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On Polynomials of K$(2,n)$ Torus Knots

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Abstract

In this paper, we introduce general formulas finding the Bracket polynomials of torus knot $K(2,n)$ and Jones polynomials of torus knots $K(2,n)$.

1. Introduction

We will use many standard terminologies and notations knot theory. (See [1] for the basic terminology of knot theory .) A knot is a simple closed curve obtained by embedding of the circle S^1 into R^3 (or S^3). A Knot is a torus knot if it is equivalent to a knot that can be drawn without any points of intersection on the trivial torus. Trivial torus is a solid T obtained by rotating around the y-axis the circle $m:(x-2)^2+y^2=1$, on the xy-plane, which as its center the point $(2,0)$, radius 1 unit [2]. The torus knot $K(p,q)$ of the type $p,\ q$ is the knot which wraps around this standard solid torus T in the longitudinal direction p times and in the meridianal direction q time, where $p,\ q$ are relatively prime.

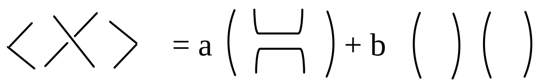
The natural question which emerges in knot theory is, given two knot diagrams, to determine if they represent the same knot. A special case is the unknotting problem: To determine if a given knot diagram represents the unknot. If the unknotting problem has a solution in some specific example, then we can prove this fact by using the appropriate Reidemeister moves (that transform our diagram into the round circle). In the case when we are given a knot diagram that we suspect represent a nontrivial knot, in order the prove its nontriviality, the standard method is to use an invariant, i.e., an assignment to each knot diagram of some algebraic object (e.g. a number or a polynomial) that depends only on the knot isotopy class, and verify that the value of this invariant for the given knot differs from that for the unknot. This method will be effective if there is a simple algorithm for computing the invariant from the knot diagram.

The famous Jones polynomial is such an invariant for knot; it will be described inhere. Due to Louis Kauffman, we begin its construction by defining the so-called Bracket polynomial of nonoriented knot diagrams.

2. The Bracket Polynomials and the Jones Polynomial

Definition 1. Let us assign to each nonoriented knot diagram K a polynomial in the variables a, b, c denoted by $\langle K \rangle$, where it is satisfies the following defining relations:

(1)



(2) < K U O > = c < K >

(3) < 0 > = 1 [3].

Here, we denote the little pictures in relation (1) by three knot diagrams K, K_A, K_B respectively. In this rotation, (1) may be rewritten in the form $\langle K \rangle = a\langle K_A \rangle + b\langle K_B \rangle$. Note that for the crossing contained in the small disk in the diagram K, no matter how it is rotated, the diagrams K_A and K_B are uniquely defined. In fact, the arcs inside the small disks of the diagrams K_A and K_B are chosen in the regions A and B respectively, i.e., when we move along the upper branch of the crossing, we first see the region A to our left, and, after the crossing, to our right (and conversely for B) (Figure 1.).

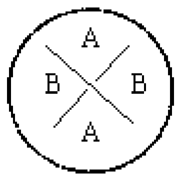
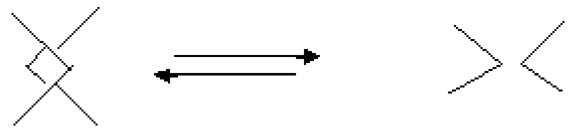
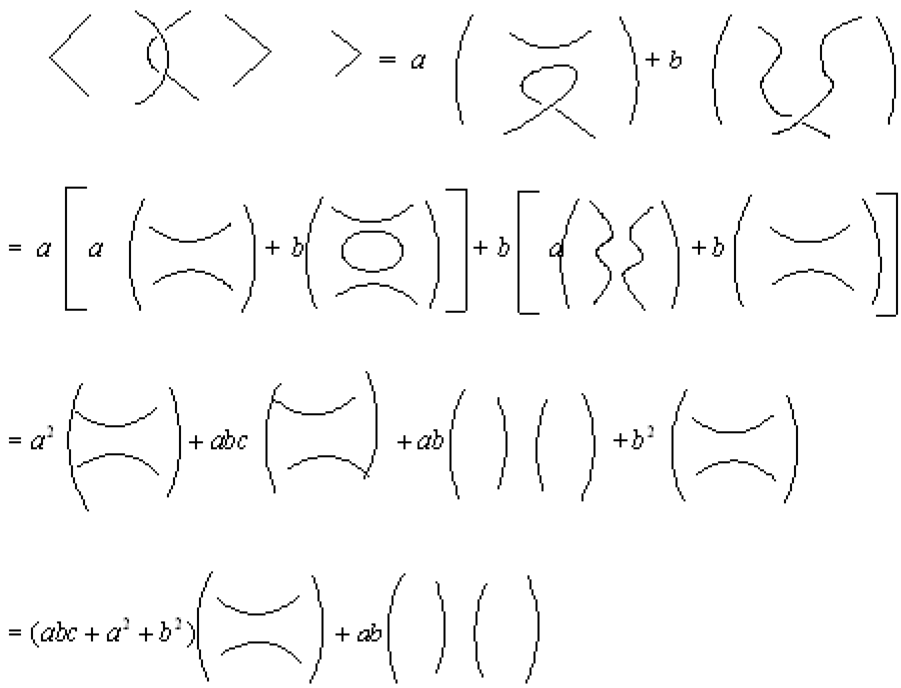


Figure 1.

Relation (2) means that the addition to the knot K of a circular component which does not intersect K that has no crossing points with K results in the polynomial being multiplied by c. Moreover, relation (3) means that the polynomial assigned to the circle is equal to 1. Consequently, we suppose that the polynomial $\langle K \rangle$ does not change under plane isotopies of the diagram K.

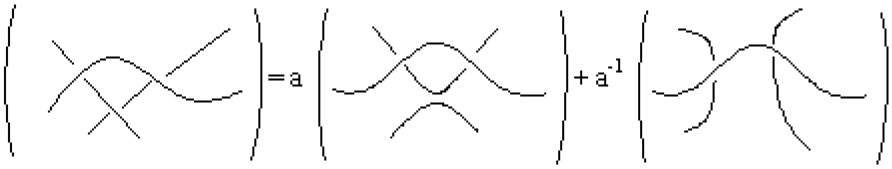
Now, let us try to impose relationships between the variables a, b, c so that the polynomial will be invariant with respect to the three Reidemeister moves. If we consider the move Ω_2, it follows that;

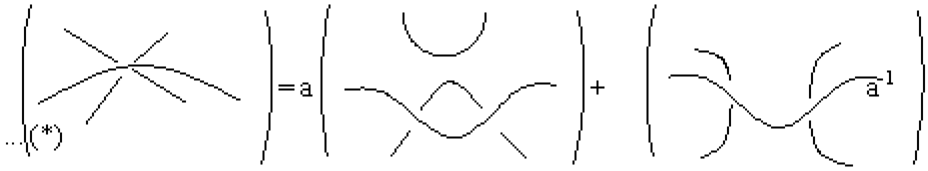




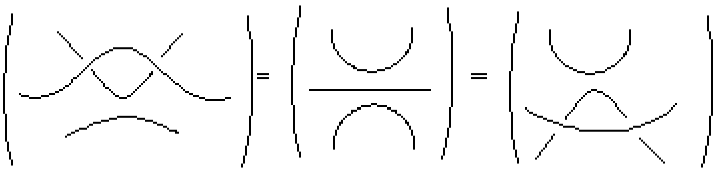
Now if we had $ab = 1$ and $a^2 + b^2 + abc = 0$, the polynomial would be invariant with respect to Ω_2. So we obtain $b = a^{-1}$ and $c = -a^2 - b^2$, thereby ensuring invariance with respect to Ω_2 of the Bracket polynomial $\langle K \rangle$.

The Bracket polynomial as above defined turns out to be invariant with respect to Ω_3 as well. By condition (1), we obtain the two following relations.



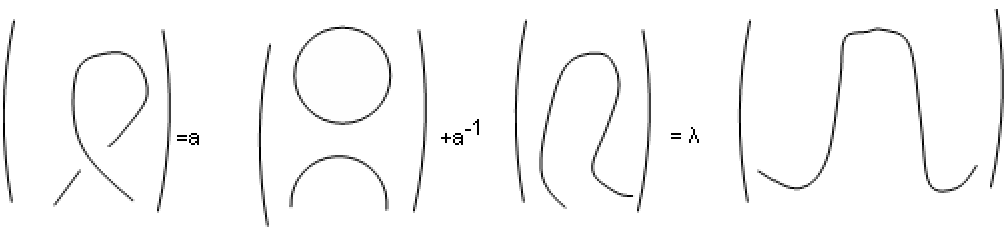


Clearly, the two diagrams that are into a^{-1} are the same. Further, applying Ω_2-invariance twice, it follows that

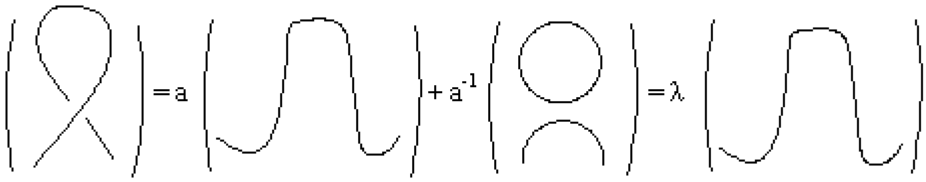


Now comparing the right-hand sides of the two formulas (*), we see that they are equal term-by term. Then so are the left-hand sides, which proves Ω_3-invariance of the Bracket polynomial.

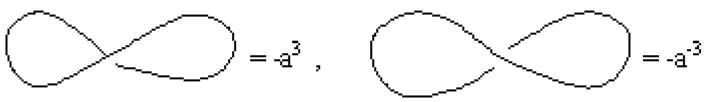
Now we turn to Ω_1. By relations (1) and (2), we have



where $\lambda = a(-a^2 - a^{-2}) + a^{-1} = -a^3$. A similar computation can be performed for the other type of little loop. Thus we have

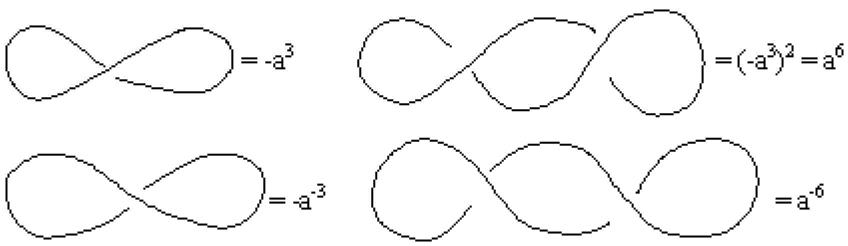


where $\lambda = a^{-1}(-a^2 - a^{-2}9 + a = -a^{-3}$. So, unless $a^3 = -1$, the Bracket polynomial is not invariant with respect to Ω_1 and therefore is not an isotopy invariant of knots. For example, we have



although both "figure eights" are diagrams of the trivial knot [3].

Example 2.1. For the simplest diagrams with one and two crossings, we have



Example 2.2. For the Hopf links Bracket polynomial is

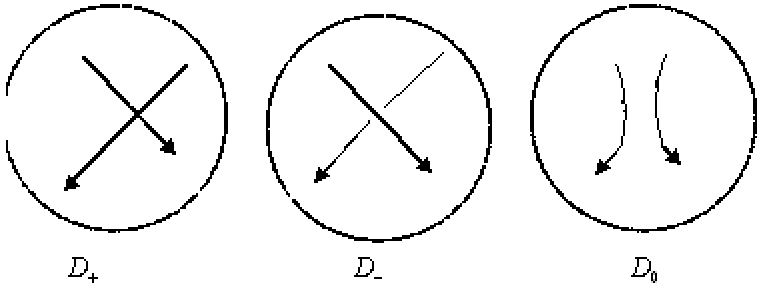
$-a^4 - a^{-4}$

2.2. Definition of the Jones Polynomial. Assume K is an oriented knot and D is a oriented regular diagram for K. Then the Jones polynomial of K, $V_K(t)$, can be defined uniquely from the following two axioms. The polynomial itself may be terms in which $\sqrt{t}$ has a negative exponent. (We assume $\left(\sqrt{t}\right)^2 = t$) The polynomial $V_K(t)$ is an invariant of K.

Axiom 1: If K is the trivial knot, then $V_K(t)=1$ [1].

Axiom 2: Suppose that D_+, D_-, D_0 are skein diagrams (see Figure 2), then the following skein relation holds [2].

$$\frac{1}{t}V_{D_+}(t) - tV_{D_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{D_0}(t).$$



Skein Diagrams

Figure 2.

The Jones polynomial satisfies the following relations:

(1') $t^{-1}V(D_+) - tV(D_-) = (t^{1/2} - t^{-1/2})V(D_0)$

(2') $V(K \cup 0) = -(t^{-1/2} + t^{1/2})V(K)$

(3') $V(0) = 1$

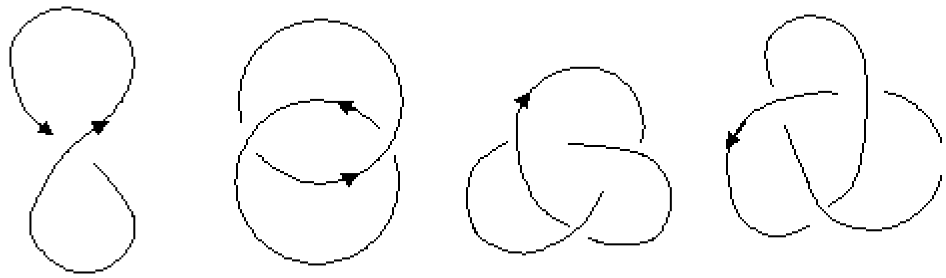
Remark. For an oriented knot diagram K, let us define its writhe number by setting

$$w(K) = \sum_i e_i$$

where the sum is taken over all crossing points and the numbers e_i are equal to ± 1 depending on the sign of the ith crossing point, which is defined as shown in Figure 3.

Figure 3

If the orientation of the knot K is reversed, then the writhe number $w(K)$ does not change. The calculating of writhe numbers for some knot examples is presented below.



$w = -1$ $w = 2$ $w = -3$ $w = 3$

Now let us define the Kauffman polynomial $\hat{P}_K(a)$ for any oriented knot K by setting

$$\hat{P}_K(a) = (-a)^{-3w(K)} P_K(a),$$

where $P_K(a)$ is the Bracket polynomial.

Let us substitute $a = t^{-1/4}$ into the Kauffman polynomial $\hat{P}_K(a)$. We then obtain a polynomial in $t^{\pm 1/4}$, denoted by $V_K(t)$ and called the Jones polynomial of the oriented knot K.

Example 2.3. The Bracket polynomial of left-hand trefoil is $a^7 - a^3 - a^{-5}$. Now we give an orientation on the left-hand trefoil. Then, the writhe number of the oriented left-hand trefoil is equal to –3. We have the Kauffman polynomial of the oriented left-hand trefoil as follows:

$$\begin{aligned} \hat{P}_K(a) &= (-a)^{-3.(-3)} P_K(a) \\ &= (-a)^9.(a^7 - a^3 - a^{-5}) \\ &= -a^{16} + a^{12} + a^4 \quad ....(**) \end{aligned}$$

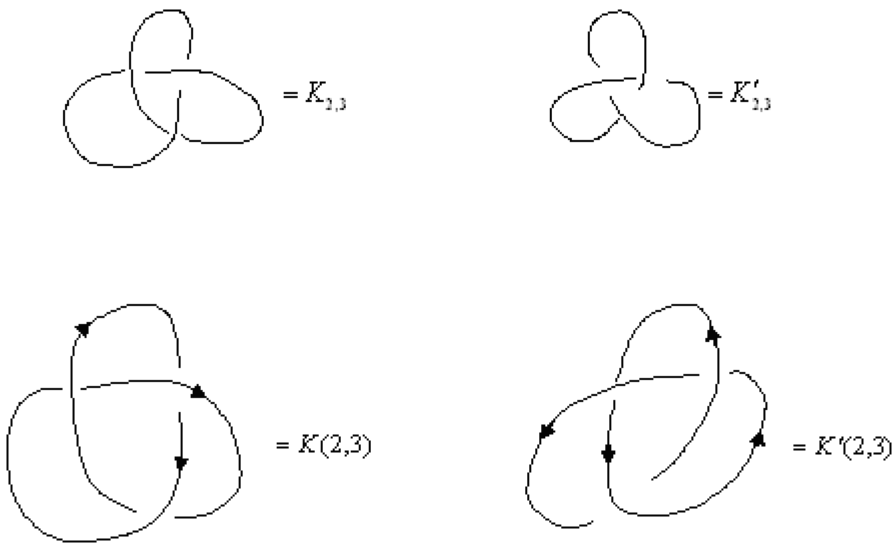
Now let us substitute $a = t^{-1/4}$ into relation (**). It follows that

$$\begin{aligned} V_K(t) &= -(t^{-1/4})^{16} + (t^{-1/4}9^{12} + (t^{-1/4})^4 \\ &= -t^{-4} + t^{-3} + t^{-1}. \end{aligned}$$

This is the Jones polynomial for the oriented left-hand trefoil.

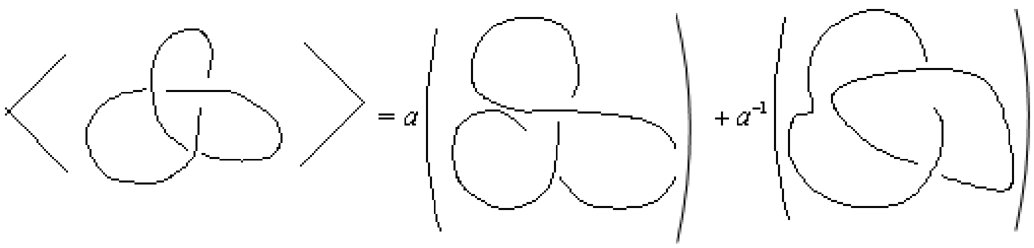
3. The Bracket Polynomial of Torus Knots $K_{2,n}$ and The Jones Polynomials of Torus Knots $K(2,n)$

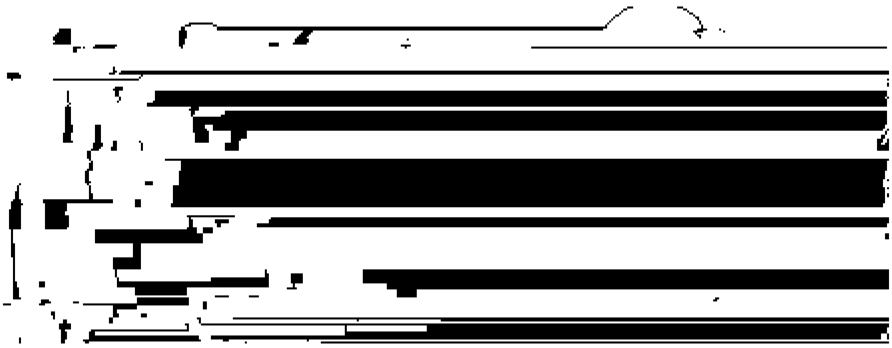
Firstly, let us label nonoriented torus knots $K_{2,n}$ and oriented torus knots $K(2,n)$ as follows. Then we use these labelled cases in our studies.

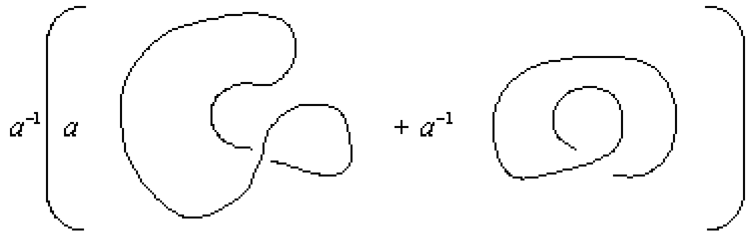


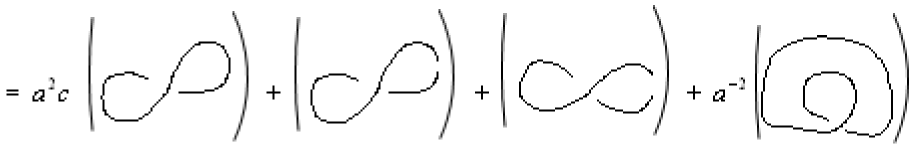
3.1. The Bracket Polynomial of Torus Knots $K_{2,n}$:

Example 1. We find the Bracket polynomial of $K_{2,3}$.

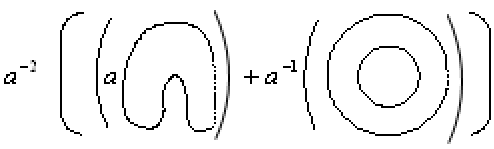








$$= a^2(-a^{-2}-a^2)\ \left[a\begin{pmatrix}O\\O\end{pmatrix}+a^{-1}(O)\right]+a\begin{pmatrix}O\\O\end{pmatrix}+a^{-1}(O)+a(OO)+a^{-1}(O)+$$



$$= (-a-a^5)(-a^{-2}-a^2)+(-a^{-1}-a^3)+a(-a^{-2}-a^2)+a^{-1}+a(-a^{-2}-a^2)+a^{-1}+$$

$$a^{-1}+a^{-3}(-a^{-2}+a^2)$$

$$= a^{-1}+a^3+a^3+a^7-a^{-1}+-a^3-a^{-1}-a^3+a^{-1}-a^{-1}-a^3+a^{-1}+a^{-1}-a^{-5}-a^{-1}$$

$$= a^7-a^3-a^{-5}$$

Thus it follows that, $P_{K_{2,3}}(a) = a^7-a^3-a^{-5}$

Example 2. For $K_{2,5}$, we obtain $P_{K_{2,5}}(a) = a^{13}-a^9+a^5-a-a^{-7}$

by the same method.

Example 3. For $K_{2,7}$, we have $P_{K_{2,7}}(a) = a^{19} - a^{15} + a^{11} - a^{7} + a^{3} - a^{-1} - a^{-9}$.

Consequence 3.1. If we carefully examine the above results, we find the following general formula for the Bracket polynomials of $K_{2,n}$

$P_{K_{2,n}}(a) = aP_{K_{2,n-1}}(a) + (-1)^{n-1}a^{-3n+2}$, where $P_{K_{2,1}}(a) = -a^{3}$.

Proof. By induction.

Example 4. For $K'_{2,3}$**,** we have $P_{K'_{2,3}}(a) = a^{-7} - a^{-3} - a^{5}$.

Example 5. For $K'_{2,5}$, we obtain $P_{K'_{2,5}}(a) = a^{-13} - a^{-9} + a^{-5} - a^{-1} - a^{7}$.

Example 6. For $K'_{2,7}$, we obtain $P_{K'_{2,7}}(a) = a^{-19} - a^{-15} + a^{-11} - a^{-7} + a^{-3} - a^{1} - a^{9}$.

Consequence 3.2. For the Bracket polynomials of $K'_{2,n}$, we find the following general formula:

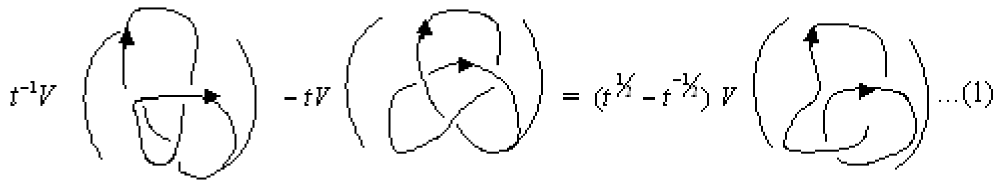
$$P_{K'_{2,n}}(a) = a^{-1}P_{K'_{2,n-1}}(a) + (-1)^{n-1}a^{3n-2}$$

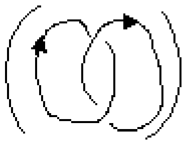
where $P_{K'_{2,1}}(a) = -a^{-3}$.

Proof. By induction.

3.2. The Jones Polynomial of Torus Knots $K(2,n)$:

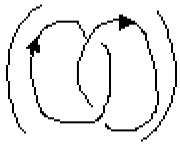
Example 7. We find the Jones polynomial of $K(2,3)$**.**



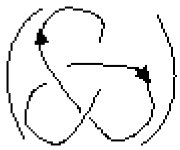
$-tV$  $= (t^{1/2} - t^{-1/2}) - t^{-1}(-t^{-1/2} - t^{1/2})$

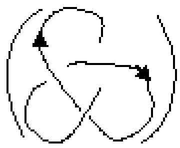
$$= t^{1/2} - t^{-1/2} + t^{-3/2} + t^{-1/2}$$

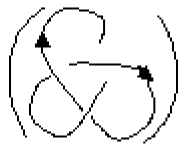
$$= t^{1/2} + t^{-3/2}$$

V  $= \dfrac{t^{1/2} + t^{-3/2}}{-t} = -t^{-1/2} - t^{-5/2}$

if we turn to relation (1)

$t^{-1} - tV$  $= (t^{1/2} - t^{-1/2})(-t^{-1/2} - t^{-5/2})$

$-tV$  $= (-1 - t^{-2} + t^{-1} + t^{-3}) - t^{-1}$

V  $= \dfrac{1}{-t}(-1 - t^{-2} + t^{-1} + t^{-3}) + t^{-2}$

$$= t^{-1} + t^{-3} - t^{-2} - t^{-4} + t^{-2}$$

$$= t^{-1} + t^{-3} - t^{-4}$$

i.e.,

$$V_{K(2,3)}(t) = t^{-1} + t^{-3} - t^{-4}.$$

Example 8. For $K(2,5)$**,** by the same method we obtain

$$V_{K(2,5)}(t) = t^{-2} + t^{-4} - t^{-5} + t^{-6} - t^{-7}$$

Example 9. For $K(2,7)$, we obtain

$$V_{K(2,7)}(t) = t^{-3} + t^{-5} - t^{-6} + t^{-7} - t^{-8} + t^{-9} - t^{-10}$$

Consequence 3.3. For the Jones polynomials of $K(2,n)$, we find the following general formula:

$$V_{K(2,n)}(t) = t^{-(\frac{n-1}{2})} + \sum_{k=1}^{n-1} (-1)^{k+1} t^{-(k+\frac{n+1}{2})}$$

Proof. By induction.

Example. For $K'(2,3)$, we obtain

$$V_{K'(2,3)}(t) = -t^{4} + t^{3} + t$$

Example 11. For $K'(2,5)$, we obtain

$$V_{K'(2,5)}(t) = -t^{7} + t^{6} - t^{5} + t^{4} + t^{2}$$

Example 12. For $K'(2,7)$, we obtain

$$V_{K'(2,7)}(t) = -t^{10} + t^{9} - t^{8} + t^{7} - t^{6} + t^{5} + t^{3}$$

Consequence 3.4. For the Jones polynomials of $K'(2,n)$, we find the following general formula:

$$V_{K'(2,n)}(t) = t^{\frac{n-1}{2}} + \sum_{k=1}^{n-1} (-1)^{k+1} t^{(k+\frac{n+1}{2})}$$

Proof. By induction.

References

[1] D.Rolfsen, Knots and Links, Publish or Perish Inc., Willmington, Delaware, DE, 1976.

[2] K.Murasugi, Knot Theory and its Applications, Birkhauser, Boston, MA, 1996.

[3] V.V. Prasolov and A.B. Sossinsky, Knots, Links, Braids and 3-Manifolds, American Mathematical Society, 1991.

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