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The Spectral Moments and Energy of Graphs

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Abstract

The energy of a graph is defined as the sum of the absolute values of its eigenvalues. In this paper, we obtain an upper bound for the energy of a graph that involves its moments.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [1]. Throught this paper, G will be always a graph of order n and size m. We use $V(G) := \{ v_1, v_2, ...v_n \}$ to denote the vertex set of G and $d(v_i)$ or d_i, where $1 \leq i \leq n$, to denote the degree of vertex v_i. For each $1 \leq i \leq n$, the 2 - degree of v_i, denoted $t(v_i)$ or t_i, is defined as the sum of degrees of the vertices adjacent to v_i, the average degree of v_i is defined as t_i/d_i, and $\sigma(v_i)$ or σ_i is defined as the sum of the 2 - degrees of vertices adjacent to v_i. We define $\Sigma_k(G)$ as $\Sigma_{i=1}^n d_i^k$. A bipartite graph $G = (X, Y; E)$ is (a, b) - semiregular if there exist two constants a and b such that each vertex in X has degree a and each vertex in Y has degree b. A bipartite graph $G = (X, Y; E)$ is (p_x, p_y) - pseudo - semiregular if there exist two constants p_x and p_y such that each vertex in X has average degree p_x and each vertex in Y has average degree p_y. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of G are called the eigenvalues of the graph G. The kth - spectral moments, denoted $M_k(G)$ or M_k, of G is defined as $\Sigma_{i=1}^n \mu_i^k$ (see [6]). The energy of a graph G, denoted $E(G)$, is defined as $E(G) = \sum_{i=1}^n |\mu_i|$. This concept was introduced by Gutman in [4] and more information and background on the energy of graphs can be found in [5]. In 1971, McClelland [11] proved that $E(G) \leq \sqrt{2mn}$, the first upper bound for $E(G)$. Since then, more upper bounds for $E(G)$ have been found and some of them can be found in the following theorems.

Theorem 1 [8]. Let G be a graph on n vertices and m edges. If $2m \geq n$, then the inequality

$$E(G) \leq \frac{2m}{n} + \sqrt{(n-1)[2m - (\frac{2m}{n})^2]} \qquad (1)$$

holds. Moreover, equality holds if and only if G is either $\frac{n}{2}K_2$, K_n, or a non-complete connected strongly regular graph with two non-trivial eigenvalues with absolute value $\sqrt{(2m - (\frac{2m}{n})^2)/(n-1)}$. If $2m \leq n$, then the inequality $E(G) \leq 2m$ holds. Moreover, equality holds if and only if G is disjoint union of edges and isolated vertices.

Theorem 2 [9]. If $2m \geq n$ and G is a bipartite graph with $n > 2$ vertices and m edges, then the inequality

$$E(G) \leq 2(\frac{2m}{n}) + \sqrt{(n-2)[2m - 2(\frac{2m}{n})^2]} \qquad (2)$$

holds. Moreover, equality holds if and only if at least one of the following statements holds:

(1) $n = 2m$ and $G = mK_2$.
(2) $n = 2t$, $m = t^2$, and $G = K_{t,t}$.
(3) $n = 2\nu$, $2\sqrt{m} < n < 2m$, and G is the incidence graph of a symmetric $2-(\nu, k, \lambda)$ - design with $k = \frac{2m}{n}$ and $\lambda = \frac{k(k-1)}{\nu-1}$.

Theorem 1 and Theorem 2 were generalized by several authors (see [13] [12]) and the latest ones are the following Theorem 3 and Theorem 4 proved by Liu, Lu, and Tian.

Theorem 3 [10]. Let G be a non-empty graph on n vertices, m edges. Then the inequality

$$E(G) \leq \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} + \sqrt{(n-1)(2m - \frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2})} \qquad (3)$$

holds. Moreover, equality in (3) holds if and only if G is either $\frac{n}{2}K_2$, K_n, or a non-bipartite connected graph satifying $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ and has three distinct eigenvalues $(p, \sqrt{\frac{2m-p^2}{n-1}}, -\sqrt{\frac{2m-p^2}{n-1}})$, where $p = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} > \sqrt{\frac{2m}{n}}$.

Theorem 4 [10]. Let $G = (X, Y)$ be a non-empty bipartite graph with $n > 2$ vertices and m edges. Then the inequality

$$E(G) \leq 2\sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2}} + \sqrt{(n-2)(2m - 2\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2})} \qquad (4)$$

holds. Moreover, equality in (4) holds if and only if G is either $\frac{n}{2}K_2$, $K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, where $r_1 r_2 = m$, or a connected bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$, and has four distinct eigenvalues $(\sqrt{p_x p_y}, \sqrt{\frac{2m-2p_x p_y}{n-2}}, -\sqrt{\frac{2m-2p_x p_y}{n-2}}, -\sqrt{p_x p_y})$, where $p_x = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$, $p_y = \frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$ and $\sqrt{p_x p_y} > \sqrt{\frac{2m}{n}}$.

Motivated by Theorem 3 and Theorem 4 above, we in this paper prove the following theorems.

Theorem 5. Let G be a non-empty graph on n vertices. If $\sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2}} \geq (\frac{M_{2k}}{n})^{\frac{1}{2k}}$, where k is a positive integer, then the inequality

$$E(G) \leq \sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2}} + (n-1)^{\frac{2k-1}{2k}}(M_{2k} - (\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2})^k)^{\frac{1}{2k}} \qquad (5)$$

holds. Moreover, equality in (5) holds if and only if G is either $\frac{n}{2}K_2$, K_n, or a non-bipartite connected graph satifying $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ and has three distinct eigenvalues $(p, (\frac{M_{2k}-p^{2k}}{n-1})^{\frac{1}{2k}}, -(\frac{M_{2k}-p^{2k}}{n-1})^{\frac{1}{2k}})$, where $p = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} > (\frac{M_{2k}}{n})^{\frac{1}{2k}}$.

Theorem 6. Let $G = (X, Y)$ be a non-empty bipartite graph with $n > 2$ vertices amd m edges. If $\sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2}} \geq (\frac{M_{2k}}{n})^{\frac{1}{2k}}$, where k is a positive integer, then the inequality

$$E(G) \leq 2\sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2}} + (n-2)^{\frac{2k-1}{2k}}(M_{2k} - 2(\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2})^k)^{\frac{1}{2k}} \qquad (6)$$

holds. Moreover, equality in (6) holds if and only if G is either $\frac{n}{2}K_2$, $K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, where $r_1 r_2 = m$, or a connected bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$, and has four distinct eigenvalues $(\sqrt{p_x p_y}, (\frac{M_{2k}-2(p_x p_y)^k}{n-2})^{\frac{1}{2k}}, -(\frac{M_{2k}-2(p_x p_y)^k}{n-2})^{\frac{1}{2k}}, -\sqrt{p_x p_y})$, where $p_x = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$, $p_y = \frac{\sigma_{s+1}}{t_{s+1}} =$

$\frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$ and $\sqrt{p_x p_y} > (\frac{M_{2k}}{n})^{\frac{1}{2k}}$.

Notice that $\sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2}} \geq (\frac{M_{2k}}{n})^{\frac{1}{2k}}$ is always true when $k = 1$ (see [10]) and $M_2 = 2m$. Thus if we let $k = 1$ in Theorem 5 and Theorem 6, then they become Theorem 3 and Theorem 4 respectively.

Let q be the number of quadrangles in a graph G. Then $M_4(G) = 2\sum_{i=1}^{n} d_i^2 - 2m + 8q$ (see [6]). Thus if we let $k = 2$ and replace M_4 by $2\sum_{i=1}^{n} d_i^2 - 2m + 8q$ in Theorem 5 and Theorem 6 then we can obtain upper bounds for general graphs and bipartite graphs which satisfy respectively the conditions in Theorem 5 and Theorem 6.

2. Lemmas

In order to prove Theorem 5 and Theorem 6, we need the following results as our lemmas. The first one is a theorem proved by Hong and Zhang in [7].

Lemma 1 [7]. Let G be a simple connected graph of order n. Then

$$\mu_1 \geq \sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2}}$$

with equality if and only if $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ or G is a bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$.

In fact, the above Hong and Zhang's theorem can be slightly strengthened to the following Lemma 1′. Notice that Lemma 1′ has been used by Liu, Lu, and Tian in [10] to obtain their results.

Lemma 1'. Let G be a non-empty simple graph of order n. Then

$$\mu_1 \geq \sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}t_i^2}}$$

with equality if and only if $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ with $\mu_1 = \frac{\sigma_1}{t_1}$ or G is a bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$ with $\mu_1 = \sqrt{p_x p_y}$, where $p_x = \frac{\sigma_1}{t_1}$ and $p_y = \frac{\sigma_n}{t_n}$.

The following Lemma 2 and Lemma 3 can be found in [3] and [2] respectively.

Lemma 2 [3]. A graph G has only one distinct eigenvalue if and only if G is an empty graph. A graph G has two distinct eigenvalues $\mu_1 > \mu_2$ with multiplicities s_1 and s_2 if and only if G is the direct sum of s_1 complete graphs of order $\mu_1 + 1$. In this case, $\mu_2 = -1$ and $s_2 = s_1\mu_1$.

Lemma 3 [2]. Let G be a graph with m edges. Then $E(G) \geq 2\sqrt{m}$ with equality if and only if G is a complete bipartite graph plus arbitrarily many isolated vertices.

3. Proofs

Proof of Theorem 5. Let G be a graph satisfying the conditions in Theorem 9. Set $\alpha := \frac{1}{2k}$ and $\beta := 1 - \alpha$. By the Hölder inequality, we have that

$$\sum_{i=2}^{n} |\mu_i| \leq (\sum_{i=2}^{n} 1^{\frac{1}{\beta}})^{\beta} (\sum_{i=2}^{n} |\mu_i|^{\frac{1}{\alpha}})^{\alpha}.$$

Therefore

$$E(G) = \sum_{i=1}^{n} |\mu_i| \leq \mu_1 + (n-1)^{\beta}(M_{2k} - \mu_1^{2k})^{\frac{1}{2k}}.$$

Consider the function $f(x) = x + (n-1)^{\beta}(M_{2k} - x^{2k})^{\frac{1}{2k}}$. It can be verified that $f(x)$ is decreasing when $(\frac{M_{2k}}{n})^{\frac{1}{2k}} \leq x \leq M_{2k}^{\frac{1}{2k}}$.

From Lemma 1′ and the assumption that

$$\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \geq (\frac{M_{2k}}{n})^{\frac{1}{2k}},$$

we have that

$$E(G) \leq f(\mu_1) \leq f(\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}}).$$

Therefore Inequality (5) is proved.

If G is $\frac{n}{2}K_2$, then $\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} = 1 = (\frac{M_{2k}}{n})^{\frac{1}{2k}}$ and both sides of Inequality (5) are equal to n.

If G is K_n, then $\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} = n-1$. From $(n-1) \geq (\frac{(n-1)^{2k}+(n-1)}{n})^{\frac{1}{2k}}$, we have $\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} = (n-1) \geq (\frac{(n-1)^{2k}+(n-1)}{n})^{\frac{1}{2k}} = (\frac{M_{2k}}{n})^{\frac{1}{2k}}$ and both sides of Inequality (5) are equal to $2(n-1)$.

If G is a non-bipartite connected graph satisfying $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ and has three distinct eigenvalues $(p, (\frac{M_{2k}-p^{2k}}{n-1})^{\frac{1}{2k}}, -(\frac{M_{2k}-p^{2k}}{n-1})^{\frac{1}{2k}})$, where $p = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} > (\frac{M_{2k}}{n})^{\frac{1}{2k}}$, then both sides of Inequality (5) are equal to $p + (n-1)(\frac{M_{2k}-p^{2k}}{n-1})^{\frac{1}{2k}}$.

Now suppose that Inequality (5) becomes an equality. Then we have that $|\mu_2| = |\mu_3| = \cdots = |\mu_n| = (\frac{M_{2k}-\mu_1^{2k}}{n-1})^{\frac{1}{2k}}$ and $\mu_1 = \sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}}$. By Lemma $1'$, we have that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ or G is a bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$. Since G is a non-empty graph, Lemma 2 implies that G has at least two distinct eigenvalues. Hence we just have the following possible cases.

Case 1. G has two distinct eigenvalues

If the two distinct eigenvalues of G have the same absolute values, then Lemma 2 implies that $\mu_1 = |\mu_2| = \cdots = |\mu_n| = 1$. Since $\sum_{i=1}^n \mu_i = 0$, the multiplicity s_1 of $\mu_1 = 1$ must be equal to $\frac{n}{2}$. Hence G is the direct sum of $s_1 = \frac{n}{2}$ complete graphs of order $\mu_1 + 1 = 2$. Namely, G is $\frac{n}{2}K_2$.

If the two distinct eigenvalues of G have different absolute values, then Lemma 2 implies that $\mu_2 = \cdots = \mu_n = -1$. Since $\sum_{i=1}^n \mu_i = 0$, $\mu_1 = n-1$ and the multiplicity s_1 of μ_1 is 1. Hence G is the direct sum of $s_1 = 1$ complete graph of order $\mu_1 + 1 = n$. Namely, G is K_n.

Case 2. G has three distinct eigenvalues

Since G has three distinct eigenvalues, there exists an integer r such that $\mu_1 > \mu_2 = \cdots = \mu_r > 0 > \mu_{r+1} = \cdots = \mu_n$ and $\mu_2 = -\mu_n$. Hence $\mu_1 \neq -\mu_n$ and G cannot be a bipartite graph. From $\mu_1 = \sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i^2}}$ and $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$, we have that $\mu_1 = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$. Since $\mu_1 > \mu_i$, for each i with $2 \leq i \leq n$, G must be connected. Set $p := \mu_1$. Then G has three distinct eigenvalues $(p, (\frac{M_{2k}-p^{2k}}{n-1})^{\frac{1}{2k}}, -(\frac{M_{2k}-p^{2k}}{n-1})^{\frac{1}{2k}})$, where $p = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} > (\frac{M_{2k}}{n})^{\frac{1}{2k}}$. □

Proof of Theorem 6. Let G be a graph satisfying the conditions in Theorem 10. Then $\mu_1 = -\mu_n$. Set $\alpha := \frac{1}{2k}$ and $\beta := 1 - \alpha$. By the Hölder inequality, we have that

$$\sum_{i=2}^{n-1} |\mu_i| \leq (\sum_{i=2}^{n-1} 1^{\frac{1}{\beta}})^{\beta} (\sum_{i=2}^{n-1} |\mu_i|^{\frac{1}{\alpha}})^{\alpha}.$$

Therefore

$$E(G) = \sum_{i=1}^{n} |\mu_i| \leq 2\mu_1 + (n-2)^{\beta}(M_{2k} - 2\mu_1^{2k})^{\frac{1}{2k}}.$$

Consider the function $f(x) = 2x + (n-2)^{\beta}(M_{2k} - 2x^{2k})^{\frac{1}{2k}}$. It can be verified that $f(x)$ is decreasing when $(\frac{M_{2k}}{n})^{\frac{1}{2k}} \leq x \leq M_{2k}^{\frac{1}{2k}}$.

From Lemma 1′ and the assumption that

$$\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \geq (\frac{M_{2k}}{n})^{\frac{1}{2k}},$$

we have that

$$E(G) \leq f(\mu_1) \leq f(\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}}).$$

Therefore Inequality (6) is proved.

If G is $\frac{n}{2}K_2$, then $\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} = 1 = (\frac{M_{2k}}{n})^{\frac{1}{2k}}$ and both sides of Inequality (6) are equal to n.

If G is $K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, where $r_1 r_2 = m$, then $\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} = \sqrt{r_1 r_2} \geq (\frac{2(r_1 r_2)^k}{n})^{\frac{1}{2k}}) = (\frac{M_{2k}}{n})^{\frac{1}{2k}}$ and both sides of Inequality (6) are equal to $2\sqrt{r_1 r_2}$.

If G is a connected bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$, and has four distinct eigenvalues $(\sqrt{p_x p_y}, (\frac{M_{2k}-2(p_x p_y)^k}{n-2})^{\frac{1}{2k}}, -(\frac{M_{2k}-2(p_x p_y)^k}{n-2})^{\frac{1}{2k}}, -\sqrt{p_x p_y})$, where $p_x = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$, $p_y = \frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$ and $\sqrt{p_x p_y} > (\frac{M_{2k}}{n})^{\frac{1}{2k}}$, then both sides of Inequality (6) are equal to $2p + (n-2)(\frac{M_{2k}-2p^{2k}}{n-2})^{\frac{1}{2k}}$, where $p = \sqrt{p_x p_y}$.

Now suppose that Inequality (6) becomes an equality. Then we have that $|\mu_2| = |\mu_3| = \cdots = |\mu_{n-1}| = (\frac{M_{2k}-2\mu_1^{2k}}{n-2})^{\frac{1}{2k}}$ and $\mu_1 = -\mu_n = \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}}$. By Lemma 1′, we have that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ or G is a bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$ with $\mu_1 = \sqrt{p_x p_y}$, where $p_x = \frac{\sigma_1}{t_1}$ and $p_y = \frac{\sigma_n}{t_n}$. Since G is a non-empty graph, Lemma 2 implies that G has at least two distinct eigenvalues. Hence we just have the following possible cases.

Case 1. G has two distinct eigenvalues with the same absolute values

Then Lemma 2 implies that $\mu_1 = -\mu_n = |\mu_2| = \cdots = |\mu_{n-1}| = 1$. Since $\sum_{i=1}^{n} \mu_i = 0$, the multiplicity s_1 of $\mu_1 = 1$ must be equal to $\frac{n}{2}$. Hence G is the direct sum of $s_1 = \frac{n}{2}$ complete graphs of order $\mu_1 + 1 = 2$. Namely, G is $\frac{n}{2}K_2$.

Case 2. G has three distinct eigenvalues

Since G is a bipartite graph, we must have that $\mu_1 = -\mu_n \neq 0$ and $\mu_2 = \cdots = \mu_{n-1} = 0$. Thus $E(G) = 2\mu_1$. From Lemma 3, we have that $2\mu_1 \geq 2\sqrt{m}$. Thus $2\mu_1^2 \geq 2m$. Notice that $2m = \sum_{i=1}^{n} \mu_i^2 = 2\mu_1^2$. Therefore $\mu_1 = \sqrt{m}$ and $E(G) = 2\sqrt{m}$. Hence by Lemma 3 G is a complete bipartite graph plus arbitrarily many isolated vertices. Namely, there exist integers $r_1 \geq 1$ and $r_2 \geq 1$ such that G is $K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, where $r_1 r_2 = m$.

Case 3. G has four distinct eigenvalues

Since $\mu_1 = -\mu_n$, $|\mu_2| = \cdots = |\mu_{n-1}|$, and G has four distinct eigenvalues, the multiplicity of μ_1 must be one. Hence we have by Lemma 1$'$ that G is a connected bipartite graph with $V = \{ v_1, v_2, ..., v_s\} \cup \{ v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$, and has four distinct eigenvalues $(\sqrt{p_x p_y}, (\frac{M_{2k} - 2(p_x p_y)^k}{n-2})^{\frac{1}{2k}}, -(\frac{M_{2k} - 2(p_x p_y)^k}{n-2})^{\frac{1}{2k}}, -\sqrt{p_x p_y})$, where $p_x = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$, $p_y = \frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$ and $\sqrt{p_x p_y} > (\frac{M_{2k}}{n})^{\frac{1}{2k}}$. $\square$

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