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Existence Result and Numerical Method for Nonlinear Third-Order Two-Point Boundary Value Problems

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Abstract

In this study, we present the existence of solution for nonlinear third-order differential equation with two-point boundary value conditions under more easily verified conditions than the conditions found in the literature. Reproducing kernel theorem play an important role in the arguments. And we give a constructive method for solving these problems, the numerical examples are given to illustrate the applicability and efficiency of the novel method.

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1 Introduction

The existence of solution for nonlinear third-order two-point boundary value problems has been paid much attention to all through. These problems can be seen as a generalized model for various physical, natural or physiological phenomena such as the flow of a thin film of viscous fluid over a solid surface [1,2], the solitary waves solution of the Korteweg-de Vries equation [3] or the thyroid-pituitary interaction [4]. In the last decade or so, several papers[5-11] have been devoted to the study of third-order differential equations with two-point boundary conditions.

Motivated greatly by the above-mentioned excellent works, in this paper, we will consider more easily verified conditions than the conditions in [5,7,8,11] for the existence of nonlinear third-order two-point boundary value problems:

$$\begin{cases} \mathbb{L}u(x) \triangleq u^{(3)}(x) = f(x, u(x), u'(x), u''(x)), \quad 0 \le x \le 1, \\ u'(0) = 0,\ u'(1) = 0,\ u(0) = 0. \end{cases} \tag{1}$$

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where $\mathbb{L} : W_2^4[0,1] \to W_2^1[0,1]$ is a bounded linear operator. The reproducing kernel space $W_2^4[0,1]$ and $W_2^1[0,1]$ are defined in the following section.

Suppose that the right-hand function $f(x, y(x), z(x), t(x))$ of $Eq.(1.1)$ satisfies condition H:

$(H1)$ $f(x, y, z, t) \in W_2^1[0,1]$, for $y = y(x)$, $z = z(x)$, $t = t(x) \in W_2^1[0,1]$.

$(H2)$ f, f_x, f_y, f_z and f_t are bounded on $[0,1] \times R^3$.

$(H3)$ $f(x, y, z, t) > 0$ on $[0,1] \times R^3$.

However, to the best of our knowledge, few papers can be found in the literature for solving these problems. We give a constructive method to obtain the exact solution of $Eq.(1.1)$ expressed by series in the reproducing kernel space. Finally, results of numerical experiments have been given, which supports the theoretical analysis of our method.

2 Preliminary

Definition 2.1. $W_2^1[0,1] = \{u(x) \mid u(x)$ *is an absolutely continuous real value function in* $[0,1]$, $u'(x) \in L^2[0,1]\}$. *The inner product and norm in* $W_2^1[0,1]$ *are given respectively by*

$$(u(x), v(x))_1 = u(0)v(0) + \int_0^1 u'(x)v'(x)dx, \ \| u \|_1 = (u(x), u(x))^{\frac{1}{2}}.$$

Theorem 2.2. *The space* $W_2^1[0,1]$ *is a reproducing kernel space. That is, for any fixed* $x \in [0,1]$, *there exists* $R1_x(y) \in W_2^1[0,1]$, $u(y) \in W_2^1[0,1]$, *such that* $u(x) = (u(y), R1_x(y))_1$. *The reproducing kernel* $R1_x(y)$ *can be denoted by*

$$R1_x(y) = \begin{cases} 1 + y, & y \leq x, \\ 1 + x, & y > x. \end{cases} \tag{2}$$

Definition 2.3. $W_2^4[0,1] = \{u(x) | u(x),\ u'(x),\ u''(x),\ u^{(3)}(x)$ *are absolutely continuous real value functions in [0,1],* $u'(0) = u'(1) = u(0) = 0$, $u^{(4)}(x) \in L^2[0,1]\}$. *The inner product and norm in* $W_2^4[0,1]$ *are given respectively by*

$$(u(x), v(x))_4 = \sum_{i=1}^{3} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(4)}(x)v^{(4)}(x)dx, \ \| u \|_4 = (u(x), u(x))^{\frac{1}{2}}. \tag{3}$$

Theorem 2.4. *The space* $W_2^4[0,1]$ *is a reproducing kernel space. The reproducing kernel* $R4_x(y)$ *can be denoted by*

$$R4_x(y) = \begin{cases} -\frac{y^2}{4717440}(-6552xy^4 + 936y^5 + 420x^3(360 - 252y - 63y^2 \\ -6y^3 + y^4) + 105x^4(360 + 60y + 15y^2 - 6y^3 + y^4) - 42x^5(360 \\ +60y + 15y^2 - 6y^3 + y^4) + 7x^6(360 + 60y + 15y^2 - 6y^3 \\ +y^4) + 504x^2(-540 + 300y + 75y^2 + 9y^3 + 5y^4)), \ y \leq x, \\ \frac{x^2}{4717440}(-936x^5 + 6552x^4y - 504(-540 + 300x + 75x^2 + 9x^3 \\ +5x^4)y^2 - 420(360 - 252x - 63x^2 - 6x^3 + x^4)y^3 - 105(360 \\ +60x + 15x^2 - 6x^3 + x^4)y^4 + 42(360 + 60x + 15x^2 \\ -6x^3 + x^4)y^5 - 7(360 + 60x + 15x^2 - 6x^3 + x^4)y^6), \ y > x. \end{cases} \tag{4}$$

The method of obtaining the reproducing kernel $R1_x(y)$, $R4_x(y)$ and the proof of Theorem 2.2 and Theorem 2.4 are given in [12, Theorem 1.3.1 and Theorem 1.3.2].

3 Notes

(1) Put $\{x_i\}$ is dense on $[0,1]$, let $\varphi_i(x) = R1_{x_i}(x)$, $\psi_i(x) = \mathbb{L}^*\varphi_i(x)$, then

$$\begin{aligned}\psi_i(x) &= (\psi_i(y), R4_x(y))_4 = (\mathbb{L}^*\varphi_i(y), R4_x(y))_4 = (\varphi_i(y), \mathbb{L}_y R4_x(y))_1 \\ &= (R1_{x_i}(y), \mathbb{L}_y R4_x(y))_1 = \mathbb{L}_y R4_x(y)|_{y=x_i},\end{aligned}$$

where $\mathbb{L}^*$ is the conjugate operator of $\mathbb{L}$. $\{\psi_i(x)\}_{i=0}^{\infty}$ is a complete system, we can have the normal orthogonal system $\{\overline{\psi}_i(x)\}_{i=0}^{\infty}$ in $W_2^4[0,1]$ by using Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=0}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=0}^{i} \beta_{ik}\psi_k(x), \tag{5}$$

where β_{ik} are orthogonalization coefficients, $\beta_{ii} > 0$, $i = 0, 1, \ldots$.

(2) $\mathbb{P}_n : W_2^4[0,1] \to Span\{\overline{\psi}_0(x), \overline{\psi}_1(x), \cdots, \overline{\psi}_n(x)\}$ is an orthogonal projection operator.

(3) $\| w \|_{C^m} = \max\limits_{0\le x\le 1}\{\sum\limits_{i=0}^{m} | w^{(i)}(x) |\}$. M and $\{M_i\}_{i=0}^{3}$ are constants.

We give the iterative formula is: for any initial value $u_0(x) \in W_2^4[0,1]$,

$$\begin{cases} \mathbb{L}v_n(x) = f(x, u_n(x), u_n'(x), u_n''(x)), \\ u_{n+1}(x) = \mathbb{P}_n v_n(x),\ n = 0, 1, \ldots \end{cases} \tag{6}$$

Lemma 3.1. *$v_n(x) \neq 0$, $x \in (0,1)$.*

Proof. Suppose there exists a $x_0 \in (0,1)$ such that $v_n(x_0) = 0$. Moreover, $v_n(x) \in W_2^4[0,1]$, then there exists a $\xi_0 \in (0, x_0)$ such that $v_n'(\xi_0) = 0$, thus there exist $\xi_1 \in (0,\xi_0)$, $\xi_2 \in (\xi_0, 1)$ such that $v_n''(\xi_1) = 0$, $v_n''(\xi_2) = 0$. It is easy to see that $v_n^{(3)}(\xi) = 0$, $\xi \in (\xi_1, \xi_2)$, so $f(\xi, u_n(\xi), u_n'(\xi), u_n''(\xi)) = \mathbb{L}v_n(\xi) = v_n^{(3)}(\xi) = 0$, which is in contradiction to $(H3)$. □

From above Lemma, $u_{n+1}(x)$ is rewritten by $u_{n+1}(x) = \alpha_n v_n(x)$, where

$$\alpha_n = \frac{\mathbb{P}_n v_n(x)}{v_n(x)},\ x \in (0,1). \tag{7}$$

4 Existence

The progress of proof is divided into four-step:

(i) The method to obtain expression for $v_n(x)$ and $u_{n+1}(x)$, $n = 0, 1, \ldots$.

(ii) $\| \alpha_n - 1 \|_{C^3} \to 0 (n \to \infty)$.

(iii) To prove that $\{u_{n+1}(x)\}$ and $\{v_n(x)\}$ are bounded in $W_2^4[0, 1]$. The application of Lemma 4.5 shows that there exists a n_k such that $| u_{n_k+1}(x) - u(x) | \to 0$, $| v_{n_k}(x) - v(x) | \to 0$, as $k \to \infty$.

If (ii) and (iii) are proved, we have

$$v(x) = \lim_{k\to\infty} v_{n_k}(x) = \lim_{k\to\infty} \frac{u_{n_k+1}(x)}{\alpha_{n_k}} = \frac{\lim\limits_{k\to\infty} u_{n_k+1}(x)}{\lim\limits_{k\to\infty} \alpha_{n_k}} = u(x).$$

(iv) By (6), $\mathbb{L}v_{n_k}(x) = f(x, u_{n_k}(x), u'_{n_k}(x), u''_{n_k}(x))$, we still need to prove $| u_{n_k}(x) - u(x) | \to 0$, which comes from the proof of $| u_{n_k+1}(x) - u_{n_k}(x) | \to 0$.

Through above discussion, inserting n_k in (6) and taking limit for k on both sides, the proof is completed.

Lemma 4.1.

$$v_n(x) = \sum_{i=0}^{\infty} \sum_{k=0}^{i} \beta_{ik} f(x_k, u_n(x_k), u'_n(x_k), u''_n(x_k)) \overline{\psi}_i(x), \tag{8}$$

$$u_{n+1}(x) = \sum_{i=0}^{n} \sum_{k=0}^{i} \beta_{ik} f(x_k, u_n(x_k), u'_n(x_k), u''_n(x_k)) \overline{\psi}_i(x). \tag{9}$$

Proof. By (6), then

$$\begin{aligned} v_n(x) &= \sum_{i=0}^{\infty} (v_n(x), \overline{\psi}_i(x))_4 \overline{\psi}_i(x) = \sum_{i=0}^{\infty} \sum_{k=0}^{i} \beta_{ik} (v_n(x), \psi_k(x))_4 \overline{\psi}_i(x) \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{i} \beta_{ik} (v_n(x), \mathbb{L}^* \varphi_k(x))_4 \overline{\psi}_i(x) = \sum_{i=0}^{\infty} \sum_{k=0}^{i} \beta_{ik} (\mathbb{L} v_n(x), \varphi_k(x))_1 \overline{\psi}_i(x) \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{i} \beta_{ik} (f(x, u_n(x), u'_n(x), u''_n(x)), R1_{x_k}(x))_1 \overline{\psi}_i(x) \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{i} \beta_{ik} f(x_k, u_n(x_k), u'_n(x_k), u''_n(x_k)) \overline{\psi}_i(x). \end{aligned}$$

Since $\mathbb{P}_n$ is an orthogonal projection operator, it follows that

$$u_{n+1}(x) = \mathbb{P}_n v_n(x) = \sum_{i=0}^{n} \sum_{k=0}^{i} \beta_{ik} f(x_k, u_n(x_k), u'_n(x_k), u''_n(x_k)) \overline{\psi}_i(x).$$

□

Lemma 4.2. $\| u_{n+1} \|_4 \leq \| v_n \|_4$

Lemma 4.3. *If* $w(x) \in W_2^4[0, 1]$, *then* $\| w \|_{C^3} \leq M \| w \|_4$

Proof. For any $x, y \in [0,1]$, $w^{(i)}(x) = (w(y), \partial_{x^i} R4_x(y))_4$. By the expression of $R4_x(y)$, $\| \partial_{x^i} R4_x(\cdot) \|_4 \leq M_i$, then

$$| w^{(i)}(x) | = | (w(y), \partial_{x^3} R4_x(y))_4 | \leq \| w \|_4 \| \partial_{x^i} R4_x(\cdot) \|_4 \leq M_i \| w \|_4, \ i = 0,1,2,3.$$

Hence $\| w \|_{C^3} \leq M \| w \|_4$, where $M = \max\{M_0, M_1, M_2, M_3\}$. □

Lemma 4.4. *A bounded set in $W_2^1[0,1]$ is a compacted set in $C[0,1]$.*

Proof. Assume $w_n(x) \in W_2^1[0,1]$ and $\| w_n \|_1 \leq M$, then

$$| w_n(x) | = | (w_n(y), R1_x(y))_1 | \leq \| w_n \|_1 \| R1_x(\cdot) \|_1 \leq M\sqrt{R1_x(x)} \leq \sqrt{2}M,$$

$\{w_n(x)\}$ is uniformly bounded in $C[0,1]$. For any x_1, $x_2 \in [0,1]$, any $n \in N^+$,

$$\begin{aligned} & | w_n(x_1) - w_n(x_2) | \\ = \ & | (w_n(y), R1_{x_1}(y) - R1_{x_2}(y))_1 | \leq \| w_n \|_1 \| R1_{x_1}(\cdot) - R1_{x_2}(\cdot) \|_1 \\ \leq \ & M\sqrt{R1_{x_1}(x_1) + R1_{x_2}(x_2) - 2R1_{x_1}(x_2)} \leq M\sqrt{| x_1 - x_2 |}. \end{aligned}$$

then for any $\varepsilon > 0$, taking $\delta = \frac{\varepsilon^2}{M^2}$, we obtain $| w_n(x_1) - w_n(x_2) | < \varepsilon$ for $| x_1 - x_2 | < \delta$, so $\{w_n(x)\}$ is equicontinuous. Therefore, $\{w_n(x)\}$ is a compact set in $C[0,1]$. □

Lemma 4.5. *A bounded set in $W_2^4[0,1]$ is a compacted set in $C^3[0,1]$.*

Proof. Suppose $w_n(x) \in W_2^4[0,1]$ and $\| w_n \|_4 \leq M$. From Lemma 4.3, $\{w_n(x)\}$ is uniformly bounded in $C^3[0,1]$. For any x_1, $x_2 \in [0,1]$, any $n \in N^+$,

$$\begin{aligned} | w_n^{(i)}(x_1) - w_n^{(i)}(x_2) | &= | (w_n(y), \partial_{x^i} R4_{x_1}(y) - \partial_{x^i} R4_{x_2}(y))_4 | \\ &\leq \| w_n \|_4 \| \partial_{x^i} R4_{x_1}(\cdot) - \partial_{x^i} R4_{x_2}(\cdot) \|_4 \\ &\leq M \| \partial_{x^{i+1}} R4_\xi(\cdot) \|_4 | x_1 - x_2 | \leq MM_0 | x_1 - x_2 |, \end{aligned}$$

where $\xi \in (x_2, x_1)(or(x_1, x_2))$, $M_0 = \max\limits_{0 \leq \xi \leq 1}\{\| \partial_{x^i} R4_\xi(\cdot) \|_4\}$, $i = 0,1,2$. Therefore, for any $\varepsilon > 0$, taking $\delta = \frac{\varepsilon}{MM_0}$, we get $| w_n^{(i)}(x_1) - w_n^{(i)}(x_2) | < \varepsilon$ for $| x_1 - x_2 | < \delta$, so $\{w_n(x)\}$ is equicontinuous. Since $\| w_n^{(3)} \|_1 = (w_n^{(3)}(0))^2 + \int_0^1 (w_n^{(4)}(x))^2 dx \leq \sum\limits_{i=0}^{3} (w_n^{(i)}(0))^2 + \int_0^1 (w_n^{(4)}(x))^2 dx = \| w_n \|_4$, then $\{w_n^{(3)}(x)\}$ is a bounded set in $W_2^1[0,1]$, thus $\{w_n^{(3)}(x)\}$ is equnicontinuous by Lemma 4.4. Consequently, $\{w_n(x)\}$ is a compact set in $C^3[0,1]$. □

Lemma 4.6. $\| \alpha_n - 1 \|_{C^3} \to 0$, *as* $n \to \infty$.

Proof. Put $\widetilde{v}_{n_k}(x) = \frac{v_{n_k}(x)}{\|v_{n_k}\|_4}$. Since $\| \widetilde{v}_{n_k} \|_4 = 1$, $\| (\mathbb{P}_{n_k} - \mathbb{I})\widetilde{v}_{n_k} \|_4 \leq \| \mathbb{P}_{n_k} - \mathbb{I} \| \to 0$, then $\{(\mathbb{P}_{n_k} - \mathbb{I})\widetilde{v}_{n_k}(x)\}$, $\{\widetilde{v}_{n_k}(x)\}$ are bounded in $W_2^4[0,1]$. From Lemma 4.5, there exists a n_{k_j} such that $\{(\mathbb{P}_{n_{k_j}}-\mathbb{I})\widetilde{v}_{n_{k_j}}(x)\}$, $\{\widetilde{v}_{n_{k_j}}(x)\}$ are convergent in $C^3[0,1]$, thus

$$\begin{aligned}\overline{\lim_{n\to\infty}}\parallel \alpha_n-1\parallel_{C^3} &= \overline{\lim_{n\to\infty}}\parallel \frac{\mathbb{P}_n v_n}{v_n}-1\parallel_{C^3}\overset{\exists n_k}{=}\lim_{k\to\infty}\parallel \frac{(\mathbb{P}_{n_k}-\mathbb{I})\widetilde{v}_{n_k}}{\widetilde{v}_{n_k}}\parallel_{C^3}\\ &= \lim_{j\to\infty}\parallel \frac{(\mathbb{P}_{n_{k_j}}-\mathbb{I})\widetilde{v}_{n_{k_j}}}{\widetilde{v}_{n_{k_j}}}\parallel_{C^3}=\parallel \frac{\lim_{j\to\infty}(\mathbb{P}_{n_{k_j}}-\mathbb{I})\widetilde{v}_{n_{k_j}}}{\lim_{j\to\infty}\widetilde{v}_{n_{k_j}}}\parallel_{C^3}=0.\end{aligned}$$

Therefore, $\parallel \alpha_n-1\parallel_{C^3}\to 0$, as $n\to\infty$. $\square$

Lemma 4.7. $\parallel v_n\parallel_4$, $\parallel u_{n+1}\parallel_4$ *are bounded.*

Proof. By $(H2)$, $\mid f(x,u_n(x),u_n'(x),u_n''(x))\mid\leq M$. From (6),

$$\begin{aligned}v_n(x) &= \int_0^x dt\int_0^t d\eta\int_0^\eta f(\xi,u_n(\xi),u_n'(\xi),u_n''(\xi))d\xi\\ &\quad -\tfrac{1}{2}x^2\int_0^1 d\eta\int_0^\eta f(\xi,u_n(\xi),u_n'(\xi),u_n''(\xi))d\xi,\end{aligned}\tag{10}$$

then

$$\begin{aligned}\mid v_n'(x)\mid &\leq \int_0^x d\eta\int_0^\eta \mid f(\xi,u_n(\xi),u_n'(\xi),u_n''(\xi))\mid d\xi\\ &\quad +\mid x\mid\int_0^1 d\eta\int_0^\eta \mid f(\xi,u_n(\xi),u_n'(\xi),u_n''(\xi))\mid d\xi\\ &\leq M\tfrac{1}{2}\mid x^2\mid+\mid x\mid M\tfrac{1}{2}\leq M.\end{aligned}$$

Using this method, $\mid v_n(x)\mid\leq \frac{5}{12}M$, $\mid v_n''(x)\mid\leq\frac{3}{2}M$, $\mid v_n^{(3)}(x)\mid\leq M$. By Lemma 4.6, $\parallel\alpha_n\parallel_{C^3}$ is bounded, then $\mid\alpha_n^{(i)}\mid$ $(i=0,1,2,3)$ are bounded, then $\mid u_{n+1}^{(i)}(x)\mid$ $(i=0,1,2,3)$ are bounded, namely $\mid u_n^{(i)}(x)\mid$ $(i=0,1,2,3)$ are bounded. Thus by $(H2)$ and (6), $\mid v_n^{(4)}(x)\mid=\mid f_x+f_yu_n'(x)+f_zu_n''(x)+f_tu_n^{(3)}(x)\mid$ is bounded. Hence, $\parallel v_n\parallel_4$ is bounded by the definition of $\parallel\cdot\parallel_4$. From Lemma 4.2, $\parallel u_{n+1}\parallel_4$ is bounded. $\square$

Lemma 4.8. *There exists a* n_k, *such that* $\mid v_{n_k+1}''(0)-v_{n_k}''(0)\mid\to 0(k\to\infty)$.

Proof. From Lemma 4.6, $\parallel\alpha_n\parallel_{C^3}\leq M_1$. Applying Lemma 4.3 and Lemma 4.7, $\parallel v_n\parallel_{C^3}\leq M_2$, then

$$\begin{aligned}\mid u_{n+1}(x)-u_n(x)\mid &= \mid\alpha_nv_n(x)-\alpha_{n-1}v_{n-1}(x)\mid\\ &\leq \mid\alpha_n(v_n(x)-v_{n-1}(x))\mid+\mid(\alpha_n-\alpha_{n-1})v_{n-1}(x)\mid\\ &\leq M_1\mid v_n(x)-v_{n-1}(x)\mid+M_2\mid\alpha_n-\alpha_{n-1}\mid\\ &\leq M_1\sum_{i=0}^{2}\mid v_n^{(i)}(x)-v_{n-1}^{(i)}(x)\mid+\rho_{n-1},\end{aligned}$$

where $\rho_{n-1}=M_2\parallel\alpha_n-\alpha_{n-1}\parallel_{C^3}$, by Lemma 4.6, $\parallel\alpha_n-\alpha_{n-1}\parallel_{C^3}\to 0$, then $\lim_{n\to\infty}\rho_{n-1}\to 0$. In the similar manner,

$$\mid u_{n+1}^{(i)}(x)-u_n^{(i)}(x)\mid\ \leq\ iM_1\sum_{i=0}^{2}\mid v_n^{(i)}(x)-v_{n-1}^{(i)}(x)\mid+i\rho_{n-1},\ i=1,2,$$

$$\sum_{i=0}^{2} \mid u_{n+1}^{(i)}(x) - u_n^{(i)}(x) \mid \leq 4M_1 \sum_{i=0}^{2} \mid v_n^{(i)}(x) - v_{n-1}^{(i)}(x) \mid +4\rho_{n-1}.$$

Since $v'_{n+1}(0) - v'_n(0) = 0$, $v'_{n+1}(1) - v'_n(1) = 0$, it follows that there exists a $\theta_n \in (0,1)$ such that

$$v''_{n+1}(\theta_n) - v''_n(\theta_n) = 0. \tag{11}$$

Since $\{\theta_n\}$ is bounded, there exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \to \theta(k \to \infty)$. In the following, the proof contains two parts:

(i) $\theta = 1$. In this case, applying Lemma 4.5 and Lemma 4.7, there exist $v(x)$ and $\overline{v}(x)$ such that $v''_{n_k}(x) \to v''(x)$, $v''_{n_k+1}(x) \to \overline{v}''(x)(k \to \infty)$(for convenience, subscripts to subsequence of $\{v_n(x)\}$ and $\{v_{n+1}(x)\}$ are denoted by n_k and $n_k + 1$, respectively). By (11),

$$\lim_{k\to\infty} \mid v''_{n_k+1}(1) - v''_{n_k}(1) \mid = \mid \overline{v}''(1) - v''(1) \mid = \lim_{k\to\infty} \mid v''_{n_k+1}(\theta_{n_k}) - v''_{n_k}(\theta_{n_k}) \mid = 0.$$

(ii) $\theta \neq 1$, namely $\theta_n < \theta < 1$(for convenience, subscript to subsequence of $\{\theta_n\}$ is denoted by n instead of n_k). When $x \in (\theta, 1]$, noting that $\mathbb{L}v_n(x) = v_n^{(3)}(x) = f(x, u_n(x), u'_n(x), u''_n(x))$, then

$$\begin{array}{l}
v''_n(x) - v''_n(\theta_n) = \int_{\theta_n}^x f(\xi, u_n(\xi), u'_n(\xi), u''_n(\xi))d\xi,\\
v'_n(x) - v''_n(\theta_n)x = \int_0^x d\eta \int_{\theta_n}^\eta f(\xi, u_n(\xi), u'_n(\xi), u''_n(\xi))d\xi,\\
v_n(x) - \frac{1}{2}v''_n(\theta_n)x^2 = \int_0^x dt \int_0^t d\eta \int_{\theta_n}^\eta f(\xi, u_n(\xi), u'_n(\xi), u''_n(\xi))d\xi,\\
v_{n+1}(x) - \frac{1}{2}v''_{n+1}(\theta_n)x^2 = \int_0^x dt \int_0^t d\eta \int_{\theta_n}^\eta f(\xi, u_{n+1}(\xi), u'_{n+1}(\xi), u''_{n+1}(\xi))d\xi,
\end{array}$$

thus by (11),

$$\begin{array}{rcl}
v_{n+1}(x) - v_n(x) & = & \int_0^x dt \int_0^t d\eta \int_{\theta_n}^\eta (f(\xi, u_{n+1}(\xi), u'_{n+1}(\xi), u''_{n+1}(\xi))\\
& & -f(\xi, u_n(\xi), u'_n(\xi), u''_n(\xi)))d\xi.
\end{array}$$

From $(H2)$, let $M_3 = \max\{f_x, f_y, f_z, f_t\}$,

$$\begin{array}{rcl}
\mid v_{n+1}(x) - v_n(x) \mid & \leq & \mid \int_0^x dt \int_0^t d\eta \int_{\theta_n}^\eta \mid f(\xi, u_{n+1}(\xi), u'_{n+1}(\xi), u''_{n+1}(\xi))\\
& & -f(\xi, u_n(\xi), u'_n(\xi), u''_n(\xi)) \mid d\xi \mid\\
& \leq & M_3 \int_0^1 dt \int_0^1 d\eta \int_0^x \sum_{i=0}^{2} \mid u_{n+1}^{(i)}(\xi) - u_n^{(i)}(\xi)d\xi\\
& \leq & M_3 \int_0^x \sum_{i=0}^{2} \mid u_{n+1}^{(i)}(\xi) - u_n^{(i)}(\xi) \mid d\xi.
\end{array}$$

By doing this,

$$\mid v_{n+1}^{(i)}(x) - v_n^{(i)}(x) \mid \;\; \leq \;\; M_3 \int_0^x \sum_{i=0}^{2} \mid u_{n+1}^{(i)}(\xi) - u_n^{(i)}(\xi)d\xi, \; i = 1, 2$$

$$\begin{aligned}\sum_{i=0}^{2} | v_{n+1}^{(i)}(x) - v_n^{(i)}(x) | &\leq 3M_3 \int_0^x \sum_{i=0}^{2} | u_{n+1}^{(i)}(\xi) - u_n^{(i)}(\xi) d\xi \\ &\leq 12M_3 (M_1 \int_0^x \sum_{i=0}^{2} | v_n^{(i)}(\xi) - v_{n-1}^{(i)}(\xi) d\xi + \rho_n) \quad (12) \\ &\leq \overline{M} \int_0^x \sum_{i=0}^{2} | v_n^{(i)}(\xi) - v_{n-1}^{(i)}(\xi) d\xi + \overline{\rho}_n,\end{aligned}$$

where $\overline{M} = 12M_1M_3$, $\overline{\rho}_n = 12M_3\rho_n$, $\lim_{n\to\infty} \overline{\rho}_n = 0$.

If $n = 1$, $\sum_{i=0}^{2} | v_2^{(i)}(x) - v_1^{(i)}(x) | \leq \overline{M}x \parallel v_1 - v_0 \parallel_{C^2} + \overline{\rho}_1$.

If $n = 2$,

$$\begin{aligned}\sum_{i=0}^{2} | v_3^{(i)}(x) - v_2^{(i)}(x) | &\leq \overline{M} \int_0^x \sum_{i=0}^{2} | v_2(i)(\xi) - v_1(i)(\xi) | d\xi + \overline{\rho}_2 \\ &\leq \overline{M} \int_0^x (\overline{M}\xi \parallel v_1 - v_0 \parallel_{C^2} + \overline{\rho}_1) d\xi + \overline{\rho}_2 \\ &\leq \frac{(x\overline{M})^2}{2} \parallel v_1 - v_0 \parallel_{C^2} + \overline{M}\overline{\rho}_1 x + \overline{\rho}_2.\end{aligned}$$

Generally,

$$\begin{aligned}&| v_{n+1}(x) - v_n(x) | + | v'_{n+1}(x) - v'_n(x) | + | v''_{n+1}(x) - v''_n(x) | \\ \leq\ & \frac{(x\overline{M})^n}{n!} \parallel v_1 - v_0 \parallel_{C^2} + \sum_{k=0}^{n} \frac{(x\overline{M})^k}{k!} \overline{\rho}_{n-k} \\ \leq\ & \frac{(\overline{M})^n}{n!} \parallel v_1 - v_0 \parallel_{C^2} + \sum_{k=0}^{[n/2]} \frac{(\overline{M})^k}{k!} \overline{\rho}_{n-k} + \sum_{k=[n/2+1]}^{n} \frac{(\overline{M})^k}{k!} \overline{\rho}_{n-k} \\ \leq\ & \frac{(\overline{M})^n}{n!} \parallel v_1 - v_0 \parallel_{C^2} + \widetilde{\rho}_n \sum_{k=0}^{[n/2]} \frac{(\overline{M})^k}{k!} + \overline{\rho} \sum_{k=[n/2+1]}^{n} \frac{(\overline{M})^k}{k!} \longrightarrow 0,\end{aligned}$$

where $[\cdot]$ denotes the integral part of " $\cdot$ ", $\widetilde{\rho}_n = \max\{\overline{\rho}_n, \overline{\rho}_{n-1}, \cdots, \overline{\rho}_{n-[n/2]}\}$, $\overline{\rho} = \max\{\overline{\rho}_0, \overline{\rho}_1, \cdots, \overline{\rho}_{n-[n/2+1]}\}$. Since $\lim_{n\to\infty} \widetilde{\rho}_n = 0$, $\parallel v_{n+1}(x) - v_n(x) \parallel_{C^2} \to 0$, as $n \to \infty$, $x \in (\theta, 1]$.

Combining (i) and (ii), we infer that there exists a n_k such that

$$\begin{cases} \parallel v_{n_k+1}(x) - v_{n_k}(x) \parallel_{C^2} \to 0, \ x \in (\theta, 1] (\theta \neq 1), \\ | v''_{n_k+1}(1) - v''_{n_k}(1) | \to 0, \ \theta = 1. \end{cases}$$

Let $\overline{v}_{n_k}(x) = v_{n_k}(1-x)$, $\overline{u}_{n_k}(x) = u_{n_k}(1-x)$, then using the above method, we can verify that

$$\begin{cases} \parallel \overline{v}_{n_k+1}(x) - \overline{v}_{n_k}(x) \parallel_{C^2} \to 0, x \in (\theta, 1] (\theta \neq 1), \\ | \overline{v}''_{n_k+1}(1) - \overline{v}''_{n_k}(1) | \to 0, \ \theta = 1, \end{cases}$$

which means that

$$\begin{cases} \parallel v_{n_k+1}(1-x) - v_{n_k}(1-x) \parallel_{C^2} \to 0, \ x \in (\theta, 1] (\theta \neq 1), \\ | v''_{n_k+1}(0) - v''_{n_k}(0) | \to 0, \ \theta = 1. \end{cases}$$

Consequently,

$$\begin{cases} \parallel v_{n_k+1}(x) - v_{n_k}(x) \parallel_{C^2} \to 0, \ x \in [0, 1-\theta) (\theta \neq 1), \\ | v''_{n_k+1}(0) - v''_{n_k}(0) | \to 0, \ \theta = 1. \end{cases}$$

Furthermore, there exists a n_k such that $| v''_{n_k+1}(0) - v''_{n_k}(0) | \to 0$, as $k \to \infty$. □

Lemma 4.9. *There exists a n_k such that $| u_{n_k+1}(x) - u_{n_k}(x) | \to 0 (k \to \infty)$.*

Proof. By (10), $v''_{n_k}(0) = -\int_0^1 d\eta \int_0^\eta f(\xi, u_{n_k}(\xi), u'_{n_k}(\xi), u''_{n_k}(\xi))d\xi$, then

$$v_{n_k}(x) = \int_0^x dt \int_0^t d\eta \int_0^\eta f(\xi, u_{n_k}(\xi), u'_{n_k}(\xi), u''_{n_k}(\xi))d\xi + \tfrac{1}{2}x^2 v''_{n_k}(0). \qquad (13)$$

Thus

$$\begin{aligned} & | v_{n_k+1}(x) - v_{n_k}(x) | \\ \leq \ & | \int_0^x dt \int_0^t d\eta \int_0^\eta (f(\xi, u_{n_k+1}(\xi), u'_{n_k+1}(\xi), u''_{n_k+1}(\xi)) \\ & - f(\xi, u_{n_k}(\xi), u'_{n_k}(\xi), u''_{n_k}(\xi)))d\xi | + \tfrac{x^2}{2} | v''_{n_k+1}(0) - v''_{n_k}(0) | \\ \leq \ & M_3 \int_0^x \sum_{i=0}^{2} | u^{(i)}_{n_k+1}(\xi) - u^{(i)}_{n_k}(\xi) | d\xi + \tfrac{1}{2}\rho_{n_k}. \end{aligned}$$

In the same manner,

$$| v^{(i)}_{n_k+1}(x) - v^{(i)}_{n_k}(x) | \ \leq \ M_3 \int_0^x \sum_{i=0}^{2} | u^{(i)}_{n_k+1}(\xi) - u^{(i)}_{n_k}(\xi) | d\xi + \rho_{n_k}, \ i = 1, 2,$$

where $M_3 = \max\{f_x, f_y, f_z, f_t\}$, $\rho_{n_k} = | v''_{n_k+1}(0) - v''_{n_k}(0) |$. By Lemma 4.8, $\lim_{k\to\infty} \rho_{n_k} \to 0$. Using the same method after (12), $| v_{n_k+1}(x) - v_{n_k}(x) | \to 0$. Lemma 4.6 and $u_{n+1}(x) = \alpha_n v_n(x)$ assure $| u_{n_k+1}(x) - u_{n_k}(x) | \to 0$. □

Theorem 4.10. *The solution of Eq.(1) exists in $W_2^4[0,1]$.*

Proof. From Lemma 4.7, $\| u_{n+1} \|_4 \leq M$. Applying Lemma 4.5, there exists a n_k such that a subsequence $\{u_{n_k+1}(x)\}$ of $\{u_{n+1}(x)\}$ converges to a function $u(x)$, a subsequence $\{v_{n_k}(x)\}$ of $\{v_n(x)\}$ converges to a function $v(x) \in W_2^4[0,1]$ in the sense of normal $\| \cdot \|_{C^3}$, thus $| u_{n_k+1}(x) - u(x) | \to 0$, $| v_{n_k}(x) - v(x) | \to 0$.

In terms of Lemma 4.6, $| \alpha_{n_k} - 1 | \to 0$, as $k \to \infty$, it holds that

$$v(x) = \lim_{k\to\infty} v_{n_k}(x) = \lim_{k\to\infty} \frac{u_{n_k+1}(x)}{\alpha_{n_k}} = \frac{\lim\limits_{k\to\infty} u_{n_k+1}(x)}{\lim\limits_{k\to\infty} \alpha_{n_k}} = u(x).$$

Due to Lemma 4.9, $| u_{n_k+1}(x) - u_{n_k}(x) | \to 0$, then $| u_{n_k}(x) - u(x) | \leq | u_{n_k+1}(x) - u_{n_k}(x) | + | u_{n_k+1}(x) - u(x) | \to 0$. In view of (6), $\mathbb{L}v_{n_k}(x) = f(x, u_{n_k}(x), u'_{n_k}(x), u''_{n_k}(x))$, taking limit for k on both sides, we have $\mathbb{L}v(x) = f(x, u(x), u'(x), u''(x))$. Hence, $\mathbb{L}u(x) = f(x, u(x), u'(x), u''(x))$, the solution of $Eq.(1)$ exists. Since $f(x, u(x), u'(x), u''(x)) \in W_2^1[0,1]$, $u(x) \in W_2^4[0,1]$. □

Corollary 4.11. *The limit function of any convergent subsequence of $\{u_n(x)\}$ is the solution to Eq.(1).*

Theorem 4.12. *If the solution of Eq.*(1) *is unique, then* $\{u_n(x)\}$ *is convergent.*

Proof. Suppose $\{u_n(x)\}$ is not convergent. By Lemma 4.7, $\| u_n \|_4$ is bounded, we can choose two subsequences $\{u_{n_k}(x)\}$, $\{u_{n_j}(x)\}$ such that $u_{n_k}(x) \to u_1(x)$, $u_{n_j}(x) \to u_2(x)$ and $u_1(x) \neq u_2(x)$, From Corollary 4.11, $u_1(x)$ and $u_2(x)$ are the solution of *Eq.*(1), this contradicts the uniqueness for solution of *Eq.*(1). □

5 Numerical examples

All computations are performed by Mathematica 5.0, and we take $u_0(x) = 0$.

Example 5.1. *Taking* $f(x, y, z, t) = -6\pi + \cos(\pi(1.5 - x)x^2) - \sin(3\pi(-1 + x)x) - \cos y - \sin z$ *in Eq.(1.1), the true solution is* $u(x) = \pi x^2(1.5 - x)$*. The numerical result are given in following Table 1, Table 2.*

Table 1: Numerical result for Example 1

Node	$u(x)$	$u_{42}(x)$	Absolute Error	Relative Error
0.1	0.0439823	0.043978	4.34201E-6	9.87217E-5
0.2	0.163363	0.163353	9.47884E-6	5.80232E-5
0.3	0.339292	0.339277	1.47661E-5	4.35203E-5
0.4	0.55292	0.5529	1.98828E-5	3.59596E-5
0.5	0.785398	0.785374	2.45757E-5	3.12907E-5
0.6	1.01788	1.01785	2.86479E-5	2.81448E-5
0.7	1.2315	1.23147	3.19506E-5	2.59444E-5
0.8	1.40743	1.4074	3.43755E-5	2.44242E-5
0.9	1.52681	1.52678	3.58507E-5	2.34807E-5

Table 2: RMS errors for the derivatives for Example 1

$\sqrt{\sum_{i=1}^{10} \frac{(u'(0.1i)-u'_{42}(0.1i))^2}{10}}$	$\sqrt{\sum_{i=1}^{10} \frac{(u''(0.1i)-u''_{42}(0.1i))^2}{10}}$	$\sqrt{\sum_{i=1}^{10} \frac{(u'''(0.1i)-u'''_{42}(0.1i))^2}{10}}$
1.50443E-8	5.13786E-8	7.10415E-7

Example 5.2. *Taking* $f(x, y, z, t) = e^{-16(2-6x+3x^2)^2} + 24(x - 1) + (1 + x)\cos((-2 + x)^2 x^2)/(1 + 16(-2 + x)^2(-1 + x)^2 x^2) - (1 + x)\cos y/(1 + z^2) - e^{-t^2}$ *in Eq.(1.1), the true solution is* $u(x) = (2x - x^2)^2$*. The numerical result are given in following Table 3, Table 4.*

Table 3: Numerical result for Example 3

Node	$u(x)$	$u_{60}(x)$	Absolute Error	Relative Error
0.1	0.0361	0.0360965	3.54375E-6	9.81649E-5
0.2	0.1296	0.129593	7.2364E-6	5.58364E-5
0.3	0.2601	0.260089	1.07918E-5	4.14911E-5
0.4	0.4096	0.409586	1.40866E-5	3.43911E-5
0.5	0.5625	0.562483	1.70015E-5	3.02248E-5
0.6	0.7056	0.705581	1.94621E-5	2.75824E-5
0.7	0.8281	0.828079	2.14175E-5	2.58634E-5
0.8	0.9216	0.921577	2.28241E-5	2.47657E-5
0.9	0.9801	0.980076	2.36614E-5	2.41418E-5

Table 4: RMS errors for the derivatives for Example 3

$\sqrt{\sum_{i=1}^{10} \frac{(u'(0.1i)-u'_{60}(0.1i))^2}{10}}$	$\sqrt{\sum_{i=1}^{10} \frac{(u''(0.1i)-u''_{60}(0.1i))^2}{10}}$	$\sqrt{\sum_{i=1}^{10} \frac{(u'''(0.1i)-u'''_{60}(0.1i))^2}{10}}$
6.53229E-9	1.93492E-8	8.75839E-9

6 Conclusion

In this paper, We have shown the easier verification conditions for the existence of nonlinear third-order two-point boundary value problems. We have developed reproducing kernel theorem for solving the nonlinear third-order two-point boundary value problems. The numerical results were as accurate as the theory predicted and it outperforms other available results. In addition, our results cannot be deduced trivially from any of the earlier published results.

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