Applied Mathematical Sciences, Vol. 3, 2009, no. 56, 2757 - 2764

A Further Symmetric Relation on the Analogue of the Apostol-Bernoulli and the Analogue of the Apostol-Genocchi Polynomials

Veli Kurt

Department of Mathematics, Akdeniz University, Antalya, 07058, Turkey
vkurt@akdeniz.edu.tr

Abstract

The main object of this paper is to investigate some relations between the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynomials. We first establish some relations between the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynomials. Furthermore we give two symmetric relations on the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynomials.

Mathematics Subject Classification: 11B68

Keywords: Bernoulli polynomials, Genocchi polynomials, Apostol-Bernoulli polynomials, Apostol-Genocchi polynomials

1 Introduction

The Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$ are defined by the following generating functions, respectively:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}, \qquad \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}, \quad |t| < 2\pi \tag{1}$$

where $B_0 = 1$. For every $n \geq 1$ the Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$ satisfy the following reccurence relations, respectively:

$$\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0, \qquad B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}.$$

For each integer $k \geq 0$

$$S_k(n) = 0^k + 1^k + 2^k + \cdots + n^k$$

is called the sum of integer powers.

$$\sum_{i=0}^{n} i^k = S_k(n)$$

is a polynomial in n of degree $k+1$.

This sum satisfies the following equation in [17]

$$S_k(n) = \sum_{i=0}^{n} i^{k-1} \binom{k}{i} \frac{n^i}{i+1} B_{k-i}.$$

On the other hand, Ronrigues [3] proved the equation

$$B_n = \frac{1}{a(1-a^n)} \sum_{k=0}^{n-1} a^k \binom{n}{k} B_k S_{n-k}(a-1).$$

Also, Tuenter [15] defined the polynomial

$$\sigma_m(a-1) = \frac{1}{m+1} \sum_{j=0}^{m} \binom{m+1}{j} B_j a^{m+1-j}.$$

He proved the following symmetric relations with respect to a, b for every pair of positive integers a and b, and all nonnegative integers m.

$$\sum_{j=0}^{m} \binom{m}{j} a^{j-1} B_j b^{m-j} \sigma_{m-j}(a-1) = \sum_{j=0}^{m} \binom{m}{j} b^{j-1} B_j a^{m-j} \sigma_{m-j}(b-1) \quad (2)$$

The following symmetric relations are given by Sheng-Liang Yang [17]

$$\sum_{i=0}^{n} \binom{n}{i} a^{i-1} B_i b^{n-i} S_{n-i}(a-1) = \sum_{i=0}^{n} \binom{n}{i} b^{i-1} B_i a^{n-i} S_{n-i}(b-1) \quad (3)$$

and

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} B_{n-k}^{(m)}(bx) \times \sum_{i=0}^{k} \binom{k}{i} S_i(a-1) B_{k-i}^{(m-1)}(ay) \quad (4)$$

$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} B_{n-k}^{(m)}(ax) \times \sum_{i=0}^{k} \binom{k}{i} S_i(b-1) B_{k-i}^{(m-1)}(by)$$

where a and b are positive integers, $n \geq 0$ and $m \geq 1$.

Symmetric properties of q-Bernoulli polynomials in q-Calculus are investigated by B.A. Kupershmidt [8]. Also T. Kim studied on q-Bernoulli polynomials and q-Euler polynomials. He wrote a lot of paper on this subject ([4], [5], [6], [7]). He gave the below symmetric relations with the help of q-Volkenborn integrals [5]:

$$\sum_{i=0}^{n} \binom{n}{i} B_i(w_2 x) S_{n-i}(w_1 - 1) w_1^{i-1} w_2^{n-i}$$
$$= \sum_{i=0}^{n} \binom{n}{i} B_i(w_1 x) S_{n-i}(w_2 - 1) w_2^{i-1} w_1^{n-i} \tag{5}$$

where w_1 and w_2 are integers.

An interpolation formula of q-Genocchi numbers are given by Cenkci, Can, Kurt [1].

Different relations and theorems on classical Bernoulli and Euler polynomials are studied by Gi-Sang Cheon [2], Srivastava, Qiu Ming Luo [9], Luo ([10], [11]), Simsek [12]. Some results on the Apostol-Bernoulli and Apostol-Euler polynomials are given by W.Wang, C. Jia, T. Wang [16].

Definition 1.1 *([9], [10], [11], [16])The Apostol Bernoulli polynomials, $B_n^{\alpha}(x, \lambda)$ of order α are defined by means of the following generating function*

$$\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{\alpha}(x, \lambda) \frac{z^n}{n!}, \quad |z + \log \lambda| \leq 2\pi, 1^{\alpha} := 1$$

where $\alpha \in Z^+, \lambda \in C, \lambda$ is a parameter, $B_n^{\alpha}(x) = B_n^{\alpha}(x, 1), B_n^{\alpha}(\lambda) = B_n^{\alpha}(\lambda, 1)$.

Definition 1.2 *([9], [16])The Apostol Euler polynomials, $\varepsilon_n^{\alpha}(x, \lambda)$ of order α are defined by means of the following generating function*

$$\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \varepsilon_n^{\alpha}(x, \lambda) \frac{z^n}{n!}, \quad |z + \log \lambda| \leq \pi, 1^{\alpha} := 1$$

where $\alpha \in Z^+, \lambda \in C, \lambda$ is a parameter, $\varepsilon_n^{\alpha}(x) = \varepsilon_n^{\alpha}(x, 1), \varepsilon_n^{\alpha}(\lambda) = \varepsilon_n^{\alpha}(\lambda, 0)$.

Definition 1.3 *We define the analogue of the Apostol-Bernoulli polynomials $B_n^{\alpha}(x, \lambda^a)$ order α and the analogue of the Apostol-Euler polynomials $\varepsilon_n^{\alpha}(x, \lambda^a)$ order α as follows, respectively:*

$$\left(\frac{z}{\lambda^a e^z - 1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{\alpha}(x, \lambda^a) \frac{z^n}{n!} \tag{6}$$

where a and α are positive integers, $\lambda \in C, |z+a\log\lambda| < 2\pi$ and

$$\left(\frac{2}{\lambda^a e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \varepsilon_n^{\alpha}(x,\lambda^a)\frac{z^n}{n!} \tag{7}$$

where a and α are positive integers, $\lambda \in C, |z+a\log\lambda| < \pi$.

From (6) and (7) we have

$$\begin{aligned}\lambda^a B_n^{\alpha}(x+1,\lambda^a) - B_n^{\alpha}(x,\lambda^a) &= nB_{n-1}^{\alpha-1}(x,\lambda^a)\\ \lambda^a \varepsilon_n^{\alpha}(x+1,\lambda^a) + \varepsilon_n^{\alpha}(x,\lambda^a) &= 2\varepsilon_n^{\alpha-1}(x,\lambda^a)\end{aligned}$$

moreover, since

$$B_n^0(x,\lambda^a) = \varepsilon_n^0(x,\lambda^a) = x^n$$

we obtain

$$\begin{aligned}\lambda^a B_n(x+1,\lambda^a) - B_n(x,\lambda^a) &= nx^{n-1}\\ \lambda^a \varepsilon_n(x+1,\lambda^a) + \varepsilon_n(x,\lambda^a) &= 2x^n\end{aligned} \tag{8}$$

2 Main Theorems

In this section, we introduce our main results. We give some theorems, definitions and corollaries which are related to the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Euler polynomials.

Theorem 2.1 *There is a following relation between the analogue of the Apostol-Bernoulli polynomials $B_n^{\alpha}(x,\lambda^a)$ and the analogue of the Apostol-Euler polynomials $\varepsilon_n^{\alpha}(x,\lambda^a)$:*

$$B_n^{\alpha}\left(kx,\lambda^{2a}\right) = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}B_k^{\alpha}(\lambda^a)\,\varepsilon_{n-k}^{\alpha}(kx,\lambda^a) \tag{9}$$

where k, a, α are positive integers, $\lambda \in C$.

Proof. From (6)

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{\alpha}\left(kx,\lambda^{2a}\right)\frac{z^n}{n!} &= \left(\frac{z}{\lambda^{2a}e^z-1}\right)^{\alpha} e^{kxz} = \frac{\left(\frac{z}{2}\right)^{\alpha}}{\left(\lambda^a e^{\frac{z}{2}}-1\right)^{\alpha}} \times \frac{2^{\alpha} e^{\frac{z}{2}(24x)}}{\left(\lambda^a e^{\frac{z}{2}}+1\right)^{\alpha}}\\ &= \sum_{n=0}^{\infty} B_n^{\alpha}(\lambda^a)\frac{z^n}{2^n n!} \times \sum_{n=0}^{\infty}\varepsilon_n^{\alpha}(2kx,\lambda^a)\frac{z^n}{2^n n!}\\ &= \sum_{n=0}^{\infty}\left(2^{-n}\sum_{k=0}^{n}\binom{n}{k}B_n^{\alpha}(\lambda^a)\,\varepsilon_{n-k}^{\alpha}(2kx,\lambda^a)\right)\frac{z^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the above equation, then (8) is obtained. ■

Corollary 2.2 *The analogue of the Apostol-Bernoulli polynomials $B_n^{\alpha}(rx,\lambda^a)$ of order α satisfy the following equalities:*

$$\begin{aligned} B_n^{\alpha}(rx,\lambda^a) &= B_n^{\alpha}\left(kx,\lambda^{2a}\right)=\sum_{m=0}^{n}\binom{n}{k}B_m^{\alpha}(x,\lambda^a)(r-1)^{n-m}x^{n-m}, \quad (10)\\ B_n^{\alpha}(rx,\lambda^a) &= B_n^{\alpha}\left(kx,\lambda^{2a}\right)=\sum_{k=0}^{n}\binom{n}{k}B_k^{\alpha-1}(rx,\lambda^a)B_{n-k}(\lambda^a) \end{aligned}$$

where $r\in Z^{+}$.

Proof. The proof of these equalities are find easily from (6). ■

Definition 2.3 *We define the analogue of the Apostol-Genocchi numbers and polynomials of order α by means of the following generating functions, respectively:*

$$\left(\frac{2z}{\lambda^a e^z+1}\right)^{\alpha}=\sum_{n=0}^{\infty}G_n^{\alpha}(\lambda^a)\frac{z^n}{n!},\qquad \left(\frac{2z}{\lambda^a e^z+1}\right)^{\alpha}e^{xz}=\sum_{n=0}^{\infty}G_n^{\alpha}(x,\lambda^a)\frac{z^n}{n!}, \quad (11)$$

where a,α are positive integers, $\lambda\in C, |z+alog\lambda|<\pi$.

From (11) we have

$$\begin{aligned} G_n^0(x,\lambda^a) &= x^n,\\ G_n^{\alpha}(x+1,\lambda^a)+G_n^{\alpha}(x,\lambda^a) &= 2nG_{n-1}^{\alpha-1}(x,\lambda^a) \end{aligned}$$

and

$$\lambda^a G_n(x+1,\lambda^a)+G_n^{\alpha}(x,\lambda^a)=2nx^{n-1}.$$

Theorem 2.4 *The analogue of the Apostol-Genocchi polynomials $G_n^{\alpha}(x,\lambda^a)$ of order α satisfy the following equation:*

$$G_n^{\alpha}(x+y,\lambda^a)=\sum_{k=0}^{n}\binom{n}{k}G_k^{\alpha}(x,\lambda^a)y^{n-k}=\sum_{k=0}^{n}\binom{n}{k}G_k^{\alpha}(y,\lambda^a)x^{n-k}. \quad (12)$$

Proof. With the help of equation (11), theorem can be proved easily. ■
We consider the following sum,

$$S=1-\lambda e^{at}+\lambda^2e^{2at}+\cdots+\left(-\lambda e^{bt}\right)^{a-1}=\begin{cases}\frac{1+\lambda^a e^{abt}}{1+\lambda e^{bt}} & ,a \text{ odd integer}\\ \frac{1-\lambda^a e^{abt}}{1-\lambda e^{bt}} & ,a \text{ even integer}\end{cases}$$

Theorem 2.5 *The analogue of the Apostol-Genocchi polynomials* $G_n^{\alpha}(x,\lambda^a)$ *of order* α *satisfy the following symmetric relation:*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\left(-\lambda^b\right)^i G_k^{\alpha}\left(bx+\frac{b}{a}i,\lambda^a\right)G_{n-k}^{\alpha-1}\left(ay,\lambda^b\right)a^k b^{n-k+1} \tag{13}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\left(-\lambda^a\right)^i G_k^{\alpha}\left(ax+\frac{a}{b}i,\lambda^b\right)G_{n-k}^{\alpha-1}\left(by,\lambda^a\right)b^k a^{n-k+1},$$

where a,b *are positive odd integers,* α *is a positive integer and* λ *is a parameter.*

Proof. We define

$$k(t)=\frac{t^{2\alpha-1}e^{abxt}\left(\lambda^{ab}e^{abt}+1\right)e^{abyt}}{\left(\lambda^a e^{at}+1\right)^{\alpha}\left(\lambda^b e^{bt}+1\right)^{\alpha}}.$$

From the above, we have

$$\begin{aligned}
k(t) &= \frac{1}{2^{2\alpha-1}a^{\alpha}b^{\alpha-1}}\left(\frac{2at}{\lambda^a e^{at}+1}\right)^{\alpha}e^{abxt}\left(\frac{\lambda^{ab}e^{abxt}+1}{\lambda^b e^{bt}+1}\right)\left(\frac{2bt}{\lambda^b e^{bt}+1}\right)^{\alpha-1}e^{abyt}\\
&= \frac{1}{2^{2\alpha-1}a^{\alpha}b^{\alpha-1}}\sum_{i=0}^{a-1}\left(-\lambda^b\right)^i e^{at\left(bx+\frac{b}{a}i\right)}\left(\frac{2at}{\lambda^a e^{at}+1}\right)^{\alpha}\left(\frac{2bt}{\lambda^b e^{bt}+1}\right)^{\alpha-1}e^{abyt}\\
&= \frac{1}{2^{2\alpha-1}a^{\alpha}b^{\alpha}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\right.\\
&\quad\left.\times\sum_{i=0}^{a-1}\left(-\lambda^b\right)^i G_k^{\alpha}\left(bx+\frac{b}{a}i,\lambda^a\right)G_{n-k}^{\alpha-1}\left(ay,\lambda^b\right)a^k b^{n-k+1}\right)\frac{t^n}{n!}.
\end{aligned} \tag{14}$$

Similarly, we obtain

$$\begin{aligned}
k(t) &= \frac{1}{2^{2\alpha-1}b^{\alpha}a^{\alpha}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\right.\\
&\quad\left.\times\sum_{i=0}^{b-1}\left(-\lambda^a\right)^i G_k^{\alpha}\left(ax+\frac{a}{b}i,\lambda^b\right)G_{n-k}^{\alpha-1}\left(by,\lambda^a\right)b^k a^{n-k+1}\right)\frac{t^n}{n!}.
\end{aligned} \tag{15}$$

Hence by (14) and (15) we have (13). ■

Corollary 2.6 *By taking* $\alpha=1$ *in* (13), *we have the following symmetric equation*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\left(-\lambda^b\right)^i G_k\left(bx+\frac{b}{a}i,\lambda^a\right)a^n b^{n-k+1}y^{n-k}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\left(-\lambda^a\right)^i G_k\left(ax+\frac{a}{b}i,\lambda^b\right)b^n a^{n-k+1}y^{n-k} \tag{16}$$

Corollary 2.7 *The analogue of the Apostol-Bernoulli polynomials $B_n^{\alpha}(x,\lambda^a)$ of order α satisfy the following symmetric relation:*

$$\sum_{i=0}^{a-1}\lambda^{bi}\sum_{k=0}^{n}\binom{n}{k}B_k^{\alpha}\left(bx+\frac{b}{a}i,\lambda^a\right)B_{n-k}^{\alpha-1}\left(ay,\lambda^b\right)a^k b^{n-k+1}$$
$$=\sum_{i=0}^{b-1}\lambda^{ai}\sum_{k=0}^{n}\binom{n}{k}B_k^{\alpha}\left(ax+\frac{a}{b}i,\lambda^b\right)B_{n-k}^{\alpha-1}\left(by,\lambda^a\right)b^k a^{n-k+1} \qquad (17)$$

where a,b are positive even integers.

Proof. We define

$$k(t)=\frac{t^{2\alpha-1}e^{abxt}\left(\lambda^{ab}e^{abt}-1\right)e^{abyt}}{\left(\lambda^a e^{at}-1\right)^{\alpha}\left(\lambda^b e^{bt}-1\right)^{\alpha}}.$$

After making necessary operation on this function we obtain (17). ■

Putting $\lambda=1$ in (17) we obtain Shen-Ling Yang's result.

ACKNOWLEDGEMENT: This work was supported by Akdeniz University Scientific Research Project Unit.

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Received: April, 2009