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# On Quasi $\alpha\psi$-Open Functions in Topological Spaces

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**Abstract**

In this paper, we introduce the notion of *Quasi* $\alpha\psi$*-open* function and investigate some of its fundamental properties and its characterisations.

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**Keywords.** $\alpha\psi$*-closed* set, $\alpha\psi$*-open* set and *Quasi* $\alpha\psi$*-function*

## 1. INTRODUCTION

Many different terms of open functions have been introduced over the course of years. Various interesting problems arise when one consider openness. Its importance is significant in various areas of Mathematics and related sciences.

The notion of $\alpha\psi$*-closed* set was introduced and studied by R.Devi et al.[2]. In this paper, we will continue the study of related functions by involving $\alpha\psi$*-open* sets. We introduce and characterize the concept of *quasi* $\alpha\psi$*-functions*.

Through out this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X, \tau) \to (Y, \sigma)$ denotes a function $f$ of a space $(X, \tau)$ into a space $(Y, \sigma)$. Let $A$ be a subset of a space $X$. The closure and the interior of $A$ are denoted by $cl(A)$ and $int(A)$, respectively.

## 2. PRELIMINARIES

Before entering to our work, we recall the following definitions, which are useful in the sequel.

**Definition 2.1.** A subset $A$ of a space $(X, \tau)$ is called

1. a *semi-open* set [3] if $A \subseteq cl(int(A))$ and a *semi-closed* set if $int(cl(A)) \subseteq A$ and
2. an $\alpha$-*open* set [4] if $A \subseteq int(cl(int(A)))$ and an $\alpha$-*closed* set if $cl(int(cl(A))) \subseteq A$.

The *semi*-closure (resp. $\alpha$-closure) of a subset $A$ of a space $(X, \tau)$ is the intersection of all *semi-closed* (resp. $\alpha$-*closed*) sets that contain $A$ and is denoted by $scl(A)$ (resp. $\alpha cl(A)$).

**Definition 2.2.** A subset $A$ of a topological space $(X, \tau)$ is called a

1. a *semi-generalized closed* (briefly *sg-closed*) set [1] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is *semi-open* in $(X, \tau)$. The complement of *sg-closed* set is called *sg-open* set,
2. a $\psi$-*closed* set [5] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is *sg-open* in $(X, \tau)$. The complement of $\psi$-*closed* set is called $\psi$-*open* set and
3. a $\alpha\psi$-*closed* set [2] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-*open* in $(X, \tau)$. The complement of $\alpha\psi$-*closed* set is called $\alpha\psi$-*open* set.

## 3. ON *QUASI* $\alpha\psi$-*OPEN* AND *QUASI* $\alpha\psi$-*CLOSED* FUNCTIONS

**Definition 3.1.** A function $f : X \rightarrow Y$is said to be *quasi* $\alpha\psi$-*open* if the image of every $\alpha\psi$-*open* set in $X$ is *open* in $Y$ .

It is evident that, the the concepts of *quasi* $\alpha\psi$-openness and $\alpha\psi$-continuity coinside if the function is a bijection.

**Theorem 3.2.** A function $f : X \rightarrow Y$ is *quasi* $\alpha\psi$-*open* if and only if for every subset $U$ of $X$, $f(\alpha\psi\text{-}int(U)) \subset int(f(U))$.

**Proof.** Let $f$ be a *quasi* $\alpha\psi$-*open* function. Now, we have $int(U) \subset U$ and $\alpha\psi\text{-}int(U)$ is a $\alpha\psi$-*open* set. Hence, we obtain that $f(\alpha\psi\text{-}int(U)) \subset f(U)$. As $f(\alpha\psi\text{-}int(U))$ is open,$f(\alpha\psi\text{-}int(U)) \subset int(f(U))$.

Conversely, assume that $U$ is a $\alpha\psi$-*open* set in $X$. then, $f(U) = f(\alpha\psi\text{-}int(U)) \subset int(f(U))$ but $int(f(U)) \subset f(U)$. Consequently, $f(U) = int(f(U))$ and hence $f$ is *quasi* $\alpha\psi$- open.

**Theorem 3.3.** If a function $f : X \to Y$ is *quasi $\alpha\psi$-open*, then $\alpha\psi$-$int(f^{-1}(G)) \subset f^{-1}(int(G))$ for every subset $G$ of $Y$.
**Proof.** Let $G$ be any arbitrary subset of $Y$. Then, $\alpha\psi$-$int(f^{-1}(G))$ is a *$\alpha\psi$-open*set in $X$ and $f$ is *quasi $\alpha\psi$-open*, then $f(\alpha\psi\text{-}int(f^{-1}(G))) \subset int(f(f^{-1}(G))) \subset int(G)$. Thus, $\alpha\psi$-$int(f^{-1}(G)) \subset f^{-1}(int(G))$.

**Definition 3.4.** A subset $A$ is said to be an *$\alpha\psi$-neighbourhood* of a point $x$ of $X$ if there exists a *$\alpha\psi$-open* set $U$ such that $x \in U \subset A$.

**Theorem 3.5.** For a function $f : X \to Y$, the following are equivalent

(i) $f$ is *quasi $\alpha\psi$-open*;

(ii) for each subset $U$ of $X$, $f(\alpha\psi\text{-}int(U)) \subset int(f(U))$;

(iii) for each $x \in X$ and each *$\alpha\psi$-neighbourhood* $U$ of $x$ in $X$, there exists a neighbourhood $V$ of $f(x)$ in $Y$ such that $V \subset f(U)$.

**Proof.** (i) $\Rightarrow$ (ii) It follows from Theorem 3.1.
(ii) $\Rightarrow$ (iii) Let $x \in X$ and $U$ be an arbitrary *$\alpha\psi$-neighbourhood* of $x \in X$. Then, there exists a *$\alpha\psi$-open* set $V$ in $X$ such that $x \in V \subset U$. Then by (ii), we have $f(V) = f(\alpha\psi\text{-}int(V)) \subset int(f(V))$ and hence $f(V)$ is open in $Y$ such that $f(x) \in f(V) \subset f(U)$.
(iii) $\Rightarrow$ (i) Let $U$ be an arbitrary *$\alpha\psi$-open* set in $X$. Then for each $y \in f(U)$, by (iii) there exists a neghbourhood $V_y$ of $y$ in $Y$ such that $V_y \subset f(U)$. As $V_y$ is a neighbourhood of $y$, there exists an open set $W_y$ in $Y$ such that $y \in W_y \subset V_y$. Thus $f(U) = \bigcup\{W_y : y \in f(U)\}$ which is an open se in $Y$. This implies that $f$ is *quasi $\alpha\psi$-open* function.

**Theorem 3.6.** A functon $f : X \to Y$ is *quasi $\alpha\psi$-open* if and only if for any subset $B$ of $Y$ and for any *$\alpha\psi$-closed* set $F$ of $X$ containing $f^{-1}(B)$, there exists a closed set $G$ of $Y$ containing $B$ such that $f^{-1}(G) \subset F$.
**Proof.** Suppose $f$ is *quasi $\alpha\psi$-open*. Let $B \subset Y$ and $F$ be a *$\alpha\psi$-closed* set of $X$ containing $f^{-1}(B)$. Now, put $G = Y - f(X - F)$. It is clear that $f^{-1}(B) \subset F \Rightarrow B \subset G$. Since $f$ is *quasi $\alpha\psi$- open*, we obtain $G$ as a closed set of $Y$. Moreover, we have $f^{-1}(G) \subset F$.
Conversely, let $U$ be a *$\alpha\psi$-open* set of $X$ and put $B = Y - f(U)$. Then $X - U$ is a *$\alpha\psi$-closed* set in $X$ containing $f^{-1}(B)$. By hypothesis, there exists a closed set $F$ of $Y$ such that $B \subset F$ and $f^{-1}(F) \subset X - U$. Hence, we obtain $f(U) \subset Y - F$. On the other hand, it follows that $B \subset F$, $Y - F \subset Y - B = f(U)$. Thus we obtain $f(U) = Y - F$ which is open and hence $f$ is a *quasi $\alpha\psi$-open* function.

**Theorem 3.7** A function $f : X \to Y$ is *quasi $\alpha\psi$-open* if and only if $f^{-1}(cl(B)) \subset \alpha\psi\text{-}cl(f^{-1}(B))$ for every subset $B$ of $Y$.

**Proof.** Suppose that $f$ is *quasi* $\alpha\psi$-*open*. For any subset $B$ of $Y$, $f^{-1}(B) \subset \alpha\psi\text{-}cl(f^{-1}(B))$. Therefore, by theorem 3.5 there exists a closed set $F$ in $Y$ such that $B \subset F$ and $(f^{-1}(F)) \subset \alpha\psi\text{-}cl(f^{-1}(B))$. Therefore, we obtain $f^{-1}(cl(B)) \subset (f^{-1}(F)) \subset \alpha\psi\text{-}cl(f^{-1}(B))$.
Conversely, let $B \subset Y$ and $F$ be a $\alpha\psi$-*closed* set of $X$ containing $f^{-1}(B)$. Put $W = cl_Y(B)$, then we have $B \subset W$ and $W$ is closed and $f^{-1}(W) \subset \alpha\psi\text{-}cl(f^{-1}(B)) \subset F$. Then by theorem 3.6., $f$ is *quasi* $\alpha\psi$-*open*.

**Theorem 3.8.** A function $f : X \to Y$ and $g : Y \to Z$ be two functions and $g \circ f : X \to Z$ is *quasi* $\alpha\psi$-*open*. If $g$ is continuous injective function, then $f$ is *quasi* $\alpha\psi$-*open*.
**Proof.** Let $U$ be a $\alpha\psi$-*open* set in $X$, then $(g \circ f)(U)$ is open in $Z$, since $g \circ f$ is *quasi* $\alpha\psi$-*open*. Again $g$ is an injective continuous function, $f(U) = g^{-1}(g \circ f(U))$ is open in Y. This shows that $f$ is *quasi* $\alpha\psi$-*open*

## 4. ON *QUASI* $\alpha\psi$-*CLOSED* FUNCTIONS

**Definition 4.1.** A function $f : X \to Y$is said to be *quasi* $\alpha\psi$-*closed* if the image of every $\alpha\psi$-*closed* set in $X$ is *closed* in $Y$ .

**Theorem 4.2.** Every *quasi* $\alpha\psi$-*closed* function is *closed* as well as $\alpha\psi$-*closed*.
**Proof.** It is obvious.

The converse of the above theorem need not be true by the following example.

**Example 4.3.** Let $X = Y = \{a, b, c\}$, $\tau = \{X,\ \{a\},\ \{b, c\},\ \phi\} = \sigma$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a,\ f(b) = b$ and $f(c) = c$. Then clearly $f$ is $\alpha\psi$-*closed* as well as *closed* but not *quasi* $\alpha\psi$-*closed*.

**Lemma 4.4.** If a function is *quasi* $\alpha\psi$-*closed*, then $f^{-1}(int(B)) \subset \alpha\psi\text{-}int(f^{-1}(B))$ for every subset $B$ of $Y$.
**Proof.** Let $B$ any arbitrary subset of $Y$. Then, $\alpha\psi\text{-}int(f^{-1}(G))$ is a $\alpha\psi$ *closed* set in $X$ and $f$ is *quasi* $\alpha\psi$-*closed*, then $f(\alpha\psi\text{-}\ int(f^{-1}(B))) \subset int(f(f^{-1}(B))) \subset int(B)$. Thus, $f(\alpha\psi\text{-}int(f^{-1}(B))) \subset f^{-1}(int(B))$.

**Theorem 4.5.** A function $f : X \to Y$ is *quasi* $\alpha\psi$-*closed* if and only if for any subset $B$ of $Y$ and for any $\alpha\psi$-*open* set $G$ of $X$ containing $f^{-1}(B)$, there exists an open set $U$ of $Y$ containing $B$ such that $f^{-1}(U) \subset G$.
**Proof** This proof is similar to that of theorem 3.6.

**Definition 4.6.** A function $f : X \to Y$ is called $\alpha\psi^*$-*closed* if the image of every $\alpha\psi$-*closed* subset of $X$ is $\alpha\psi$-*closed* in $Y$.

**Theorem 4.7.** If $f : X \to Y$ and $g : Y \to Z$ be any *quasi $\alpha\psi$-closed* functions, then $g \circ f : X \to Z$ is a *quasi $\alpha\psi$-closed* function.
**Proof.** It is obvious.

**Theorem 4.8.** Let $f : X \to Y$ and $g : Y \to Z$ be any two functions, then

(i) If $f$ is *$\alpha\psi$-closed* and $g$ is *quasi $\alpha\psi$-closed*, then $g \circ f$ is *closed*;

(ii) If $f$ is *quasi $\alpha\psi$-closed* and $g$ is *quasi $\alpha\psi$-closed*, then $g \circ f$ is *$\alpha\psi^*$-closed*;

(iii) If $f$ is *$\alpha\psi^*$-closed* and $g$ is *quasi $\alpha\psi$-closed*, then $g \circ f$ is *quasi $\alpha\psi$- closed.*

**Proof.** It is obvious.

**Theorem 4.9.** Let $f : X \to Y$ and $g : Y \to Z$ be any two functions such that $g \circ f : X \to Z$ is *quasi $\alpha\psi$-closed.*

(i) If $f$ is *$\alpha\psi$-irresolute* surjective, then $g$ is is *closed*;

(ii) If $g$ is *$\alpha\psi$-continuous* injective, then $f$ is *$\alpha\psi^*$-closed.*

**Proof.** (i) Suppose that $F$ is an arbitrary closed set in $Y$. As $f$ is *$\alpha\psi$-irresolute*, $f^{-1}(F)$ is *$\alpha\psi$-closed* in $X$. Since $g \circ f$ is *quasi $\alpha\psi$-closed* and $f$ is surjective, $(g \circ f)(f^{-1}(F)) = g(F)$, which is closed in $Z$. This implies that $g$ is a closed function.
(ii) Suppose $F$ is any *$\alpha\psi$-closed* set in $X$. Since $g \circ f$ is *quasi $\alpha\psi$-closed*, $(g \circ f)(F)$ is closed in $Z$. Again $g$ is a *$\alpha\psi$-continuous* injective function, $g^{-1}(g \circ f(F)) = f(F)$, which is *$\alpha\psi$-closed* in $Y$. This shows that $f$ is *$\alpha\psi^*$-closed.*

**Theorem 4.10.** Let $X$ and $Y$ be topological spaces. Then the function $f : X \to Y$ is a *quasi $\alpha\psi$-closed* if and only if $f(X)$ is closed in $Y$ and $f(V) - f(X - V)$ is open in $f(X)$ whenever $V$ is *$\alpha\psi$-open* in $X$.
**Proof.** Necessity: Suppose $f : X \to Y$ is a *quasi $\alpha\psi$-closed* function. Since $X$ is *$\alpha\psi$-closed*, $f(X)$ is closed in $Y$ and $f(V) - f(X - V) = f(V) \cap f(X) - f(X - V)$ is open in $f(X)$ when $V$ is *$\alpha\psi$-open* in $X$.
Sufficiency: Suppose $f(X)$ is closed in $Y$, $f(V) - f(X - V)$ is open in $f(X)$ when $V$ is *$\alpha\psi$-open* in $X$ and let $C$ be closed in $X$. Then $f(C) = f(X) - (f(C - X) - f(C))$ is closed in $f(X)$ and hence closed in $Y$.

**Corollary 4.11.** Let $X$ and $Y$ be topological spaces. Then a surjective function $f : X \to Y$ is *quasi $\alpha\psi$-closed* if and only if $f(V) - f(X - V)$ is open in $Y$ whenever $U$ is *$\alpha\psi$-open* in $X$.
**Proof.** It is obvious.

**Theorem 4.12.** Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be *$\alpha\psi$-continuous* and *quasi $\alpha\psi$-closed* surjective function. Then the topology on $Y$ is $\{f(V) - f(X - V) : V$ is *$\alpha\psi$-open* in $X\}$.
**Proof.** Let $W$ be open in $Y$. Let $f^{-1}(W)$ is *$\alpha\psi$-open* in $X$, and $f(f^{-1}(W)) - f(X - f^{-1}(W)) = W$. Hence all open sets an $Y$ are of the form $f(V) - f(X - V)$, $V$ is *$\alpha\psi$-open* in $X$. On the other hand, all sets of the form $f(V) - f(X - V)$. $V$ is *$\alpha\psi$-open* in $X$, are open in $Y$ from corollary 4.11.

**Definition 4.13.** A topological space $(X, \tau)$ is said to be *$\alpha\psi$-normal* if for any pair of disjoint *$\alpha\psi$-closed* subsets $F_1$ and $F_2$ of $X$, there exists disjoint open sets $U$ and $V$ such that $F_1 \subset U$ and $F_2 \subset V$.

**Theorem 4.14.** Let $X$ and $Y$ be topological spaces with $X$ is *$\alpha\psi$-normal*. If $f : X \to Y$ is *$\alpha\psi$-continuous* and *quasi $\alpha\psi$-closed* surjective function. Then $Y$ is *normal*.
**Proof.** Let $K$ and $M$ be disjoint closed subsets of $Y$. Then $f^{-1}(K)$, $f^{-1}(M)$ are disjoint *$\alpha\psi$-closed* subsets of $X$. Since $X$ is *$\alpha\psi$-normal*, there exists disjoint open sets $V$ and $W$ such that $f^{-1}(K) \subset V$, $f^{-1}(M) \subset W$. Then $K \subset f(V) - f(X - V)$ and $M \subset f(W) - f(X - W)$, further by corollary 4.11, $f(V) - f(X - V)$ and $f(W) - f(X - W)$ are open sets in $Y$ and clearly $(f(V) - f(X - V)) \cap (f(W) - f(X - W)) = \phi$. This shows that $Y$ is *normal*.

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