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Generalized Universal Theorem of Isometric Embedding

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Abstract

Universal theorem is an important theorem, in this paper, Figiel's lemma that concerns the isometric embedding from a Banach space X into the continuous space $C(L)$ are extended. That is a form of extension of universal theorem.

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1 Introduction

In 1932, Mazur-Ulam [1] established the following problem: *if X, Y are normed linear spaces and $f : X \to Y$ is an surjective isometric mapping, then f is an affine mapping*. The "surjection" is a necessary condition. As a matter of fact, there is an example[2] which shows that the above result may not be true without this condition. When is this affinity result true without the "surjection" ? Many authors today investigate the cases not restricted to this surjection condition.

In 1968, T.Figiel[3,4] extended Mazur-Ulam's theorem and proved the following theorem.

Theorem 1.1. *Suppose that X, Y are both real Banach spaces.If $F : X \to Y$ is an isometry mapping and $F(0) = 0$, then there exists a continuous linear mapping $f : \overline{spanF(X)} \mapsto X$, which satisfies $f \circ F = i_X$, and that f is unique and $||f||_{span\, F(x)}|| = 1$.*

In part 2 we introduce Figiel's lemma and establish an extension of it, and we finish the proof in part 3.

2 Extension of Universal Theorem

Universal theorem[4] is very important in Banach space:
(1). *Any separable Banach space X can be isometrically embedded to $C[0,1]$.*
(2). *Any Banach space X can be isometrically embedded to $C(K)$. (Here K is the unit ball of X^*, which is w^*- compact set.)*
The following two properties come from Figiel[4].

Property 2.1[4] *Suppose that X, Y are two real Banach spaces a is one of smooth points of the spheres $\{x \in X, ||x|| = ||a||\}$, and $F : X \to Y$ is an isometry mapping, such that $F(0) = 0$, and besides there exists $f \in Y^*, ||f|| = 1$, such that for any $r \in \mathbf{R}$, $f(F(ra)) = r||a||$, then $f \circ F = f_a$, where f_a is a support function at a.*

Property 2.2.[4] Suppose that Y is a real normed linear space, $\mathbf{R}$ is a real space, and $F : \mathbf{R} \to \mathbf{Y}$ is an isometric embedding. Then there exists $f \in Y^*, ||f|| = 1$ such that $f \circ F = id_{\mathbf{R}}$. Here, $id_{\mathbf{R}}$ is an identity operator on $\mathbf{R}$.

Now,we extended **Property 2.2,** as follows:

Theorem 2.3. *Suppose that X, Y are two real Banach spaces, $F : X \to Y$ is an isometry mapping and $F(0) = 0$. Then,*
i) There exists $f \in Y^*, ||f|| = 1$, such that $f \circ F \in S(X^*)$.
ii)$f \circ F = id_{\mathbf{R}}$ as $X = \mathbf{R}$.
iii) There exists a compact L_X^Y(w^*-topology), such that $U : X \mapsto C(L_X^Y)$, $U(x) = F(x)|_{L_X}$ is an linear isometric mapping. This iii) can be said to **generalized of Universal theorem.**

Remark 2.4[4]. If $X = Y$, then **ii)** in **universal theorem** is obvious by Theorem 2.3.

3 Proof of Theorem

The proof **Theorem 2.3.** need **Mazur's Theorem**[5]: " *Suppose that A is a solid closed convex set of a separable Banach space. Then the smooth points set $sm(A)$ of A, is the residual set of a bounded $\partial(A)$, and $sm(A)$ is dense in $\partial(A)$.*"

Thanks to R.Villa [5] we give **Proof of Theorem 2.3.**
Proof:

Section 1: Suppose X is a separable space. Let $a \in X$ be the smooth point of $\{x \in X, ||x|| = ||a||\}$, and f_a be the support function at a, i.e. $||f_a|| = 1$, $f_a(a) = ||a||$. Then for $\forall n \in \mathbf{N}$, by Hahn-Banach Theorem, there exists

$g_n \in Y^*, ||g_n|| = 1$, so that

$$g_n(F(na) - F(-na)) = ||F(na) - F(-(na))|| = 2n||a||,$$

and for $\forall r, |r| \leq n$, we obtain

$$\begin{aligned} 2n||a|| &\leq |g_n(F(na) - F(ra))| + |g_n(F(ra) - F(-na))| \\ &\leq ||F(na) - F(ra)|| + ||F(ra) - F(-na)|| \\ &= |n - r| \cdot ||a|| + |n + r| \cdot ||a|| = 2n||a||. \end{aligned}$$

This implies that

$$(n - r)||a|| = |g_n(F(na) - F(ra))|. \tag{1}$$

If $r = 0$, then $|g_n(F(na))| = n||a||$. Let us take $\varepsilon_n \in \{-1, 1\}$ such that

$$\varepsilon_n g_n(F(na)) = n||a||. \tag{2}$$

Moreover, for any $|r| \leq n$, we claim that

$$\varepsilon_n g_n(F(ra)) = r||a||.^1 \tag{3}$$

Indeed,
$\varepsilon_n(g_n(F(na)) - g_n(F(ra))) = n||a|| - \varepsilon_n g_n F(ra) \geq n||a|| - |g_n F(ra)| \geq (n - |r|)||a|| \geq 0$.

From (1),
$\varepsilon_n(g_n(F(na)) - g_n(F(ra))) = |(g_n(F(na)) - g_n(F(ra)))| = (n - r)||a||$, and from (2) we show that (3)is true.

Because $B(Y^*)$ is w^*-compact, there is $g_a \in B(Y^*)$ which is a cluster point of $\{g_n\}$ such that

$$g_a(F(ra)) = r||a|| \quad (\forall r \in R). \tag{4}$$

By Property 2.1, we can show that $g_a \circ F = f_a$, and we denote that $f = g_a$. Thus complete the proof of i) under the condition of X is separable.

For ii), we can let $\|a\| = 1$, then by (4) $g_1 \circ F(r) = r$, this also extend Property 2.2.

Next, we will prove that iii). Let

$$L_X^Y = \{g \in B(Y^*),\ g \circ F \in B(X^*)\} \tag{5}$$

[1] In Lemma 9.4.4 of [4]-p404, we can find similarity result of the space of reals R, In [4]-p407 give a general result but no proof of it.

then L_X^Y is a w^*-compact. Let

$$U : X \to C(L_X^Y), \;\; U(x) = F(x)|_{L_X^Y} \;\; (x \in X). \tag{6}$$

Obviously, $||U(x)|| \le ||F(x)|| = ||x||$ and U is a continuous linear mapping on X.

In fact, for any $x, y \in X, \alpha, \beta \in \mathbf{R}, g \in L_X^Y$,

$U(\alpha x + \beta y)(g) = F(\alpha x + \beta y)(g) = g \circ F(\alpha x + \beta y) = \alpha g \circ F(x) + \beta g \circ F(y) = \alpha U(x)(g) + \beta U(y)(g)$.

Since X is separable, then there exists $g_a \in S(Y^*)$ such that $|g_a(F(a))| = \|a\|$ for any smooth point $a \in X$. Hence $\|U(a)\| = \|a\|$ and again Mazur's theorem implies that the smooth points set of X is dense. Thus

$$||U(x)|| = ||x||. \tag{7}$$

Thus complete the proof of iii) under the condition of X is separable.

Section 2: If X is not separable, let $\Xi = \{X_\gamma : X_\gamma (\subset X) \; is \; separable\}$. Again

$$U_\gamma : X_\gamma \to C(L_{X_\gamma}^Y), \;\; U_\gamma(x) = F(x)|_{L_{X_\gamma}^Y} \;\; (x \in X_\gamma).$$

If $X', X'' \subset X$ are two separable subspaces, then $\overline{X' + X''}$ is also separable, and

$$L_{\overline{X'+X''}} \subset L_{X'} \cap L_{X''}.$$

Then the w^*-closed set family $\{L_{X_\gamma}^Y : X_\gamma \subset X, \;\; X_\gamma$ is separable $\}$ have finite intersection property. Because that $B(Y^*)$ is w^*- compact, $L_X^Y = \{g \in B(Y^*), \; g \circ F \in B(X^*)\}$, so

$$\cap \{L_{X_\gamma}^Y : X_\gamma \subset X, X_\gamma \; is \; separable\} \neq \emptyset$$

and

$$L_X^Y = \cap \{L_{X_\gamma}^Y : X_\gamma \subset X, X_\gamma \; is \; separable\}.$$

In fact, if $g \in L_X^Y$, then $g \circ F \in X^*$, so $g \circ F \in X_\gamma^*$ for any subsets $X_\gamma \subset X$.

Conversely, if $g \in L_{X_\gamma}^Y$ for any γ, then $g \circ F$ is a continuous linear functional on any X_γ, thus for any $x_1, x_2 \in X$, we set $X_1 = \overline{span\{x_1, x_2\}}$.
We easily prove that $g \circ F$ is a linear functional on X_1, furthermore, $g \circ F \in X^*$.

Next, we claim that

$$||F(x)|_{L_X^Y}|| = ||x||, \; \forall x \in X.$$

In fact, let $x \in X$, and set

$$\Xi' = \{X_\gamma : x \in X_\gamma, X_\gamma \; is \; separable\}.$$

Then Ξ' is a direction set which is non-empty, by (6) and (7) above **Section 1**

$$||U_{X_\gamma}(x)|| = ||x||.$$

For any $X_\gamma \in \Xi'$ there exists $g_\gamma \in L^Y_{X_\gamma}$ such that

$$|g_\gamma \circ F(x)| = ||x||.$$

so

$\{g_\gamma : X_\gamma \in \Xi'\}$ is a net in $B(Y^*)$. Then there is a w^*- closed point $g_0 \in B(Y^*)$.

Hence for any V which is a w^*- neighborhood of g_0, there exists $X_\beta \in \Xi$, such that $X_\beta \supseteq X_\gamma$ and $g_\beta \in V$.

Now, we will prove that $g_0 \in L^Y_X$. In fact, for any $X_\gamma \in \Xi, g_0 \in L^Y_{X_\gamma}$. Otherwise,

assume that $\widetilde{X_0} \in \Xi$, but $g_0 \notin L^Y_{\widetilde{X_0}}$. Because $L^Y_{\widetilde{X_0}}$ is w^*-closed set, then we can find a w^*-neighborhood V_0 of g_0 such that

$$V_0 \cap L^Y_{\widetilde{X_0}} = \emptyset.$$

Since g_0 is a w^*-closed point of $\{g_\gamma : X_\gamma \in \Xi'\}$, then there exists $X_\gamma \in \Xi'$ such that $X_\gamma \supseteq \widetilde{X_0}$ and $g_\gamma \in V_0$, therefore $g_\gamma \in L^Y_{X_\gamma} \subset L^Y_{\widetilde{X_0}}$, which implies that $g_0 \in L^Y_{\widetilde{X_0}}$, leading to a contradiction.

According to (6), (7), and $|g_0 \circ F(x)| = ||x||$, then $||F(x)|_{L^Y_X}|| = ||x||$, U is an isometry and $||g_0 \circ F|| = 1$. Completing the proof.

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