K,L), \tag{10}$$

**Solow-neutral technology**

$$Y = F(A(t)K,L), \tag{11}$$

**Harrod-neutral technology**

$$Y = F(K,A(t)L). \tag{12}$$

**Analysis of the Augmented Model**

We can summarize the three types of t.p. in the simple equation

$$Y = A^{\gamma} K^{\alpha} L^{1-\alpha}, \tag{13}$$

referring to $\gamma = 1$ (Hicks), $\gamma = \alpha$ (Solow) and $\gamma = 1 - \alpha$ (Harrod). The absence of t.p. is also contemplating in (13), puting $\gamma = 0$.

Replacing (3) and (9) into (13), we obtain

$$Y = K^{\alpha} A_0^{\gamma} L_0^{1-\alpha} e^{[\gamma g + (1-\alpha) n] t} \tag{14}$$

and the initial value problem is now

$$\begin{cases} K' = s K^{\alpha} A_0^{\gamma} L_0^{1-\alpha} e^{[\gamma g + (1-\alpha) n] t} - \delta K \\ K(0) = K_0 \end{cases} \tag{15}$$

which solution is given by

$$K(t) = e^{-\delta t} \left\{ K_0^{1-\alpha} + \frac{(1-\alpha)\, s A_0^{\gamma} L_0^{1-\alpha}}{(1-\alpha)(\delta + n) + \gamma g} \left[ e^{[(1-\alpha)(n+\delta) + \gamma g] t} - 1 \right] \right\}^{\frac{1}{1-\alpha}}. \tag{16}$$

The asymptotical behaviour of $K$ becomes

$$\left( \frac{K'}{K} \right)_{\infty} = n + \frac{\gamma}{1-\alpha} g. \tag{17}$$

We still have a long run balanced growth path for capital when any kind of t.p. is considered

Deriving (13) and dividing both sides by $Y$, we obtain

$$\frac{Y'}{Y} = \gamma \frac{A'}{A} + \alpha \frac{K'}{K} + (1-\alpha) \frac{L'}{L}. \tag{18}$$

Puting (17) into (18), the system in the two economic variables, capital and output, follow a long run balanced growth path

$$\left( \frac{K'}{K} \right)_{\infty} = \left( \frac{Y'}{Y} \right)_{\infty} = n + \frac{\gamma}{1-\alpha} g. \tag{19}$$

Combining (13) and (16), the capital output ratio is now

$$\begin{aligned} \frac{K}{Y} &= e^{-[(1-\alpha)(\delta+n)+\gamma g] t} \left\{ \left( \frac{K_0}{L_0} \right)^{1-\alpha} A_0^{-\gamma} - \frac{(1-\alpha)\, s}{(1-\alpha)(\delta+n) + \gamma g} \right\} \\ &\quad + \frac{(1-\alpha)\, s}{(1-\alpha)(\delta+n) + \gamma g}\ , \end{aligned} \tag{20}$$

and its behaviour in a long run analysis is

$$\left( \frac{K}{Y} \right)_{\infty} = \frac{(1-\alpha)\, s}{(1-\alpha)(\delta+n) + \gamma g}\ . \tag{21}$$

## 2.3 Conclusions

We can summarize all the results obtained in a systematic way. Let $\mu = \left(\frac{K'}{K}\right)_\infty$ be the long run capital growth rate. Analysing the solution (16), we conclude the following:

**I - absence of t.p.,** $\gamma = 0$

$$\mu_0 = n$$

**II - introduction of t.p.**

i) Hicks, $\gamma = 1$

$$\mu_I = n + \frac{1}{1-\alpha} g = n + c_I g$$

where $c_I = \dfrac{1}{1-\alpha}$.

ii) Solow, $\gamma = \alpha$

$$\mu_S = n + \frac{\alpha}{1-\alpha} g = n + c_S g$$

where $c_S = \dfrac{\alpha}{1-\alpha}$.

iii) Harrod, $\gamma = 1 - \alpha$

$$\mu_H = n + g = n + c_H g.$$

where $c_H = 1$.

**III - coefficients** $c_I$, $c_S$, and $c_H$

Introducing t.p. under condition (9), the model is affected by a kind of displacement on its long run capital growth rate $\mu$. It jumps from its stacionary state $n$ to another stacionary state $n + c_x g$. The meanning of $x$ is important as we shall see. Although $x$ only reflects on the manner which the economic variables $Y$, $K$ and $L$ are affected by the introduction of t.p., it allows to state an order relationship between the different three $\mu_x$. Therefore we have

$$\begin{aligned} \text{if } 0 &< \alpha < \frac{1}{2} \text{ then } \mu_S < \mu_H < \mu_I, \\ \text{if } \alpha &= \frac{1}{2} \text{ then } \mu_S = \mu_H < \mu_I, \\ \text{if } \frac{1}{2} &< \alpha < 1 \text{ then } \mu_H < \mu_S < \mu_I. \end{aligned}$$

We can observe that depending on the value of $\alpha$, the three multiplicator terms, $c_I = \frac{1}{1-\alpha}$, $c_S = \frac{\alpha}{1-\alpha}$, and $c_H = 1$, of the t.p. growth rate, $g$, have two opposite actions on $g$: an accelerator action and a delayed one. If $\frac{1}{2} < \alpha < 1$, then $c_I$ and $c_S$ are accelerators coefficients of $g$. If $0 < \alpha < \frac{1}{2}$ then $c_S$ is a delayed coefficient of $g$. Finally, if $\alpha = \frac{1}{2}$ then $c_S = c_H = 1$. It is curious to note that, no matter what value of $\alpha$ is, the biggest long run growth rate is $\mu_I$. This is related with the original idea of Solow expressed in [3].

# 3 An Overview with Dynamical System Theory

In this section we propose a new approach to the study of Solow model, introducing the methods of dynamical system theory. In order to do this, we will construct a nonlinear system of differential equations in the capital and output variables. Our primary goal is to release the model of previous assumptions on $A(t)$ and $L(t)$. We pretend to investigate what kind of conclusions we get on $K(t)$ and $Y(t)$ variables, when hypothesis around $A(t)$ and $L(t)$ are as much general as possible.

## 3.1 The Dynamical System of Solow Model

Let $F(K(t), L(t))$ be some production function defined on the open subset $E$ of $\Re^2$, $E = (0, \infty) \times (0, \infty)$, twice differentiable, increasing and concave on $K$ and $L$. We also assume that $F$ is an homogeneous function of degree $\alpha + \beta$, with $\alpha, \beta > 0$. Let $Y(t)$, $K(t)$, $A(t)$ and $L(t)$ be real functions of real variable, differentiables on $\Re$.

The first equation is then

$$Y(t) = A(t)F(K(t), L(t)), \tag{22}$$

and the second one is the fundamental equation of Solow model

$$K'(t) = sY(t) - \delta K(t). \tag{23}$$

For the time being, we are going to say nothing about growth rates of t.p. and labour force.

Deriving the equation (22) in order to time we obtain

$$Y' = A'F + A\left[F'_K K' + F'_L L'\right].$$

Dividing both sides by $Y$, puting (22) into the second member, and using (23), we then obtain a nonlinear system of ODEs on $K(t)$ and $Y(t)$

$$\begin{cases} K' = -\delta K + sY \\ Y' = \left(\frac{A'}{A} + s\frac{F'_K}{F}Y - \delta\frac{F'_K}{F}K + \frac{F'_L}{F}L'\right)Y \end{cases} \tag{24}$$

We will refer to this system as **Dynamical System of Solow Model, DSSM**.

## 3.2 Equilibrium points of DSSM

The Hartman-Grobman Theorem, wich can be seen for example in [2], establishes that the qualitative behaviour of the solution set of a nonlinear system of ODEs near a hyperbolic equilibrium point (which means that none of the eigenvalues of the linearized system have zero real part), is the same as the qualitative behaviour of the solution set of the corresponding linearized system near the equilibrium point. So, by definition the equilibrium points of (24) are the solutions of

$$\begin{cases} -\delta K + sY = 0 \\ \left(\frac{A'}{A} + s\frac{F'_K}{F}Y - \delta\frac{F'_K}{F}K + \frac{F'_L}{F}L'\right)Y = 0. \end{cases} \tag{25}$$

We exclude the economically meaningless equilibrium solutions $Y = K = 0$. Then we can state the following

**Proposition 1** *A necessary and sufficient condition for the existence of equilibrium points for the DSSM, consists of*

$$\frac{A'}{A} + \frac{F'_L}{F}L' = 0. \tag{26}$$

**Corollary 2** *Under the above conditions, the equilibrium points of DSSM are*

$$Y = \frac{\delta}{s}K. \tag{27}$$

This equality is accordingly with the long run analysis in (21), once considered $1-\alpha=\beta$, $\gamma=1$, and the result (26), which is here given by $g+\beta n=0$. The equilibrium points (27) consist of a straight line in the $KOY$ phase plane.

After linearization the DSSM, we compute its eigenvalues. They obey

$$\lambda^2+\left(\delta-\delta K\frac{F'_K}{F}\right)\lambda-\delta K\frac{L'}{F^2}\left(FF''_{LK}-F'_LF'_K\right)=0. \tag{28}$$

In order to proceed with the stability analysis of the equilibrium points, it is convenient to make some assumptions on the production function.

## 3.3 DSSM with Cobb-Douglas Production Function

Let insert $F(K(t),L(t))=K^\alpha(t)L^\beta(t)$ with $\alpha,\ \beta>0$, in (24), and let $F$ be an homogeneous function of degree $\alpha+\beta$. This allows that the production function can exhibit increasing returns to scale if $\alpha+\beta>1$, constant if $\beta=1-\alpha$, and decreasing if $\alpha+\beta<1$. We then obtain the system

$$\begin{cases} K'=-\delta K+sY \\ Y'=\left(\frac{A'}{A}+s\alpha\frac{Y}{K}-\delta\alpha+\beta\frac{L'}{L}\right)Y. \end{cases} \tag{29}$$

**Proposition 3** *A necessary and sufficient condition for the existence of equilibrium points for the DSSM (29), consists of*

$$\frac{A'}{A}+\beta\frac{L'}{L}=0. \tag{30}$$

The equilibrium points are the same, $Y=\frac{\delta}{s}K$.

If we want to get an economic interpretation of this equation, we must abandon the usual assumptions on the growth rates $A'/A=g$ and $L'/L=n$, because $\beta$ is positive. We can state that in order to guarantee the existence of equilibrium points for the DSSM, it is no longer possible to consider $A$ and $L$ with simultaneously exponential growth. This is an open question at the moment.

Turning back to the system (29), we add the two initial conditions

$$\begin{aligned} K(0) &= K_0 \\ Y(0) &= Y_0=A(0)L^\beta(0)K_0^\alpha, \end{aligned}$$

and then we obtain the solution

$$\begin{cases} K(t) = e^{-\delta t}\left\{K_0^{1-\alpha} + (1-\alpha)s\int_0^t A(u)L^{\beta}(u)e^{\delta(1-\alpha)u}du\right\}^{\frac{1}{1-\alpha}} \\ Y(t) = A(t)L^{\beta}(t)e^{-\delta\alpha t}\left\{K_0^{1-\alpha} + (1-\alpha)s\int_0^t A(u)L^{\beta}(u)e^{\delta(1-\alpha)u}du\right\}^{\frac{\alpha}{1-\alpha}} . \end{cases} \tag{31}$$

Accordingly with (29), the growth rates of capital and product are

$$\begin{aligned} \frac{K'}{K} &= -\delta + s\frac{Y}{K} \\ \frac{Y'}{Y} &= \frac{A'}{A} + \alpha\frac{K'}{K} + \beta\frac{L'}{L}. \end{aligned} \tag{32}$$

### 3.3.1 $A(t)$ and $L(t)$ verify the *n.s.c.* (30)

Then

$$\frac{Y'}{Y} = \alpha\frac{K'}{K}. \tag{33}$$

### 3.3.2 $A(t)$ and $L(t)$ don't verify the *n.s.c.* (30)

Then there are no equilibrium points for the DSSM, except the $(0,0)$ equilibrium that we have neglected by meaningless economic reasons. From (32) and considering the solution (31), we have

$$\begin{aligned} \lim_{t\to\infty}\frac{K'}{K} &= \lim_{t\to\infty}\left\{-\delta + sA(t)L^{\beta}(t)e^{\delta(1-\alpha)t}\left\{K_0^{1-\alpha} + (1-\alpha)s\int_0^t A(u)L^{\beta}(u)e^{\delta(1-\alpha)u}du\right\}^{\frac{\alpha-1}{1-\alpha}}\right\} \\ &= \lim_{t\to\infty}\left\{-\delta + \frac{sA(t)L^{\beta}(t)e^{\delta(1-\alpha)t}}{K_0^{1-\alpha} + (1-\alpha)s\int_0^t A(u)L^{\beta}(u)e^{\delta(1-\alpha)u}du}\right\} \\ &= \lim_{t\to\infty}\left\{-\delta + s\,e^{\delta(1-\alpha)t}\,\frac{A'(t)L^{\beta}(t) + \beta A(t)L^{\beta-1}(t)L'(t) + \delta\,(1-\alpha)\,A(t)L^{\beta}(t)}{(1-\alpha)\,sA(t)L^{\beta}(t)}\right\} \\ &= \lim_{t\to\infty}\left\{-\delta + \tfrac{1}{1-\alpha}\left(\tfrac{A'}{A} + \beta\tfrac{L'}{L}\right) + \delta\right\} \\ &= \frac{g+\beta n}{1-\alpha} . \end{aligned}$$

Substituing this into the second equation of (32), we have for the product

$$\lim_{t\to\infty}\frac{Y'}{Y} = g + \beta n + \alpha \lim_{t\to\infty}\frac{K'}{K} = \frac{g+\beta n}{1-\alpha}\ .$$

We want to point out that if we don't consider the existence of equilibrium points for the DSSM, which means that the neoclassical assumptions are allowed, then we get

$$\left(\frac{Y'}{Y}\right)_{\infty} = \left(\frac{K'}{K}\right)_{\infty} = \frac{g+\beta n}{1-\alpha}\ , \tag{34}$$

which replies the behaviour of the OSM (19), taking $\gamma = 1$ and $\beta = 1-\alpha$ into account.

## 3.4 Stability Analysis of DSSM with Cobb-Douglas Production Function

With the production function considered above, $F = K^{\alpha}L^{\beta}$, we then have $FF''_{LK} - F'_L F'_K = 0$. According to (28), the two eigenvalues are

$$\lambda = 0 \quad \vee \quad \lambda = (\alpha - 1)\,\delta. \tag{35}$$

We can no longer use the Hartman-Grobman Theorem because the equilibrium points, $\bar{X} = \left(K, \frac{\delta}{s}K\right)$, are nonhyperbolics. So, we are looking for a certain Liapunov function that allow us to decide about their stability. This method due to Liapunov can be seen in [2].

Let $E$ be an open subset of $\Re^2$ containing $\bar{X} = \left(K, \frac{\delta}{s}K\right)$. Let

$$f(K,Y) = \begin{bmatrix} -\delta K + sY \\ \left(\frac{A'}{A} + s\alpha\frac{Y}{K} - \delta\alpha + \beta\frac{L'}{L}\right)Y \end{bmatrix}$$

with $f\left(\bar{X}\right) = 0$.

**Proposition 4** *The function $V \in C^1(E)$, defined by $V(K,Y) = \left(K - \frac{s}{\delta}Y\right)^2$ is a Liapunov function.*

**Proof.** According to the definition in [2], the function $V$ must verify

i) $V\left(\bar{X}\right) = 0$ and ii) $V(K,Y) > 0$ for $(K,Y) \neq \bar{X}$, which is easy to check. ■

The Liapunov method consists of apply the following criterion, once a suitable Liapunov $V$ function is found.

a) if $\overset{\bullet}{V}(K,Y) \leq 0 \ \ \forall_{(K,Y)\in E}$ then $\bar{X}$ is stable,

b) if $\overset{\bullet}{V}(K,Y) < 0 \ \ \forall_{(K,Y)\in E\setminus\left\{\bar{X}\right\}}$ then $\bar{X}$ is asymptotically stable,

c) if $\overset{\bullet}{V}(K,Y) > 0 \ \ \forall_{(K,Y)\in E\setminus\left\{\bar{X}\right\}}$ then $\bar{X}$ is unstable.

Here, dot means the total derivative with respect to time.

Considering the Liapunov function $V(K,Y) = \left(K - \frac{s}{\delta}Y\right)^2$, we compute

$$\overset{\bullet}{V}(K,Y) = V_K'K' + V_Y'Y' = 2\left(K - \frac{s}{\delta}Y\right)K' - 2\frac{s}{\delta}\left(K - \frac{s}{\delta}Y\right)Y' = 2\left(K - \frac{s}{\delta}Y\right)\left(K' - \frac{s}{\delta}Y'\right).$$

In order to decide the sign of this expression it is needed to substitute $K'$ and $Y'$ by (29) and consider (30). We then obtain

$$\overset{\bullet}{V}(K,Y) = -\frac{2}{\delta}\left(sY - \delta K\right)^2\left(1 - \alpha\frac{s}{\delta}\frac{Y}{K}\right). \tag{36}$$

Now we can see how the stability analysis is deeply conditional on the three economic elements, depreciation, $\delta$, saving rate, $s$, and $\alpha$.

## 4 DSSM versus OSM

We want to evaluate the results obtained by two different ways. One way is concerned to a single differential equation, the capital accumulation equation, and the other way consists of a differential equation system, the DSSM. The stability analysis is our priority issue. In order to compare the results, it's indispensable that all previous assumptions are the same in the two ways. As the final result (36) obtained with DSSM, was established under (22) and (30) conditions, we now have to consider them in (15).

The nonlinear differential equation $K' = sY - \delta K$ with $Y = AK^\alpha L^{1-\alpha}$ is

$$\begin{cases} K' = sAK^\alpha L^{1-\alpha} - \delta K \\ K(0) = K_0 \end{cases}$$

Set $K' = f(K)$. We look for the equilibrium solutions, that is $f(K) = 0$. The unique positive equilibrium solution, denoted by $\bar{K}$, is

$$\bar{K} = \left(\frac{\delta}{sA} L^{\alpha-1}\right)^{\frac{1}{\alpha-1}}.$$

After linearization we obtain

$$f'(K) = sA\alpha K^{\alpha-1} L^{1-\alpha} - \delta.$$

To study the stability of $\bar{K}$ equilibrium solution we compute

$$f'\left(\bar{K}\right) = sA\alpha \left(\frac{\delta}{sA} L^{\alpha-1}\right)^{\frac{\alpha-1}{\alpha-1}} L^{1-\alpha} - \delta$$

which gives

$$f'\left(\bar{K}\right) = \delta\left(\alpha - 1\right).$$

As $f'\left(\bar{K}\right)$ is always negative then the $\bar{K}$ equilibrium solution is asymptotically stable for $K$ in a neighborhood of $\bar{K}$. Having (35) in mind we realize that the DSSM is a good generalization of OSM, and the most important advantage achieved with the construction of the DSSM, is to allow the system to have a more rich long run dynamic behaviour. Because a zero eigenvalue appears instability can occur at the same time with asymptotic stability, depending on the sign of (36).

Therefore, turning back to the DSSM and following the Liapunov criterion, we can state that if the capital-output ratio verifies

$$\frac{K}{Y} < \alpha \frac{s}{\delta}$$

then the positive equilibrium solution is unstable, otherwise it is asymptotically stable. This fact is not possible to observe in the study of the single capital accumulation equation.

# 5 Conclusion

In this paper we have shown that, contrary to Solow's original result, saving not only determines the accumulation but also plays a crucial role in the stability analysis of the model. The three economic elements, $\alpha$, $s$ and $\delta$, are the key elements in a future bifurcation analysis.

Finally, it is important to remember that these results depend on the condition expressed in (30), which means that the assumptions usually made on the two growth rates $n$ and $g$, shouldn't consider to be simultaneously constant. It is convenient to assume that the rate of technological progress $g$ (and/or $n$) must reflect on the accumulation of some kind of knowledge.

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